Algebraic Curves

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Lecture 40

KdV and $\wp$.

Let $K = k((x))$ and $D = \mathrm{d}/\mathrm{d}x$. Suppose $u, a, b \in K$ and

$$L = D^2 - 2u, \quad M = D^3 - 3aD - 3b/2.$$

Then $LM = ML$ if and only if

$$u' = a', \;\; b' = a''$$
$$12aa' = a'''. \qquad (*)$$

The last equation is called the **stationary KdV**. If $\wp(x)$ is the the Weierstrass P-function associated to

$$Y^2 = 4X^3 - g_2 X - g_3$$

then $\wp(x)$ is a solution of (*) as is $1/x^2$.

Dressing opeators and commuting subalgebras

If A is a commuting k-subalgebra of $k((D^{-1}))$ such that

$$A \cap k[[D^{-1}]] = k \;\text{ and }\; \dim_k \frac{k((D^{-1}))}{A + k[[D^{-1}]]} < \infty$$

generated by elements

$$f = D^2 + O(D^{-1}) \;\text{ and } g = D^3 + O(D^{-1}).$$

Suppose there exists a dressing operator

$$\Psi = 1 + \sum_{i=1}^{\infty} c_i D^{-1} \quad (c_i \in K)$$

for A. Then

$$\Psi f \Psi^{-1} = D^2 - 2c_1' \quad \text{and } \Psi g \Psi^{-1} = D^3 - 3c_1 D - 3c_1'' - 3c_2'.$$

Hence c_1' is a solution of (*).

Baker Functions

Let $K\{\{z\}\}$ be the ring

$$\sum_{i \in \mathbf{Z}} c_i z^i \quad (c_i \in K)$$

where $\operatorname{ord}_x c_i$ is bounded below and tends to ∞ as $i \to -\infty$. Then $k((z)) \subset K\{\{z\}\}$ and

$$e^{xz^{-1}} \in K\{\{z\}\}.$$

Suppose W is a k-subspace of $k((z))$ such that

$$\dim_k (W \cap k[[z]]) = \dim_k \left(\frac{k((z))}{W + k[[z]]}\right) < \infty.$$

Example. Suppose $g_2^3 - 27g_3^2 \neq 0$ and W is the k-subalgebra generated by

$$f = z^{-2} \quad \text{and } g = z^{-3}(1 - (g_2/4)z^4 - (g_3/4)z^6))^{1/2}.$$

If $\bar{W}$ is the intersection of $K((z))$ with the closure of $e^{xz^{-1}} KZ$ then $\bar{W} \cong K((z))/K[[z]]$ and $D(e^{xz^{-1}} \bar{W}) \subset e^{xz^{-1}} \bar{W}$. Moreover, $e^{xz^{-1}} \bar{W}$ is a free $K[D]$-module of rank 1. In particlar, there exists a unique series

$$\psi_W(z) = 1 + c_1 z + c_2 z^2 + \cdots \in z\bar{W}.$$

This is the **Baker Function** associated to W. Put $\Psi_W = \psi_W(D^{-1})$. Then, if $f(z) \in k((z))$,

$$\Psi_W f(D^{-1}) \Psi_W^{-1} \in K[D] \iff f(z)W \subseteq W.$$

The key point is if

$$ze^{-xz^{-1}} h(D) z^{-1} e^{xz^{-1}} f(z) = g(z)$$

then $g(D^{-1}) = h(D) f(D^{-1})$.