DUALITY FOR THE de RHAM COHOMOLOGY OF AN ABELIAN SCHEME

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In this paper we will demonstrate the equality of three different pairings between the first de Rham cohomology group of an abelian scheme over a base flat over $\mathbb{Z}$ and that of its dual. Two of these pairings have been discussed explicitly in the literature, for example in [BBM] and [D], the third is implicit in [MM] and all three, as well as their equality, are used in [C-UVB]. That these pairings are equal is a fact undoubtedly well known to the experts, but we feel that a proof of their equality should be made accessible to non-experts like the author of this article.

In the last section we deduce a generalization to arbitrary characteristic of Serre's formula for the pairing on the first de Rham cohomology group of a curve over a field of characteristic zero.

1.1 The definitions of the pairings

Fix a scheme S either reduced or flat over $\mathbb{Z}$. We will suppress S in the notation. For example, if $f : X \to S$ is a smooth scheme over S, we will set $\Omega^i_X = \Omega^i_{X/S}$, $\mathcal{H}^i_{DR}(X) = \mathbb{R}^i_{f_*}(\Omega_{X/S})$ and $\mathbb{R}^i = \mathbb{R}^i_{f_*}$ in general. Let $\pi : A \to S$ be an Abelian scheme over S. Let $\hat{\pi} : \hat{A} \to S$ be its dual. Then it is well known that there is a canonical perfect pairing between $\mathcal{H}^1_{DR}(A)$ and $\mathcal{H}^1_{DR}(\hat{A})$ onto $\mathcal{O}_S$. (See *Theorie de Dieudonné Cristalline II*, §5 [BBM].) For an $\mathcal{O}_S$-module $\mathcal{S}$, we set $\mathcal{S}^\vee = \mathrm{Hom}_{\mathcal{O}_S}(\mathcal{S}, \mathcal{O}_S)$. We will now define three pairings between $\mathcal{H}^1_{DR}(\hat{A})^\vee$ and $\mathcal{H}^1_{DR}(A)^\vee$ onto $\mathcal{O}_S$.

First, let $\mathcal{P}$ denote the Poincaré sheaf on $\hat{A} \times A$. Let $c(\mathcal{P})$ denote its Chern class in the global sections of $\mathcal{H}^2_{DR}(\hat{A} \times A)$. Then identifying $\mathcal{H}^2_{DR}(\hat{A} \times A)$ with $\mathcal{H}^2_{DR}(\hat{A}) \oplus \big(\mathcal{H}^1_{DR}(\hat{A}) \otimes \mathcal{H}^1_{DR}(A)\big) \oplus \mathcal{H}^2_{DR}(A)$ using the Kunneth formula, it is easy to see that $c(\mathcal{P})$ is actually a global section of $\mathcal{H}^1_{DR}(\hat{A}) \otimes \mathcal{H}^1_{DR}(A)$. This yields our first pairing, $(\ ,\ )_{\mathrm{Chern}}$.

Second, let G and $\hat{G}$ denote the universal vectorial extensions of A and $\hat{A}$ over S. We will also let π and $\hat{\pi}$ denote the structural morphisms from G and $\hat{G}$ to S. Let $\mathcal{P}^{\#}$ denote the pull-back of $\mathcal{P}$ to $\hat{G} \times G$. Then $\mathcal{P}^{\#}$ has a canonical connection whose curvature form is an invariant two-form ω on $\hat{G} \times G$ (see Deligne, Hodge IIII 10.2.7.2 [D] as well as the discussion below). Identifying the Lie algebras, $\mathrm{Lie}(G)$ and $\mathrm{Lie}(\hat{G})$, of G and $\hat{G}$ over S, with $\mathcal{H}^1_{DR}(A)^\vee$ and $\mathcal{H}^1_{DR}(\hat{A})^\vee$ (see Theorem 1.2.2(i) below) and $\mathrm{Lie}(\hat{G} \times G)$ with $\mathrm{Lie}(\hat{G}) \oplus \mathrm{Lie}(G)$, our second pairing $(\ ,\ )_{\mathrm{Curv}}$ is described by

$$(v, w)_{\mathrm{Curv}} = \omega((v, 0), (0, w))$$

for v a local section of $\mathrm{Lie}(\hat{G})$ and w a local section of $\mathrm{Lie}(G)$.

Third and finally, $\mathrm{Lie}(\hat{G})$ is canonically isomorphic to $\mathcal{H}^1_{DR}(A)$ (see §1.4 and Mazur-Messing §4) and, as we will show below (Theorem 1.2.2(i)), the coherent sheaf on S of invariant one-forms of $\hat{G}$ over S, $\mathrm{Inv}(\Omega^1_{\hat{G}})$, is naturally isomorphic to $\mathcal{H}^1_{DR}(\hat{A})$. Thus we get a perfect pairing between $\mathcal{H}^1_{DR}(\hat{A})$ and $\mathcal{H}^1_{DR}(A)$ and so a third pairing, $(\ ,\ )_{\mathrm{Vec}}$ between $\mathcal{H}^1_{DR}(\hat{A})^\vee$ and $\mathcal{H}^1_{DR}(A)^\vee$.

The main result of this paper will be

Theorem 1.1.1. $(\ ,\ )_{\mathrm{Chern}} = (\ ,\ )_{\mathrm{Curv}} = (\ ,\ )_{\mathrm{Vec}}$.

Remarks. Because of the non-degeneracy of $(\ ,\ )_{\mathrm{Vec}}$, once we show these pairings are the same, we will have verified their non-degeneracy (the non-degeneracy of $(\ ,\ )_{\mathrm{Chern}}$ is also verified in [BBM] §5.1). We suspect this is true for an arbitrary base (see the remarks at the end of §1.4).

Before we embark on the proof we will need to establish some facts about vectorial extensions.

1.2 Universal vectorial extensions

We first recall the definition of a vector group over S. If $\mathcal{S}$ is a quasi-coherent sheaf on S, then we let $V(\mathcal{S}) = \mathrm{Spec}S(\mathcal{S})$ where $S(\mathcal{S})$ is the symmetric $\mathcal{O}_S$-algebra of $\mathcal{S}$. Then

$V(\mathcal{S})$ is naturally a group scheme over S, but it has more structure. Namely, $V(\mathcal{O}_S)$ is a ring scheme over S and $V(\mathcal{S})$ is a $V(\mathcal{O}_S)$-module over S. Any S scheme V which is a $V(\mathcal{O}_S)$ module is called a vector group over S and every such V is of the form $V(\mathcal{S})$ where $\mathcal{S} = \mathrm{Hom}_{V(\mathcal{O}_S)}(V, V(\mathcal{O}_S))$. Although $\mathbb{G}_a$ usually just denotes the additive group scheme, we will also use it to denote the vector group $V(\mathcal{O}_S)$, when convenient, since the two are canonically isomorphic as group schemes.

