Algebraic Curves

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Lecture 39

The basic set up

Proposition. *(i) The quotient $\mathcal{C}/(\mathcal{I}+\mathcal{J})$ has dimension 1 over k and is generated by any element $\begin{pmatrix} f \\ 0 \end{pmatrix}\begin{pmatrix} 0 \\ f\omega \end{pmatrix}$ where $Res f\omega = 1$. (ii) The following sequence is exact:*

$$0 \to k \to \mathcal{C}/\mathcal{J} \stackrel{c\mapsto(j\mapsto jc)}{\longrightarrow} Hom_{\mathcal{C}}(\mathcal{J}:\mathcal{C}/\mathcal{J}) \to 0$$

Theorem. *The following sequence is exact:*

$$0 \to \Omega(C) \stackrel{\omega\mapsto(\omega,\omega)}{\longrightarrow} \Omega_{K(C)} \oplus \Omega_{\mathcal{O}_C} \stackrel{(\omega,\nu)\mapsto\omega-\nu}{\longrightarrow} \Omega_{\mathbf{A}_C} \stackrel{Res}{\longrightarrow} k \to 0.$$

Corollary. *Let $c = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $d = \begin{pmatrix} 0 \\ \omega \end{pmatrix}$. Let $B^* := B \cup \{c,d\}$ be a basis for $U := \begin{pmatrix} K \\ \Omega_K \end{pmatrix}$ and $E^* =: E \cup \{c,d\}$ a basis for $W := \begin{pmatrix} \mathcal{O} \\ \Omega_{\mathcal{O}} \end{pmatrix}$. Let $a = \begin{pmatrix} f \\ 0 \end{pmatrix}$ and $b = \begin{pmatrix} 0 \\ f\omega \end{pmatrix}$ where $Res(f\omega) = 1$. Then $B \cup E \cup \{a,b,c,d\}$ is a basis for $\begin{pmatrix} \mathbf{A}_K \\ \Omega_{\mathbf{A}_K} \end{pmatrix}$.*

If S is a finite subset of B or E choose an ordering of S and let $e(S)$ be the product of the elements of S in this order in the Clifford algebra $\mathcal{C}$. Then the elements $e(S)a^ib^jc^md^ne(T)$ where $S \subset B$, $T \subset E$, S, T are finite and $i,j,m,n \in \{0,1\}$ is a basis for C. Now the elements $e(S)a^ib^je(T)$, where $S \subset B^*$, $T \subset E^*$ and $S \cup T \neq \emptyset$, span $\mathcal{I}+\mathcal{J}$. Using

$$a^ib^jc = c(-a)^i(-b)^j + i(-b)^j$$

and

$$a^ib^jd = d(-a)^i(-b)^j + j(-a)^i$$

for $i, j \in \{0,1\}$, we see that the elements $e(S)a^i b^j e(T) \cup \{1, a, b\}$ where $S \subset B, T \subset E^*$ and $S \cup T \neq \emptyset$ is a basis for $\mathcal{I} + \mathcal{J}$. It follows that ab is a basis for $\mathcal{C}/(\mathcal{I}+\mathcal{J})$.

We will now prove (ii). Consider the pairing $U \times W \to k$,

$$\left(\begin{pmatrix} f \\ \eta \end{pmatrix}, \begin{pmatrix} a \\ \nu \end{pmatrix} \right) = q\left(\begin{pmatrix} f \\ \eta \end{pmatrix} + \begin{pmatrix} a \\ \nu \end{pmatrix} \right) = \operatorname{Res}(f\nu + a\eta).$$

It follows that if $u \in U$ and $w \in W$.

$$e(u)e(w) = -e(w)e(u) + (u, w).$$

Theorem. *The pairing $(\ ,\ )_{res}$ is perfect and the orthogonal complement of $\Omega_{\mathbf{A}_C}(D)$ is $\mathbf{A}_C(-D)$. It induces perfect pairings,*

$$K/k \times \Omega_{\mathcal{O}_C}/\Omega(C) \to k \quad \text{ and } \quad \mathcal{O}_C/k \times \Omega_K \to k.$$

In fact, the statement $\Omega_K^\perp = K$ together with the first line of this theorem implies all the other statements of this and the previous theorem. If $Z = U \cap W = \begin{pmatrix} k \\ k\omega \end{pmatrix}$, this pairing induces a perfect pairing $U/Z \times W/Z \to k$.

We now suppose B and E are dual bases modulo Z. If

$$x = \sum r_{i,j}(S)e(S)a^i b^j,$$

where S ranges over finite subsets of B, and $T \subset E$,

$$e(T)x \equiv \sum r_{i,j}(T' \cup R)e(R)a^i b^j \bmod \mathcal{J}.$$

where T' is the "dual" set to T and R runs over finite subsets of $B - T'$.

If $\mathcal{J}x \subset \mathcal{J}$, then we see $r_{i,j}(S) = 0$ for all $S \neq \emptyset$. Now $ca^i b^j = \pm(a^i b^j c + i(-b)^j)$ and so $r_{i,j}(\emptyset) = 0$, if $i = 1$ and similarly if $j = 1$. Thus $x \in k$.

Suppose now $e(T)e(T') \equiv 1 \bmod \mathcal{J}$ for T a subset of E. Also suppose that a and b anti-commute with any element of $B \cup E$. Then if x is as above and $S \subset B$,

$$(ab(-d)^j(-c)^i e(S'))x \equiv r_{i,j}(S)ab \bmod \mathcal{I} + \mathcal{J}. \qquad (*)$$

Let $h \in Hom_{\mathcal{C}}(\mathcal{J}, \mathcal{C}/\mathcal{J})$. Suppose for $S \subset B$

$$h(ab(-d)^j(-c)^i e(S')) = x_{i,j}(S)ab \bmod \mathcal{I} + \mathcal{J}.$$

Then, $h(r) = ry \bmod \mathcal{J}$ where

$$y = \sum x_{i,j}(S)e(S)a^i b^j \bmod \mathcal{J}.$$

The point is, we see from $(*)$ that if $r \notin \mathcal{J}$, $\exists s \in \mathcal{C}$ such that $sr \notin \mathcal{I} + \mathcal{J}$.