Algebraic Curves

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Lecture 36

De Rham Cohomology

Assume C is a curve over k with function field K such that $\bar{k} \cap K = k$.

Example. Suppose $\mathrm{Char} k = 3$ and $C: y^2 = x^3 + x$.

We were proving,

Theorem. *$H^0_{DR}(C)$ and $H^2_{DR}(C)$ both have dimension 1 and $H^1_{DR}(C)$ has dimension $2g$. Moreover, there is a natural perfect pairing*

$$(\ , \ )_C : H^1_{DR}(C) \times H^1_{DR}(C) \to H^2_{DR}(C).$$

Proof. Suppose $\{U, V\}$ is an ope cover of C by proper opens. Then a 0 cycle if a pair f_U, f_V such that

$$\mathrm{d} f_U = 0, \mathrm{d} f_V = 0 \ \text{ and } f_U - f_V = 0.$$

A 2-cocycle is corresponds to a form ω on $U \cap V$. It is a coboundary if and only if there exist forms η_U on U and η_V on V such that $\eta_U - \eta_V = \omega$.

Claim: The map $\Omega^1(U \cap V) \to k$,

$$\omega \mapsto \sum_{P \in U'} \mathrm{Res}_P \omega$$

induces an isomorphism $H^1_{DR}(C) \to k$. We saw If $U' = \{P_1, \ldots, P_n\}$ the sequence

$$0 \to \Omega^1(C) \to \Omega^1(U) \to \bigoplus_i \Omega_{K_{P_i}} / \Omega_{\hat{\mathcal{O}}_{P_i}} \xrightarrow{\sum_i \mathrm{Res}_{P_i}} k \to 0$$

is exact.

Let $H^1(\mathcal{O}_C) = \mathcal{O}(U \cap V) / \{f - g : f \in \mathcal{O}(U), g \in \mathcal{O}(V)\}$.

Lemma. $\dim_k H^1(\mathcal{O}_C) = g$ *and the following sequence is exact:*

$$0 \to \Omega(C) \to H^1_{DR}(C) \to H^1(\mathcal{O}_C) \to 0.$$

Proof.

If $(\omega_U, \omega_V, f_{U,V})$ and $(\nu_U, \nu_V, g_{U,V})$ are one-cocycles representing ω and ν, $(\omega, \nu)_C$ is the class of

$$(U, V) \mapsto \omega_U g_{U,V} - f_{U,V} \nu_V.$$

Lemma. *If* $P \neq Q \in V$ *and* $n \in \mathbf{Z}$ *there exists an* $f \in \mathcal{O}(V)$ *such that* $\operatorname{ord}_P f \geq n$ *and* $\operatorname{ord}_Q(f - 1) \geq n$.