Let A and G be as above. A vectorial extension of A is a group scheme over S which is an extension of A by a group scheme with the structure of a vector group. A morphism of vectorial extensions of A is called vectorial if it is a morphism of extensions of group schemes and its restriction to the vector groups is a $V(\mathcal{O}_S)$-module homomorphism. The group G is an extension of A by the vector group $W = V(R^1(\mathcal{O}_A)^\vee\ )$. Being universal means that if E is an extension of A by vector group V there exists a unique vectorial morphism from G to E such that E is the pushout of corresponding morphism from W to V.

Let $\mathrm{Inv}(\Omega^i_G)$ denote the sheaf of invariant i-forms on G over S.

Lemma 1.2.1. *The sections of* $\mathrm{Inv}(\Omega^i_G)$ *are closed.*

Proof. First, we suppose η is a section of $\mathrm{Inv}(\Omega^1_G)$ and ∂_1 and ∂_2 are invariant derivations on G over S, then (η, ∂_i) is a section of $\mathcal{O}_S$ (where, here, $(\ ,\ )$ is the natural pairing between one-forms and derivations) and so $d\eta(\partial_1, \partial_2) = \partial_1(\eta, \partial_2) - \partial_2(\eta, \partial_1) = 0$, where $\eta(\ ,\ )$ denotes the alternating form on derivations associated to η. It follows that $d\eta = 0$ since the invariant derivations of G over S span the sheaf of all derivations of G over S over $\mathcal{O}_G$. The lemma follows from this as $\mathrm{Inv}(\Omega^i_G) = \Lambda^i \mathrm{Inv}(\Omega^1_G)$. □

In particular, there is a natural map from $\mathrm{Inv}(\Omega^i_G)$ into $\mathcal{H}^i_{DR}(G)$. Let g denote the natural morphism from G to A.

Theorem 1.2.2. (i) *There exists a canonical isomorphism* $\mathcal{H}^i_{DR}(A) \to \mathrm{Inv}(\Omega^i_G)$ *such that the following diagram commutes:*

$$\begin{array}{ccc} \mathcal{H}^i_{DR}(A) & \longrightarrow & \mathrm{Inv}(\Omega^i_G) \\ & & \downarrow \\ & & \mathcal{H}^i_{DR}(G)\ . \end{array}$$

(ii) *If S is flat over $\mathbb{Z}$, the map from $\mathcal{H}^i_{DR}(A)$ into $\mathcal{H}^i_{DR}(G)$ is an injection and an isomorphism if S is a $\mathbb{Q}$ scheme.*

PROOF. To prove (i) we may suppose S is affine equal to $\mathrm{Spec}(R)$. Let $\mathcal{C}$ be an ordered affine open covering of A. Suppose $\omega = (\{\omega_U\}, \{f_{U,V}\})$ is a hyper-one-cocycle for the complex $\Omega^{\cdot}_A$ with respect to $\mathcal{C}$. Let H denote the $\mathbb{G}_a$-torseur over A obtained by gluing together the affines $\ddot{U} := U \times \mathrm{Spec}(R[x_U])$ with respect to the gluing data $x_U - x_V = f_{U,V}$ above $U \cap V$. It follows that H is an extension of A by $\mathbb{G}_a$ (see [M-M] chap.1 §1). Let $h : H \to A$ denote the projection. Then $v = \{\ddot{U} \mapsto h^*\omega_U - dx_U\}$ is a global section of Ω^1_H invariant under $\mathbb{G}_a$. The map from H into global sections of Ω^1_H which sends a point a on H to $T_a^*v - v$ (here T_a denotes translation by a) factors through A and so must be constant and hence zero (see Lemma 1.8 of [M-M]). It follows that v is an element of $\mathrm{Inv}(\Omega^i_H)$ which pulls back to dx on $\mathbb{G}_a$. Now by the universal property of G there exists a unique vectorial homomorphism $w : W \to \mathbb{G}_a$ such that H is the pushout of G by w. Let $\tilde{w} : G \to H$ denote the induced homomorphism and let $v(\omega) = \tilde{w}^*(v)$. Then by its construction $v(\omega)$ lies in $\mathrm{Inv}(\Omega^1_G)$ and, considered as a hyper-chain for $\Omega^{\cdot}_G$, differs from $g^*\omega$ by a hyper-coboundary. We claim that the map $\omega \mapsto v(\omega)$ induces an isomorphism from $\mathcal{H}^1_{DR}(A)$ onto $\mathrm{Inv}(\Omega^1_H)$.

First we check that ν is a homomorphism. Suppose ω' is another hyper-one-cocycle. Let (H', v') and (H'', v'') be the pairs of $\mathbb{G}_a$-torseurs over A and invariant differentials corresponding to ω' and $\omega + \omega'$. Then H'' is the Baer sum of H and H' so is the quotient of $G =: H \times_A H'$ by the action of $\mathbb{G}_a$. Let p_1, p_2 and t be the maps from G to H, H' and H''. The additivity of ν follows from the fact that

$$t^*v'' = p_1^*v + p_2^*v'$$

which follows from the fact that, in an evident notation, $t^*x''_U = p_1^*x_U + p_2^*x'_U$.

Now, suppose $\omega = (\{df_U\}, \{f_U - f_V\})$, the hyper-coboundary of the 0-chain $\{f_U\}$ for $\Omega^{\cdot}_A$ with respect to $\mathcal{C}$. Then $\{U \mapsto h^*f_U - x_U\}$ is a global section of $\mathcal{O}_H$ which gives a splitting of

$$0 \to \mathbb{G}_a \to H \to A \to 0 \ .$$

It follows that $w = 0$ and that $\ddot{w} = (g, 0)$. Since $v = dx$, $v(\omega) = 0$. This implies that $\omega \mapsto v(\omega)$ induces a homomorphism from $\mathcal{H}^1_{DR}(A)$ onto $\mathrm{Inv}(\Omega^1_H)$. Call this homomorphism

v.

To show that v is an isomorphism we will describe its inverse. Let η be a global section of $\mathrm{Inv}(\Omega^1_H)$. It follows that the pullback of η to W is equal to dx for a unique vectorial homomorphism $x : W \to \mathbb{G}_a$. Let $\mathcal{C}$ be be an ordered affine covering of A such that for each $U \in \mathcal{C}$, there exists a section $s_U : U \to G$ of g above U. It follows that for $U, V \in \mathcal{C}$, $F_{U,V} := s_U - s_V$ is a morphism from $U \cap V$ into W. Set $\omega_U = s_U^*\eta$ and $f_{U,V} = F_{U,V}^* x$. Using the fact that η is closed we see that $(\{\omega_U\}, \{f_{U,V}\})$ is a hyper-onecocycle for $\Omega^{\cdot}_A$ with respect to $\mathcal{C}$. Let $w(\eta)$ denote the class of this hyper-cocycle. It is easy to check that w is the inverse to v which establishes our claim. Hence the natural map $\mathcal{H}^1_{DR}(A)$ into $\mathcal{H}^1_{DR}(G)$ factors through an isomorphism into $\mathrm{Inv}(\Omega^1_G)$.

