Algebraic Curves

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Lecture 37

Clifford Algebras

Suppose V is a vector space over k and $q\colon V \to k$ is a quadratic form on V. I.e. (i) $q(cv) = c^2q(v)$ and (ii) $(v,w) \mapsto q(v+w) - q(v) - q(w)$ is bilinear. Then the Clifford algebra $\mathcal{C}(V,q)$ is the quotient of the tensor algebra of V over k by the two sided ideal generated by the elements

$$e \otimes e - q(e).$$

Examples. (i) If $q = 0$, $\mathcal{C}(V,q)$ is the exterior algebra. (ii) If $V = k^n \oplus k^n$ and $q(u,w) = u \cdot w$, $\mathcal{C} := \mathcal{C}(V,q)$ is isomorphic to $M_{2^n}(k)$. Suppose $n = 1$. Then $\mathcal{C}$ is isomorphic to the algebra generated by $e = (1,0)$ and $f = (0,1)$ where

$$e^2 = f^2 = 0 \ \text{ and } (e+f)^2 = 1.$$

In general, if B is a basis for V and for each finite subset S of B, e_S is a product of the elements of S (in some order), then $\{e_S, S \subseteq B, S \text{ is finite}\}$ is a basis for $\mathcal{C}(V,q)$.

Anderson's Construction

Suppose X is a genus one curve over k with function field K and suppose ω is a non-zero global differential on X. Let $\mathcal{C}$ be the Clifford algebra associated to

$$\begin{pmatrix} \mathbf{A}_K \\ \Omega_{\mathbf{A}_K} \end{pmatrix}, \quad q(\begin{pmatrix} f \\ \nu \end{pmatrix}) = \mathrm{Res}(f\nu) := \sum_{P \in C} \mathrm{Res}_P(f_P\nu_P).$$

Let $\mathcal{I}$ be the right ideal generated by the elements in $\begin{pmatrix} K \\ \Omega_K \end{pmatrix}$ and $\mathcal{J}$ be the left ideal generated by the elements in $\begin{pmatrix} \mathcal{O} \\ \Omega_{\mathcal{O}} \end{pmatrix}$. Let $V = \mathcal{C}/\mathcal{J}$.

Proposition. *(i) The quotient $\mathcal{C}/(\mathcal{I}+\mathcal{J})$ has dimension 1 over k and is generated by any element* $\begin{pmatrix} f \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ f\omega \end{pmatrix}$ *where $Resf\omega = 1$. (ii) If $c \in \mathcal{C}$ and $\mathcal{J}c \subseteq \mathcal{J}$, then $c \in k + \mathcal{J}$.*

If $\chi \in \mathbf{A}$ there exists a unique k-derivation $\chi^{\#}$ on $\mathcal{C}$ such that

$$\chi^{\#} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \chi a \\ -\chi b \end{pmatrix},$$

for $a, b \in \mathbf{A}_K$ and there exists a k-linear map $\bar{\chi}: V \to V$ such that

$$\bar{\chi}(cv) = (\chi^{\#} c)v + c\bar{\chi}(v).$$

It follows that $\bar{\chi}$ is unique up to a scalar addend (i.e. up to the addition of a k-linear multiple of the identity, $\mathbf{1}$). Let $\bar{A} = \{\bar{\chi}: \chi \in \mathbf{A}\}$. Then $\mathfrak{a}$ is a Lie algebra and if $a, b \in A$,

$$[a, b] \in k\mathbf{1}.$$

Let $\bar{K}$ be the subset of $\mathfrak{a}$ consisting of all $\bar{\chi}$, $\chi \in K$ and $\bar{\mathcal{O}}$ the subset of $\mathfrak{a}$ consisting of all $\bar{\chi}$, $\chi \in \mathcal{O}$. These are Lie subalgebras of $\mathfrak{a}$. Now let H be the subspace of V which is the image of $\mathcal{I}$ and L the image of k.

Lemma. *$\mathfrak{b}H \subseteq H$ and $\mathfrak{g}L \subseteq L$.*

Suppose $\alpha_{\neq 0} \in V^*$ such that $\alpha(H) = 0$, $\beta_{\neq 0} \in L$ and $\chi \in \mathbf{A}$. Let

$$\tau(\alpha, \beta, \bar{\chi})(x) = \sum_{n=0}^{\infty} (\alpha \bar{\chi}^n \beta) \frac{x^n}{n!}$$

Proposition. *The Laurent series*

$$\gamma_\rho(x) := \frac{-d^3 \log \tau(\alpha, \beta, \bar{\chi})(x)}{\mathrm{d}x^3}$$

only depends on $\rho := Res\chi\omega$ (and, of course, X and ω).

Theorem. *$\gamma_1(x) = \wp'(x)$, where $\wp(x)$ is the formal expansion at 0 of the Weierstrass P function of the Jacobian of X expanded in the parameter associated to ω.*