Algebraic Curves

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Lecture 38

Anderson's Construction continued

For a curve C over k, let $\Omega(C)$ be the global differentials on C, $\Omega_{\mathbf{A}_C}$ the restricted product of $\Omega_{K(C)_P}$, $\Omega_{\mathcal{O}_C}$ the image in $\Omega_{\mathbf{A}_C}$ of $\prod_{P\in C}\Omega_{R_P}$ and $\Omega_C(D)$, $\mathbf{A}_C(D)\Omega_{\mathcal{O}_C}$.

Suppose X is a genus one curve over k with function field K and suppose ω is a non-zero global differential on X. Let $\mathbf{A} = \mathbf{A}_K$ Let $\mathcal{C}$ be the Clifford algebra associated to

$$\begin{pmatrix} \mathbf{A} \\ \Omega_{\mathbf{A}} \end{pmatrix}, \quad q(\begin{pmatrix} f \\ \nu \end{pmatrix}) = (f,\nu)_{res} := \ \mathrm{Res} f\nu := \sum_{P\in X} \mathrm{Res}_P(f_P\nu_P).$$

$\mathcal{I} = U\mathcal{C}$ where $U := \begin{pmatrix} K \\ \Omega_K \end{pmatrix}$ and $\mathcal{J} = \mathcal{C}W$ where $W := \begin{pmatrix} \mathcal{O} \\ \Omega_{\mathcal{O}} \end{pmatrix}$. Let $V = \mathcal{C}/\mathcal{J}$.

Proposition. *(i) The quotient $\mathcal{C}/(\mathcal{I}+\mathcal{J})$ has dimension 1 over k and is generated by any element $\begin{pmatrix} f \\ 0 \end{pmatrix}\begin{pmatrix} 0 \\ f\omega \end{pmatrix}$ where $\mathrm{Res} f\omega = 1$. (ii) If $c \in \mathcal{C}$ and $\mathcal{J}c \subseteq \mathcal{J}$, then $c \in k + \mathcal{J}$.*

If $\chi \in \mathbf{A}$ there exists a unique k-derivation $\chi^{\#}$ on $\mathcal{C}$ such that

$$\chi^{\#}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \chi a \\ -\chi b \end{pmatrix},$$

for $a, b \in \mathbf{A}_K$ and there exists a k-linear map, unique up to a scalar addend, $\bar{\chi}: V \to V$ such that

$$\bar{\chi}(cv) = (\chi^{\#}c)v + c\bar{\chi}(v).$$

We made Lie algebras $\mathfrak{g} = \{\bar{\chi}: \chi \in \mathbf{A}\}$, $\mathfrak{a} = \{\bar{\chi}: \chi \in K\}$ and $\mathfrak{b} = \{\bar{\chi}: \chi \in \mathcal{O}\}$.

Assuming the above proposition, we saw if $a, b \in \mathfrak{g}$, $[a,b] \in k\mathbf{1}$ and if $H = (\mathcal{I}+\mathcal{J})/\mathcal{J}$ and $L = (k+\mathcal{J})/\mathcal{J}$, $\mathfrak{a}H \subseteq H$ and $\mathfrak{b}L \subseteq L$.

Suppose $\alpha_{\neq 0} \in V^*$ such that $\alpha(H) = 0$, $\beta_{\neq 0} \in L$ and $\chi \in \mathbf{A}$. Let

$$\tau(\alpha, \beta, \bar{\chi})(x) = \sum_{n=0}^{\infty} (\alpha \bar{\chi}^n \ beta) \frac{x^n}{n!}$$

Proposition. *The Laurent series*

$$\gamma_\rho(x) := \frac{-d^3 \log \tau(\alpha, \beta, \bar{\chi})(x)}{\mathrm{d}x^3}$$

only depends on $\rho := \ Res \chi\omega$ *(and, of course,* X *and* ω*).*

Proof. First for $X, Y \in \mathfrak{g}$ such that $[X, Y] = \lambda \mathbf{1}$,

$$e^{x(X+Y)} = e^{-\lambda x^2/2} e^{xX} e^{xY}.$$

Theorem. $\gamma_1(x) = \wp'(x)$, *where* $\wp(x)$ *is the the Weierstrass P function of the Jacobian of* X *expanded in the parameter associated to* ω*.*

Nuts and bolts

Suppose C is arbitrary and $K = K(C)$.

Theorem. *The following sequence is exact:*

$$0 \to \Omega(C) \overset{\omega \mapsto (\omega,\omega)}{\longrightarrow} \Omega_{K(C)} \oplus \Omega_{\mathcal{O}_C} \overset{(\omega,\nu) \mapsto \omega - \nu}{\longrightarrow} \Omega_{\mathbf{A}_C} \overset{Res}{\longrightarrow} k \to 0.$$

Theorem. *The pairing* $(\ ,\ )_{res}$ *is perfect and the orthogonal complement of* $\Omega_{\mathbf{A}_C}(D)$ *is* $\mathbf{A}_C(-D)$*. It induces perfect pairings,*

$$K/k \times \Omega_{\mathcal{O}_C}/\Omega(C) \to k \quad \text{and} \quad \mathcal{O}_C/k \times \Omega_K \to k.$$

Proof.