Now (i) follows from the compatibilities of the isomorphisms $\mathcal{H}^i_{DR}(A) \cong \Lambda^i \mathcal{H}^1_{DR}(A)$ and $\mathrm{Inv}(\Omega^i_G) \cong \Lambda^i \mathrm{Inv}(\Omega^1_G)$ with the appropriate maps.

Finally, if S is flat over $\mathbb{Z}$, then $h^i(A)$ injects into $h^1(A_{\mathbb{Q}}/S_{\mathbb{Q}})$ so we can assume S is a $\mathbb{Q}$-scheme. Then $\mathbb{R}^0_{g_*}(\Omega^{\cdot}_{G/A}) = \mathcal{O}_A$ and $\mathbb{R}^i_{g_*}(\Omega^{\cdot}_{G/A}) = 0$, $i > 0$ since for opens U of A over which g has a section, $G_U \cong U \times_S \mathbb{A}^n_S$. Hence (ii) follows from the Leray spectral sequence for $\mathbb{R}^{\cdot}_{\pi_*}(\Omega^{\cdot}_A)$ over A. $\square$

In particular, $\mathrm{Lie}(\hat{G})$ is canonically isomorphic to $\mathcal{H}^1_{DR}(\hat{A})^\vee$ and we will henceforth identify these two sheaves on S.

Remark. Part (ii) of the statement of this theorem is false in general. In fact, if S is the spectrum of a finite field, one can show that the kernel of the map from $\mathcal{H}^i_{DR}(A)$ into $\mathcal{H}^i_{DR}(G)$ is equal to the kernel of Vershiebung and the cokernel is infinite dimensional.

Lemma 1.2.3. *Suppose $f : X \to S$ is a scheme over S. Suppose $g : Y \to X$ is a $\mathbb{G}_a$-torseur over X corresponding to a global section ϕ of $R^1(\mathcal{O}_X)$ and $h = f \circ g$. Let i be an integer and suppose that the sequence*

$$R^{i-1}_{f_*}(\mathcal{O}_X) \longrightarrow R^i_{f_*}(\mathcal{O}_X) \longrightarrow R^{i+1}_{f_*}(\mathcal{O}_X),$$

where the arrows are $x \mapsto \phi \cup x$, is exact, that the image of $R^{i-1}_{f_*}(\mathcal{O}_X) \to R^i_{f_*}(\mathcal{O}_X)$ is divisible and the image of $R^i_{f_*}(\mathcal{O}_X) \to R^{i+1}_{f_*}(\mathcal{O}_X)$ is $\mathbb{Z}$-torsion free. Then the sequence

$$R^{i-1}_{f_*}(\mathcal{O}_X) \longrightarrow R^i_{f_*}(\mathcal{O}_X) \longrightarrow R^i_{h_*}(\mathcal{O}_Y) \longrightarrow 0$$

is exact where the second arrow is the natural one and the first is $y \in R^{i-1}_{f_*}(\mathcal{O}_X) \mapsto \phi \cup y$.

Proof. We may suppose S is affine. Let $\mathcal{C}$ be an affine open cover of X and $e = \{e_{U,V}\}$ be a one-cocycle with values in $\mathcal{O}_X$ representing ϕ. Then Y is obtained by gluing the affine schemes $\ddot{U} := U \times \mathrm{Spec}(\mathbb{Z}[x_U])$ according to the data, $x_U - x_V = e_{U,V}$ above $U \cap V$. Let x denote the 0-chain $\{x_U\}$ and x^j the cup product of x with itself j times.

Since Y is affine over X, $R^j_{h_*}(\mathcal{O}_Y)$ is isomorphic to $R^j_{f_*}(g_*\mathcal{O}_Y)$ under the natural map.

Suppose k is an m-chain on X with coefficients in $g_*\mathcal{O}_Y$ with respect to $\mathcal{C}$. Then we may write

$$k = \sum_{j=0}^{n} k_j \cup x^j$$

where k_j is an m-chain on X with coefficients in $\mathcal{O}_X$ (the products here and following are cup products). We say k is of degree at most n. Then

$$(*) \qquad \delta k = \delta k_n \cup x^n + (\delta k_{n-1} + (-1)^m ne \cup k_n) \cup x^{n-1} + (\text{terms of lower degree})$$

where we take $x^{-1} = 0$. Let κ be a class in $R^i_{f_*}(g_*\mathcal{O}_Y)$. Suppose $m = i$, k is a cocycle representing κ and n is the minimal degree of such a cocycle. Then it follows that k_n is a cocycle with coefficients in $\mathcal{O}_X$ and $ne \cup k_n$ is a coboundary. If $n > 0$ our hypothesis imply that $k_n = (n+1)e \cup r + \delta s$ for some $(i-1)$-cocycle r and some $(i-1)$-chain s, and so,

$$k + \delta(r \cup x^{n+1} + s \cup x^n)$$

has lower degree than k. Thus $n = 0$ and $R^i_{f_*}(\mathcal{O}_X) \to R^i_{h_*}(\mathcal{O}_Y)$ is surjective.

Now, to complete the proof, we must determine the kernel of $R^i_{f_*}(\mathcal{O}_X) \to R^i_{h_*}(\mathcal{O}_Y)$. Suppose α is an i-cocycle of degree 0 and $\alpha = \delta k$ for an $(i-1)$-cochain k. Then if the degree of k is greater than 1, we may, as above, change k by a coboundary to decrease the degree. Hence we suppose k has degree at most 1 so equals $k_1 \cup x + k_0$ and then it follows from $(*)$ that $\delta k_1 = 0$ and $\alpha = \delta k_0 - e \cup k_1$. Thus α represents an element in the image of $R^{i-1}_{f_*}(\mathcal{O}_X)$. □

Corollary 1.2.4. *If S is flat over $\mathbb{Z}$, $\pi_*(\mathcal{O}_G) = \mathcal{O}_S$.*

Proof. The scheme G may be expressed as a successive extention of non-trivial $\mathbb{G}_a$ torseurs starting from A. The corollary follows from the lemma in the case $i = 0$ by induction on the number of $\mathbb{G}_a$ torseurs using the facts that $\pi_*(\mathcal{O}_A) = \mathcal{O}_S$ and $R^1(\mathcal{O}_A)$ is a locally free $\mathcal{O}_S$-module. $\square$

It follows that $H^0(G, \Omega^1_G)_d$ injects into $h^1(G)$ and so if S is a $\mathbb{Q}$-scheme is equal to $\mathrm{Inv}(\Omega^{\cdot}_G)$. Since, $\Omega^i_{G/S}$ injects into $\Omega^i_{G_{\mathbb{Q}}/S_{\mathbb{Q}}}$ if S is flat over $\mathbb{Z}$,

Corollary 1.a. *If S is flat over $\mathbb{Z}$, $H^0(G, \Omega^1_G)_d = \mathrm{Inv}(\Omega^{\cdot}_G)$.*

For any morphism $t : Z \to S$, let $R_{t_*}(\mathcal{O}_Z)$ denote the sheaf of graded algebras $\oplus_i R^i(\mathcal{O}_Z)$.

