A Ramified Weight Character Computation

Robert Coleman William Stein

July 15, 2001

NOTE: At present this document is really just email between myself and Robert made precise and organized.

1 The Problem

Set $p = 2$ and $\Lambda^* := \mathbf{Z}_2[[1+4\mathbf{Z}_2]]$. For any integer $m \geq 1$, let $W^*_{2,m}$ be the set of imaginary quadratic integers (i.e., they satisfy a quadratic monic polynomial with coefficients in $\mathbf{Z}$) in $\mathbf{Z}_2$ that are congruent to $1 \bmod 4$ and have norm 2^m. I don't think $W^*_{2,m}$ is finite because I think that

$$W^*_{2,m} = \left\{ \gamma = \frac{-a+\sqrt{a^2-4\cdot 2^m}}{2} \;:\; a \equiv -1 \pmod 4, \quad \sqrt{a^2-4\cdot 2^m} \equiv -a \pmod 8 \right\}.$$

For $\gamma \in W^*_{2,m}$, let $\mathcal{O}_\gamma$ be the set of orders in the ring of integers of $\mathbf{Q}(\gamma)$ that contain γ.

The set $\mathcal{O}_\gamma$ can be computed by listing the discriminants of its elements. You can find the γ by looking at these numbers. Suppose $N^2 D = t^2 - 4\cdot 2^m$ where D is square free. Then the orders containing $\gamma = (t+N\sqrt{D})/2$ correspond to the factors of N.

Let

$$B_m = \sum_{\gamma \in W^*_{2,m}} \sum_{\mathcal{O} \in \mathcal{O}_\gamma} h(\mathcal{O}) \cdot \frac{[\gamma]}{\gamma^2 - 2^m}.$$

Note that B_m an element of Λ^* since γ is a unit in $\mathbf{Z}_2$. Here for $\gamma \in 1+4\mathbf{Z}_2$, $[\gamma]$ is the element in the group part of Λ^*, so the above could be viewed as an expression of B_m as a $\mathbf{Z}_2$-linear combination on the formal vector space generated by the elements of the additive group of $1+4\mathbf{Z}_2$.

Let $Q(T)$ be the series in $\Lambda^*[[T]]$ such that

$$T\,\mathrm{d}\log Q(T) = \sum_{m\geq 1} B_m T^m.$$

I *suspect* that

$$Q(T) = \exp\left(\sum_{m \geq 1} \frac{B_m}{m} T^m \right).$$

If $X = [5] - 1$, then we can identify any element of Λ^* as a series in X. Explicitly,

$$[\gamma] = (1+X)^{\log\gamma/\log 5}.$$

It then makes sense to consider the reduction modulo 2 of $Q(T)$ as a power series $\sum a_n(X)T^n$ over $\mathbf{F}_2[[X]]$. I (RC) conjecture that $\operatorname{ord}_X a_n(X) = \binom{n}{2}$. I (RC) checked this for $n \leq 4$.

At present, I don't even know if $a_n(X) \neq 0$ for large n.

2 A Remark

I know that $Q(T)$ is entire and it follows from Emerton's work that it has one zero of valuation -1. (This means, in particular, that $v(a_1) = 1$ and $v(a_n) > n$.) Using this, he constructed a family of eigenforms

$$f(X)(q) = \sum_{n\geq 1} b_n(X)q^n$$

where where the $b_n(X)$ are analytic functions bounded by 1 on $1 > |X| > |8|$ and $f(X)$ has weight-character determined by

$$5 \mapsto (1+X), \qquad -1 \mapsto 1,$$

so is classical of weight k if $1 + X = \zeta 5^{k-1}$ for $k \geq 2$ and ζ is a root of unity.

There seems to be some confusion (at least in my mind) whether a level one weight k modular form corresponds to the character $\gamma \mapsto \gamma^k$ or $\gamma \mapsto \tau^k(\gamma)\gamma^{k-1}$.

Also, $v(a_2(X)) = v(X)$. (See the main theorem in Chapter 4, only I have to translate the notation. In particular, $u = a_2(X), t = 5^{14}(X+1) - 1$.)

Now one can also reduce the $b_n(X)$ modulo 2 and get elements in $\mathbf{F}_2[[X]]$. Let $F(X)$ be the reduction of $f(X)$ mod 2. It follows from Proposition 5.8 that

$$F(X)(q) \equiv q + Xq + Xq^2 + Xq^3 + Xq^5 \bmod (X^2, q).$$

It would be nice to know something mod X^3. I don't believe that $b_9(X)$ mod 2 is divisible by X (because $\Delta \equiv f(5^{11})$ mod 8).