Corollary 1.2.6. *If S is a $\mathbb{Q}$-scheme and the left annihilator of ϕ in $R_{f_*}(\mathcal{O}_X)$ is $R_{f_*}(\mathcal{O}_X) \cup \phi$, then $R_{h_*}(\mathcal{O}_Y) \cong R_{f_*}(\mathcal{O}_X)/(\phi)$.*

Proof. Clearly, in these circumstances, the hypotheses of the lemma are satisfied for all i. $\square$

Corollary 1.2.7. *If S is a $\mathbb{Q}$ scheme, then $R_{\pi_*}(\mathcal{O}_G) = \mathcal{O}_S$.*

Proof. This follows from the previous corollary by induction using the facts that $\pi_*(\mathcal{O}_A) = \mathcal{O}_S$, $R^1(\mathcal{O}_A)$ is a free O_S-module and $R^i(\mathcal{O}_A) = \Lambda^i R^1(\mathcal{O}_A)$. $\square$

Remark. One can use the arguments in the lemma to show that the conclusions of Corollaries 1.2.4 and 1.2.7 do not hold for arbitrary S.

1.3 The Equality of $(\ ,\ )_{\mathrm{Chern}}$ and $(\ ,\ )_{\mathrm{Curv}}$

Now we will show that $(\ ,\ )_{\mathrm{Chern}} = (\ ,\ )_{\mathrm{Curv}}$ when S is flat over $\mathbb{Z}$ (which we do not yet assume). First we will describe $(\ ,\ )_{\mathrm{Curv}}$ more precisely. Let $\mathcal{P}^\#$ denote the pullback of $\mathcal{P}$ to $\hat{G} \times G$. For a product $X \times X$, $p_{X,i}$ will denote the projection onto the i-th factor. If X is a group, μ_X will denote the addition law. By the theorem of the square, there are isomorphisms

$$v_{\hat{G}} : (\mathrm{id} \times p_{G,1})^* \mathcal{P}^\# \otimes (\mathrm{id} \times p_{G,2})^* \mathcal{P}^\# \to (\mathrm{id} \times \mu_G)^* \mathcal{P}^\# \qquad \text{on } \hat{G} \times G \times G$$

and

$$v_G : (p_{\hat{G},1} \times \mathrm{id})^* \mathcal{P}^\# \otimes (p_{\hat{G},2} \times \mathrm{id})^* \mathcal{P}^\# \to (\mu_{\hat{G}} \times \mathrm{id})^* \mathcal{P}^\# \qquad \text{on } \hat{G} \times \hat{G} \times G\ .$$

Using the universal property of $\hat{G}$ there is a canonical integrable connection relative to $\hat{G}$, $\nabla_{\hat{G}} : \mathcal{P}^{\#} \to \Omega^1_{\hat{G}\times G/\hat{G}} \otimes \mathcal{P}^{\#}$ such that both v_G and $v_{\hat{G}}$ are horizontal. Similarly, there is a canonical integrable connection $\nabla_G : \mathcal{P}^{\#} \to \Omega^1_{\hat{G}\times G/G} \otimes \mathcal{P}^{\#}$ such that both v_G and $v_{\hat{G}}$ are horizontal. Since $\Omega^1_{\hat{G}\times G}$ is naturally isomorphic to the direct sum of $\Omega^1_{\hat{G}\times G/\hat{G}}$ and $\Omega^1_{\hat{G}\times G/G}$ there is a connection $\nabla : \mathcal{P}^{\#} \to \Omega^1_{\hat{G}\times G} \otimes \mathcal{P}^{\#}$ which restricts to both $\nabla_{\hat{G}}$ and ∇_G and is such that v_G and $v_{\hat{G}}$ are horizontal. Let ω denote the curvature form of ∇. It follows that on $\hat{G} \times \hat{G} \times G \times G$,

$$(\mu_{\hat{G}} \times \mu_G)^*\omega = (p_{\hat{G},1} \times p_{G,1})^*\omega + (p_{\hat{G},1} \times p_{G,2})^*\omega + (p_{\hat{G},2} \times p_{G,1})^*\omega + (p_{\hat{G},2} \times p_{G,2})^*\omega$$

and since $\nabla_{\hat{G}}$ and ∇_G are integrable the images of ω in $\Omega^2_{\hat{G}\times G/\hat{G}}$ and $\Omega^2_{\hat{G}\times G/G}$ are zero. We deduce that $(\mu_{\hat{G}} \times \mu_G)^*\omega - (p_{\hat{G},1} \times p_{G,1})^*\omega$ is zero when considered as a relative differential with respect to $p_{\hat{G},2} \times p_{G,2}$. In other words, ω is an invariant 2-form on $\hat{G} \times G$ and hence gives a pairing on $\mathrm{Lie}(\hat{G} \times G) \cong \mathrm{Lie}(\hat{G}) \oplus \mathrm{Lie}(G)$. The pairing $(\ ,\ )_{\mathrm{Curv}}$ is now defined as above. Henceforth $\mathcal{P}^{\#}$ will denote the pair $(\mathcal{P}^{\#}, \nabla)$.

Since ω maps to zero in $\Omega^2_{\hat{G}\times G/\hat{G}}$ and $\Omega^2_{\hat{G}\times G/G}$,

$$(v, w')_{\mathrm{Curv}} - (v', w)_{\mathrm{Curv}} = \omega((v, w), (v', w'))$$

where v and v' are sections of $\mathrm{Lie}(\hat{G})$ and w and w' are sections of $\mathrm{Lie}(G)$. Moreover, ω is a global section of the image of $\mathrm{Inv}(\Omega^1_{\hat{G}}) \otimes \mathrm{Inv}(\Omega^1_G)$. It is clear that $(\ ,\ )_{\mathrm{Curv}}$ is also the pairing corresponding to this tensor.

Suppose $X \to Y$ is a morphism of S-schemes. Then invertible sheaves on X relative to Y with connection are represented by global sections of $\mathbb{R}^1(\mathcal{O}^*_X \xrightarrow{d\log} \Omega^1_{X/Y})$ while invertible sheaves on X with integrable connection relative to Y are represented by global sections of $\mathbb{R}^1(\Omega^*_{X/Y})$ where $\Omega^*_{X/Y}$ denotes the multiplicative de Rham complex,

$$\mathcal{O}^*_X \xrightarrow{d\log} \Omega^1_{X/Y} \xrightarrow{d} \Omega^2_{X/Y} \xrightarrow{d} \dots\ .$$

Let $X = \hat{G} \times G$. Taking hyper-cohomology of the commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & F^2(\Omega^{\cdot}_X) & \longrightarrow & \Omega^*_X & \longrightarrow & (\mathcal{O}^*_X \xrightarrow{d\log} \Omega^1_X) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & F^1(\Omega^{\cdot}_X) & \longrightarrow & \Omega^*_X & \longrightarrow & \mathcal{O}^*_X & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & F^1(\Omega^{\cdot}_{\hat{A}\times A}) & \longrightarrow & \Omega^*_{\hat{A}\times A} & \longrightarrow & \mathcal{O}^*_{\hat{A}\times A} & \longrightarrow & 0
\end{array}$$

in which the rows are exact, yields a commutative diagram:

$$\begin{array}{ccccc}
\mathbb{R}^1(\mathcal{O}_X^* \xrightarrow{d\log} \Omega_X^1) & \longrightarrow & \mathbb{R}^2(F^2(\Omega_X^{\cdot})) & \longrightarrow & \mathcal{H}^2_{DR}(X) \\
\downarrow & & \downarrow & & \downarrow \\
R^1(\mathcal{O}_X^*) & \longrightarrow & \mathbb{R}^2(F^1(\Omega_X^{\cdot})) & \longrightarrow & \mathcal{H}^2_{DR}(X) \\
\uparrow & & \uparrow & & \uparrow \\
R^1(\mathcal{O}^*_{\hat{A}\times A}) & \longrightarrow & \mathbb{R}^2(F^1(\Omega^{\cdot}_{\hat{A}\times A})) & \longrightarrow & \mathcal{H}^2_{DR}(\hat{A}\times A)
\end{array}$$

where for a complex $\mathcal{S}$, $F^i(\mathcal{S})$ denotes the subcomplex, such that $F^i(\mathcal{S})^j = 0$ for $j < i$ and $F^i(\mathcal{S})^j = \mathcal{S}^j$ for $j \geq i$.

Regard, for now, $\mathcal{P}$ as a global section of $R^1(\mathcal{O}^*_{\hat{A}\times A})$ and $\mathcal{P}^\#$ as a global section of $\mathbb{R}^1(\mathcal{O}_X^* \xrightarrow{d\log} \Omega_X^1)$. The fact that as a sheaf $\mathcal{P}^\#$ is the pullback of $\mathcal{P}$ implies that the image in $R^1(\mathcal{O}_X^*)$ of $\mathcal{P}$ is equal to the image of $\mathcal{P}^\#$. Moreover the image of $\mathcal{P}$ in $\mathcal{H}^2_{DR}(\hat{A}\times A)$ is its Chern class while the image of $\mathcal{P}^\#$ in $\mathbb{R}^2(F^2(\Omega_X^*))$ is the image is its curvature under the injection from $\mathrm{Inv}(\Omega_X^2)$. The equality of $(\ ,\ )_{\mathrm{Curv}}$ with $(\ ,\ )_{\mathrm{Chern}}$ when S is flat over $\mathbb{Z}$ now follows from Theorem 1.2.2 which implies that the maps $\mathrm{Inv}(\Omega_X^2) \to \mathcal{H}^2_{DR}(X)$ and $\mathcal{H}^2_{DR}(\hat{A}\times A) \to \mathcal{H}^2_{DR}(X)$ are injections, and so the curvature is the image of the Chern class.

1.4 The Equality of $(\ ,\ )_{\mathrm{Curv}}$ and $(\ ,\ )_{\mathrm{Vec}}$

The pairings $(\ ,\ )_i$, where i equals Curv or Vec, induce homomorphisms from $\mathrm{Lie}(\hat{G})$ to $\mathcal{H}^1_{DR}(A)$ which we will denote by h_i. We will now show $(\ ,\ )_{\mathrm{Curv}} = (\ ,\ )_{\mathrm{Vec}}$ when S is flat over $\mathbb{Z}$ by showing $h_{\mathrm{Curv}} = h_{\mathrm{Vec}}$. The argument was motivated by the proof of Théorème 5.1.6 of [BBM], which is a similar but weaker result.

By a pointed invertible sheaf on A, we mean an invertible sheaf $\mathcal{L}$ such that $e^*\mathcal{L}$ is trivial where $e : S \to A$ is the origin. The scheme $\hat{G}$ classifies isomorphism classes of pointed invertible sheaves on A algebraically equivalent to zero with integrable connection (see [M-M]). If S if flat over $\mathbb{Z}$, the existence of an integrable connection on an invertible sheaf implies that it is algebraically equivalent to zero. I.e., in this case, for an S scheme T, $\hat{G}(T)$ is the group of isomorphism classes of pointed invertible sheaves on $T \times \hat{G}$ with integrable connection relative to T. It follows that when S is flat over $\mathbb{Z}$,

$$\hat{G}(T) = \mathrm{Ker}\big(\mathbb{R}^1(\Omega^*_{T\times A/T}) \to R^1(T, \mathcal{O}^*_T)\big) \ .$$

Let $S[\varepsilon]$ equal the scheme of dual numbers over S and $A[\varepsilon] = S[\varepsilon]\times_S A$. For $u \in \mathrm{Lie}_{\hat{G}}(S)$, let $f_u : S[\varepsilon] \to \hat{G}$ denote the corresponding element of $\hat{G}(S[\varepsilon])$.

Taking hyper-cohomology of the exact sequence

$$0 \longrightarrow \Omega^1_{S[\varepsilon]} \otimes \Omega^{\cdot}_A \longrightarrow \Omega^*_{A[\varepsilon]} \longrightarrow \Omega^*_{A[\varepsilon]/S[\varepsilon]} \longrightarrow 0$$

we get a map

$$(*) \qquad \mathbb{R}^1(\Omega^*_{A[\varepsilon]/S[\varepsilon]}) \longrightarrow \mathbb{R}^2(\Omega^1_{S[\varepsilon]} \otimes \Omega^{\cdot}_A) \cong \Omega^1_{S[\varepsilon]} \otimes \mathcal{H}^1_{DR}(A) \ .$$

Let $\partial/\partial\varepsilon$ denote the derivation on $S[\varepsilon]$, $a + b\varepsilon \mapsto b$, and also, when appropriate, the corresponding homomorphism $\Omega^1_{S[\varepsilon]} \to \mathcal{O}_S$. The image of the element of $\mathbb{R}^1(\Omega^*_{A[\varepsilon]/S[\varepsilon]})$ corresponding to f_u under the composition of $(*)$ with $\partial/\partial\varepsilon$ is, by definition, $h_{\mathrm{Vec}}(u)$.

Let $(\mathcal{L}, \nabla)$ denote the invertible sheaf with connection on $\hat{G} \times A$ corresponding to the identity element in $\hat{G}(\hat{G})$. Then $\mathcal{L}$ is the pullback of $\mathcal{P}$ to $\hat{G} \times A$. Let ζ denote the global section of $\mathbb{R}^1(\Omega^*_{\hat{G}\times A/\hat{G}})$ corresponding to $(\mathcal{L}, \nabla)$. If T is an S scheme and $f \in \hat{G}(T)$, f corresponds to $(\mathrm{id} \times f)^*(\mathcal{L}, \nabla)$ on $T \times A$ or equivalently to $(\mathrm{id} \times f)^*\zeta$ in $\mathbb{R}^1(\Omega^*_{T\times A/T})$.

In particular,

Lemma 1.4.1. *The image of* ζ *under the composition*

$$\mathbb{R}^1(\Omega^*_{\hat{G}\times A/\hat{G}}) \overset{(f_u\times\mathrm{id})^*}{\longrightarrow} \mathbb{R}^1(\Omega^*_{A[\varepsilon]/S[\varepsilon]}) \longrightarrow \Omega^1_{S[\varepsilon]} \otimes \mathbb{R}^1(\Omega^{\cdot}_A)$$

is $d\varepsilon \otimes h_{\mathrm{Vec}}(u)$.

Let $X = \hat{G} \times G$. Then invertible sheaves on X with connections whose relativations over G and over $\hat{G}$ are integrable correspond to global sections of $\mathbb{R}^1(\mathcal{S})$ where $\mathcal{S}$ denotes the complex

$$\mathcal{O}^*_X \overset{d\log}{\longrightarrow} \Omega^1_X \longrightarrow \Omega^2_{X/G} \oplus \Omega^2_{X/\hat{G}} \overset{d\oplus d}{\longrightarrow} \Omega^3_{X/G} \oplus \Omega^3_{X/\hat{G}} \overset{d\oplus d}{\longrightarrow} \ldots$$

of sheaves on X where the map from Ω^1_X to $\Omega^2_X/G \oplus \Omega^2_{X/\hat{G}}$ is the exterior derivation composed with the direct sum of the natural maps from Ω^2_X to $\Omega^2_{X/G}$ and $\Omega^2_{X/\hat{G}}$. Now $\mathcal{S}$ sits in an exact sequence

$$0 \longrightarrow F^1(\Omega^{\cdot}_{\hat{G}}) \otimes F^1(\Omega^{\cdot}_G) \longrightarrow \Omega^*_X \longrightarrow \mathcal{S} \longrightarrow 0\ .$$

Let $\mathcal{P}^{\#}$ be the sheaf with connection on X as above. Then the relativation of $\mathcal{P}^{\#}$ with respect to $\hat{G}$ is the pullback of $(\mathcal{L}, \nabla)$. Let β denote the global section of $\mathbb{R}^1(\mathcal{S})$ corresponding to $\mathcal{P}^{\#}$. Then the image of β in $\mathbb{R}^2(F^1(\Omega^{\cdot}_{\hat{G}}) \otimes F^1(\Omega^{\cdot}_G))$ is the class of ω (the curvature form of ∇). We will now suppose S is flat over $\mathbb{Z}$. We know $\mathbb{R}^2(F^1(\Omega^{\cdot}_{\hat{G}}) \otimes F^1(\Omega^{\cdot}_G))$ is canonically isomorphic to $\mathrm{Inv}(\Omega^1_{\hat{G}}) \otimes \mathrm{Inv}(\Omega^1_G)$ by Corollary 1.a. If $u \in \mathrm{Lie}_{\hat{G}}$ then the image of β under the composition

$$\mathbb{R}^1(\mathcal{S}) \longrightarrow \mathrm{Inv}(\Omega^1_{\hat{G}}) \otimes \mathrm{Inv}(\Omega^1_G) \xrightarrow{u \otimes \mathrm{id}} \mathrm{Inv}(\Omega^1_G)$$

is $h_{\mathrm{Curv}}(u)$.

Let $\mathcal{S}[\varepsilon]$ denote the complex

$$\mathcal{O}^*_{G[\varepsilon]} \xrightarrow{d\log} \Omega^1_{G[\varepsilon]} \longrightarrow \Omega^2_{G[\varepsilon]/S[\varepsilon]} \xrightarrow{d} \Omega^3_{G[\varepsilon]/S[\varepsilon]} \xrightarrow{d} \dots$$

where the map $\Omega^1_{G[\varepsilon]} \to \Omega^2_{G[\varepsilon]/S[\varepsilon]}$ is the exterior derivation composed with the natural map. Since $F^1(\Omega^{\cdot}_{S[\varepsilon]}) = (0, \Omega^1_{S[\varepsilon]}, 0, \dots)$ we have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & F^1(\Omega^{\cdot}_{\hat{G}}) \otimes F^1(\Omega^{\cdot}_G) \arrow[r] \arrow[d] & \Omega^*_X \arrow[r] \arrow[d, "(f_u \times \mathrm{id})^*"] & \mathcal{S} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \Omega^1_{S[\varepsilon]} \otimes F^1(\Omega^{\cdot}_G) \arrow[r] & \Omega^*_{G[\varepsilon]} \arrow[r] & \mathcal{S}[\varepsilon] \arrow[r] & 0
\end{tikzcd}$$

in which the rows are exact. Taking hyper-cohomology yields the commutative diagram

$$\begin{tikzcd}
\mathbb{R}^1(\mathcal{S}) \arrow[r] \arrow[d] & \mathbb{R}^2(F^1(\Omega^{\cdot}_{\hat{G}}) \otimes F^1(\Omega^{\cdot}_G)) \cong \mathrm{Inv}(\Omega^1_{\hat{G}}) \otimes \mathrm{Inv}(\Omega^1_G) \arrow[d, "(f_u \times \mathrm{id})^*"] \\
\mathbb{R}^1(\mathcal{S}[\varepsilon]) \arrow[r] & \mathbb{R}^2(\Omega^1_{S[\varepsilon]} \otimes F^1(\Omega^{\cdot}_G)) \cong \Omega^1_{S[\varepsilon]} \otimes \mathrm{Inv}(\Omega^1_G)
\end{tikzcd}$$

as $\mathbb{R}^1(F^1(\Omega^{\cdot}_G)) \cong \mathrm{Inv}(\Omega^1_G)$ by Corollary 1.a. From the commutativity of

$$\begin{tikzcd}
\mathrm{Inv}(\Omega^1_{\hat{G}}) \arrow[rr, "f_u^*"'] \arrow[dr, "u"] & & \Omega^1_{S[\varepsilon]} \arrow[dl, "\partial/\partial\varepsilon"'] \\
& \mathcal{O}_S &
\end{tikzcd}$$

we see that the diagram

$$\begin{tikzcd}
\mathbb{R}^1(\mathcal{S}) \arrow[r] \arrow[d] & \mathrm{Inv}(\Omega^1_{\hat{G}}) \otimes \mathrm{Inv}(\Omega^1_G) \arrow[r, "u\otimes\mathrm{id}"'] \arrow[d] & \mathrm{Inv}(\Omega^1_G) \arrow[dl, "\partial/\partial\varepsilon \times \mathrm{id}"] \\
\mathbb{R}^1(\mathcal{S}[\varepsilon]) \arrow[r] & \Omega^1_{S[\varepsilon]} \otimes \mathrm{Inv}(\Omega^1_G) &
\end{tikzcd}$$

commutes. Hence,

Lemma 1.4.2. *The image of β via the composition*

$$\mathbb{R}^1(\mathcal{S}) \longrightarrow \mathbb{R}^1(\mathcal{S}[\varepsilon]) \longrightarrow \Omega^1_{S[\varepsilon]} \otimes \mathrm{Inv}(\Omega^1_G)$$

is $d\varepsilon \otimes h_{\mathrm{Curv}}(u)$.

Taking hyper-cohomology of the natural commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & \Omega^1_{S[\varepsilon]} \otimes F^1(\Omega^{\cdot}_G) \arrow[r] \arrow[d] & \Omega^*_{G[\varepsilon]} \arrow[r] \arrow[d] & \mathcal{S}[\varepsilon] \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \Omega^1_{S[\varepsilon]} \otimes \Omega^{\cdot}_G \arrow[r] & \Omega^*_{G[\varepsilon]} \arrow[r] & \Omega^*_{G[\varepsilon]/S[\varepsilon]} \arrow[r] & 0 \\
0 \arrow[r] & \Omega^1_{S[\varepsilon]} \otimes \Omega^{\cdot}_A \arrow[r] \arrow[u] & \Omega^*_{A[\varepsilon]} \arrow[r] \arrow[u] & \Omega^*_{A[\varepsilon]/S[\varepsilon]} \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

we obtain the right-hand side of the commutative diagram:

$$\begin{tikzcd}
\mathbb{R}^1(\mathcal{S}) \arrow[r] \arrow[d] & \mathbb{R}^1(\mathcal{S}[\varepsilon]) \arrow[r] \arrow[d] & \Omega^1_{S[\varepsilon]} \otimes \mathrm{Inv}(\Omega^1_G) \arrow[d] \\
\mathbb{R}^1(\Omega^{\cdot}_{\hat{G}\times G/\hat{G}}) \arrow[r] & \mathbb{R}^1(\Omega^{\cdot}_{G[\varepsilon]/S[\varepsilon]}) \arrow[r] & \Omega^1_{S[\varepsilon]} \otimes \mathcal{H}^1_{DR}(G) \\
\mathbb{R}^1(\Omega^{\cdot}_{\hat{G}\times A/\hat{G}}) \arrow[r] \arrow[u] & \mathbb{R}^1(\Omega^{\cdot}_{A[\varepsilon]/S[\varepsilon]}) \arrow[r] \arrow[u] & \Omega^1_{S[\varepsilon]} \otimes \mathcal{H}^1_{DR}(A) \arrow[u]
\end{tikzcd}$$

The commutativity of the left-hand side is automatic. Moreover, the arrows in the last column are injections by Theorem 1.2.2 (ii). Since the image of β in the global sections of $\mathbb{R}^1(\Omega^{\cdot}_{\hat{G}\times G/\hat{G}})$ is equal to the image of ζ by definition, it follows from Theorem 1.2.2 (i), Lemma 1.4.1 and Lemma 1.4.2 that $h_{\mathrm{Curv}} = h_{\mathrm{Vec}}$. This concludes the proof of Theorem 1.1.1 when S is flat over $\mathbb{Z}$.

To deduce from this the theorem in the case when S is reduced, observe that it suffices to prove the result when S is a spectrum of a field since $\mathcal{H}^1_{DR}(A)$ is locally free and so commutes with arbitrary base change. Next, any Abelian scheme over a field can be lifted to an Abelian scheme over a ring flat over $\mathbb{Z}$ [NO, Cor.3.2]. So the theorem follows by again using the fact that $\mathcal{H}^1_{DR}(A)$ commutes with base change. $\square$

What we need to do, in general, but can't, is lift an Abelian scheme over a local Artinian ring to an Abelian scheme over a base flat over $\mathbb{Z}$. We can, however, lift it to a formal Abelian scheme over such a base and it seems possible that one could set up the foundations necessary to make our arguments above go through.

1.5 Curves and Jacobians

Suppose X is a smooth proper curve over S and J is its Jacobian. Then $\mathcal{H}^1_{DR}(X)$ is canonically isomorphic to $\mathcal{H}^1_{DR}(J)$ and J is canonically isomorphic to $\hat{J}$. Hence the pairing described in the previous sections gives rise to a nondegenerate pairing $(\ ,\ )_X : \mathcal{H}^1_{DR}(X) \times \mathcal{H}^1_{DR}(X) \to \mathcal{O}_S$. We will describe this pairing in terms of the geometry of X in this section.

Suppose Y is a smooth proper connected scheme of dimension n over S. Let $t_Y : \mathcal{H}^n_{DR}(Y) \to \mathcal{O}_S$ be the trace isomorphism. Let $\cup : \mathcal{H}^1_{DR}(X) \times \mathcal{H}^1_{DR}(X) \to \mathcal{H}^2_{DR}(X)$ denote the cup product.

Theorem 1.5.1. *$(\omega, v)_X = t_X(\omega \cup v)$ for ω and v sections of $\mathcal{H}^1_{DR}(X)$.*

Proof. Let $a : X \to J$ be an Albanese morphism. Then identifying J with $\hat{J}$, and letting $(\ ,\ )_J$ denote the resulting pairing on $\mathcal{H}^1_{DR}(J)$ it suffices to prove

$$(\alpha, \beta)_J = t(a^*\alpha \cup a^*\beta)$$

for sections α and β of $\mathcal{H}^1_{DR}(J)$.

For γ a section of $\mathcal{H}^1_{DR}(X)$ let v_γ denote the section $\delta \mapsto t_X(\gamma \cup \delta)$ of $\mathcal{H}^1_{DR}(X)^\vee$. Let p_1 and p_2 denote the first and second projections of $X \times X$ onto X. Then

$$c(\mathcal{P})(a^*v_\gamma, a^*v_\delta) = c((a \times a)^*\mathcal{P})(v_\gamma, v_\delta) = -t_{X\times X}(c((a \times a)^*\mathcal{P}) \cup p_1^*\gamma \cup p_2^*\delta)\ .$$

Here we use the compatibility of the trace isomorphisms with the Kunneth formula. Now $(a\times a)^*\mathcal{P}$ is isomorphic to the invertible sheaf associated to a divisor of the form $\Delta - E$ where Δ is the diagonal and E is the sum of a horizontal and a vertical divisor on $X\times X$. It follows that

$$c((a\times a)^*\mathcal{P})\cup p_1^*\gamma\cup p_2^*\delta = c(\Delta)\cup p_1^*\gamma\cup p_2^*\delta\ .$$

Hence

$$-t_{X\times X}(c((a\times a)^*\mathcal{P})\cup p_1^*\gamma\cup p_2^*\delta) = -t_X(c(\Delta)\cup p_1^*\gamma\cup p_2^*\delta) = -t_X(\gamma\cup\delta) = v_\delta(\gamma)\ .$$

Here we use the projection formula applied as the diagonal map of X into $X\times X$ and the compatibility of the Chern class of a divisor and its de Rham homology class (see Thm.7.5 and Prop.7.7.1 of [H, §II] for the case in which the base is a field). If h denotes the canonical isomorphism from $\mathcal{H}^1_{DR}(J)^\vee$ onto $\mathcal{H}_{DR}(J)$ we can summarize the above as

$$h(a^*v_\gamma)(a^*v_\delta) = v_\delta(\gamma)$$

and, in particular, $h^{-1}(a^*\alpha) = a_*v_{a^*\alpha}$ for α a section of $\mathcal{H}^1_{DR}(J)$. Hence if α and β are sections of $\mathcal{H}^1_{DR}(J)$, we have

$$(\alpha,\beta)_J = a^*v_{a^*\alpha}(\beta) = v_{a^*\alpha}(a^*\beta) = t_X(a^*\alpha\cup a^*\beta)$$

which completes the proof. □

Suppose $S=\mathrm{Spec}(K)$ where K is an algebraically closed field. As a corollary, we establish Serre's formula for the cup product of two de Rham cohomology classes on a curve X over S when K has characteristic zero and prove an analogue when K has positive characteristic. Let p denote the characteristic of K if $\mathrm{char}(K)>0$ and ∞ otherwise.

For a divisor D on X over K, $|D|$ will denote the support of D considered as a divisor on X and $D(+1)$ will denote $D+|D|$. Suppose D is a non-special effective divisor such that $\mathrm{ord}_P D\le p-1$ for all P in $X(K)$. Let $\Omega^1_X(D(+1))_0$ denote the sub-sheaf of $\Omega^1_X(D(+1))$ consisting of differentials with zero residues along D. Then the inclusion map of the de Rham complex $\Omega^{\cdot}_X$ into the complex $d:\mathcal{O}_X(D)\to\Omega^1_X(D(+1))_0$ is a quasi-isomorphism. Let $H^1_{DR}(X)$ denote $\mathcal{H}^1_{DR}(X)(\mathrm{Spec}(K))$. Since $H^1(X,\mathcal{O}_X(D))=0$ we conclude that $H^1_{DR}(X)$ is canonically isomorphic to $H^0(X,\Omega^1_X(D(+1)_0)/dH^0(X,\mathcal{O}_X(D))$. In particular, if E is an effective divisor such that $\mathrm{ord}_P E \leq p-1$ for all P, we have a natural map of $H^0(X,\Omega^1_X(E(+1)_0)$ into $H^1_{DR}(X)$. Moreover, if $E' \leq E$, the obvious diagram commutates.

Denote by $\mathcal{W}$ the space of differentials ω on X with zero residues such that $\mathrm{ord}_P(\omega) \geq -p$. By the previous discussion we may associate to each such differential a well defined class $[\omega]$ in $H^1_{DR}(X)$. Explicitly, if ω is regular on an open U, there exists a rational function f on X with poles of order strictly less than p such that $v := \omega - df$ is regular in a neighborhood V of $X-U$. Then $[\omega]$ is the class of the hyper-cocycle $(\{\omega, v\}, \{f\})$ with respect to the covering $\{U, V\}$.

Corollary 1.5.1.1. *Suppose ω_1 and ω_2 are elements of $\mathcal{W}$ and for each $P \in X(K)$ let $g_{i,P}$ be a Laurent series in a local parameter at P with pole of order strictly less than p such that $\omega_1 - d\lambda_{1,P}$ is regular at P. Then*

$$([\omega_1],[\omega_2])_X = \sum_{P \in X(K)} \mathrm{Res}_P(\lambda_{1,P}\omega_2 - \lambda_{2,P}\omega_1 + \lambda_{2,P} d\lambda_{1,P}) \ .$$

In particular, if $d\lambda_{1,P} = \omega_1$ for all P or ω_2 is holomorphic, then

$$([\omega_1],[\omega_2])_X = \sum_{P \in X(K)} \mathrm{Res}_P(\lambda_{1,P}\omega_2) \ .$$

PROOF. There exists a covering $\{U, V\}$ of X, $v_1, v_2 \in \Omega^1_{X/K}(V)$ and rational functions $g_1, g_2 \in \mathcal{O}_X(U \cap V)$ such that $\omega_i \in \Omega^1_{X/K}(U)$, g_i has poles of order $< p$ on X and $\omega_i - v_i = dg_i$ on $U \cap V$ for $i = 1$ or 2. It follows $[\omega_1] \cup [\omega_2]$ equals the class of the hyper-cocycle

$$U \cap V \mapsto g_1 v_2 - g_2 \omega_1$$

in $H^2_{DR}(X)$. The trace of the class of this hyper-cocycle is

$$\sum_{P \in D} \mathrm{Res}_P(g_1 v_2 - g_2 \omega_1)$$

where $D = (X - U)(K)$. Now

$$\begin{aligned}\mathrm{Res}_P(g_1 v_2 - g_2\omega_1) &= \mathrm{Res}_P(\lambda_{1,P} v_2 - g_2\omega_1) = \mathrm{Res}_P(\lambda_{1,P}\omega_2 - \lambda_{1,P} dg_2 - g_2\omega_1) \\ &= \mathrm{Res}_P(\lambda_{1,P}\omega_2 - \lambda_{1,P} dg_2 - g_2 d\lambda_{1,P} + g_2(d\lambda_{1,P} - \omega_1)) \\ &= \mathrm{Res}_P(\lambda_{1,P}\omega_2 + \lambda_{2,P}(d\lambda_{1,P} - \omega_1) - d(g_2\lambda_{1,P}) \\ &= \mathrm{Res}_P(\lambda_{1,P}\omega_2 - \lambda_{2,P}\omega_1 + \lambda_{2,P} d\lambda_{1,P})\end{aligned}$$

for $P \in D$ using the fact that $(g_1 - \lambda_{1,P})v_2$ and $(g_2 - \lambda_{2,P})(d\lambda_{1,P} - \omega_1))$ are regular at such P. On the other hand, if P is not an element of D (i.e., an element of $U(K)$), then $\mathrm{Res}_P(\lambda_{1,P}\omega_2 - \lambda_{2,P}\omega_1 + \lambda_{2,P} d\lambda_{1,P}) = 0$ since both ω_1 and ω_2 are regular on U. This proves the corollary. □

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