Grössencharakter L-functions of real quadratic fields twisted by modular symbols

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§1. Introduction

Let $K = \mathbb{Q}[\sqrt{D}]$ be a real quadratic extension of discriminant $D > 0$. Hecke (in 1918) [He] was the first to introduce the notion of a *Grössencharakter* on ideals of K. Actually, Hecke defined Grössencharakters for an arbitrary algebraic number field, but we shall not need this here. A Grössencharakter ψ is defined on principal fractional ideals (β) of K by

$$\psi((\beta)) = \left|\frac{\beta}{\beta'}\right|^{\frac{\pi i k}{\log \epsilon}}.$$

Here β' is the image of β under the non-trivial automorphism of $K/\mathbb{Q}$ and $\epsilon > 1$ is a fundamental unit of O_K, the ring of integers of K. (Note that $\psi((\beta))$ is independent of the generator β.) Then ψ is extended to all ideals $\mathfrak{j}$ as follows: If $\mathfrak{j}^h = (\beta)$, define $\psi(\mathfrak{j})$ to be an h^{th} root of $\psi((\beta))$ so that

$$\psi(\mathfrak{j}^h) = \psi((\beta)) = \left|\frac{\beta}{\beta'}\right|^{\frac{\pi i k}{\log \epsilon}}.$$

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Let $\mathfrak{b}$ be a fractional ideal of K. The Hecke L-function with Grössencharakter ψ associated to the ideal class A of $\mathfrak{b}^{-1}$ is defined to be

$$\begin{aligned} L(s,\psi,A) &= \sum_{\mathfrak{a}\in A} \frac{\psi(\mathfrak{a})}{(\mathbb{N}\mathfrak{a})^s} \\ &= \frac{(\mathbb{N}b)^s}{\psi(\mathfrak{b})} \sum_{0\neq(\beta)\subseteq\mathfrak{b}} \frac{\psi((\beta))}{|\mathbb{N}(\beta)|^s}, \end{aligned}$$

where $\mathbb{N}$ denotes the norm from K to $\mathbb{Q}$. Hecke [He] then showed that $L(s,\psi,A)$ has a meromorphic continuation to all s with at most a simple pole at $s=1$ and satisfies a functional equation in $s\mapsto 1-s$.

Siegel [Si] found another proof of the functional equation by considering the hyperbolic Fourier expansion of the real analytic Eisenstein series

$$E(z,s) = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \mathrm{Im}(\gamma z)^s.$$

for the full modular group $\Gamma = SL_2(\mathbb{Z})$. Here

$$\Gamma_\infty = \left\{ \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} : m\in\mathbb{Z} \right\}$$

is the stabilizer of the cusp ∞.

Let $f(z) = \sum_{n\geq 1} a_n e^{2\pi i n z}$ be a weight two cuspform for $\Gamma_0(N)$, normalized so that $a_1 = 1$. Define the modular symbol

$$\langle \gamma, f\rangle = -2\pi i \int_\tau^{\gamma\tau} f(z)dz$$

for $\gamma\in\Gamma_0(N)$ and $\tau\in\mathbb{H}^* = \mathbb{H}\cup\mathbb{Q}\cup\{i\infty\}$, where $\mathbb{H}$ denotes the upper half plane. Note that the modular symbol does not depend on the choice of $\tau\in\mathbb{H}^*$, and by writing

$$\left\langle \begin{pmatrix} * & * \\ c & d \end{pmatrix}, f \right\rangle = -2\pi i \int_{-d/c}^{i\infty} f(z)dz,$$

we may extend the definition of the modular symbol to matrices which are not necessarily in $\Gamma_0(N)$.

In a series of papers ([Go1],[Go2],[O'S],[D-O'S]) the Eisenstein series twisted by modular symbols were introduced and studied. These Eisenstein series are defined by

$$E^*_{\mathfrak{a}}(z,s) = \sum_{\gamma \in \Gamma_{\mathfrak{a}} \backslash \Gamma} \langle \gamma, f \rangle \operatorname{Im}(\sigma_{\mathfrak{a}}^{-1} \gamma z)^s,$$

where $\mathfrak{a} \in \mathbb{Q} \cup \{i\infty\}$ is a cusp of $\Gamma = \Gamma_0(N)$,

$$\Gamma_{\mathfrak{a}} = \{\gamma \in \Gamma_0(N) : \gamma \mathfrak{a} = \mathfrak{a}\}$$

is the stabilizer of $\mathfrak{a}$ in Γ, and $\sigma_{\mathfrak{a}} \in SL_2(\mathbb{R})$ is uniquely determined by the conditions

$$\sigma_{\mathfrak{a}}^{-1} \mathfrak{a} = \infty, \;\; \sigma_{\mathfrak{a}}^{-1} \Gamma_{\mathfrak{a}} \sigma_{\mathfrak{a}} = \Gamma_\infty.$$

The $E^*_{\mathfrak{a}}(z,s)$ are not automorphic, but for all $\gamma \in \Gamma$, they satisfy the relation

$$E^*_{\mathfrak{a}}(\gamma z, s) = E^*_{\mathfrak{a}}(z,s) - \langle \gamma, f \rangle E_{\mathfrak{a}}(z,s),$$

where

$$E_{\mathfrak{a}}(z,s) = \sum_{\gamma \in \Gamma_{\mathfrak{a}} \backslash \Gamma} \operatorname{Im}(\sigma_{\mathfrak{a}}^{-1} \gamma z)^s$$

is the ordinary real analytic Eisenstein series for Γ associated to the cusp $\mathfrak{a}$.

The Eisenstein series $E_{\mathfrak{a}}(z,s)$ has a meromorphic continuation in s to the entire complex plane and the column vector

$$\mathcal{E}(z,s) = {}^t(E_{\mathfrak{a}_1}(z,s), E_{\mathfrak{a}_2}(z,s), \ldots)$$

(with the $\mathfrak{a}_i$ running over all inequivalent cusps) satisfies the functional equation

$$\mathcal{E}(z,s) = \Phi(s)\mathcal{E}(z, 1-s).$$

If $\Gamma_0(N)$ has r inequivalent cusps, then the so-called *scattering matrix* $\Phi(s)$ is an $r \times r$ matrix with entries $\phi_{\mathfrak{a}\mathfrak{b}}$ indexed by pairs of cusps of $\Gamma_0(N)$. These entries may be given explicitly in terms of divisor sums and Gamma functions, see e.g. [Hej]. Similar properties hold for $E^*_{\mathfrak{a}}(z,s)$. In particular, $E^*_{\mathfrak{a}}(z,s)$ has a meromorphic continuation to $\mathbb{C}$ and the column vector

$$\mathcal{E}^*(z,s) = {}^t(E^*_{\mathfrak{a}_1}(z,s), E^*_{\mathfrak{a}_2}(z,s), \ldots)$$

satisfies

$$\mathcal{E}^*(z,s) = \Phi(s)\mathcal{E}^*(z,1-s) + \Phi^*(s)\mathcal{E}(z,1-s) \tag{1}$$

where again, $\Phi^*(s)$ is an $r \times r$ matrix with entries $\phi^*_{\mathfrak{ab}}$ indexed by pairs of cusps of $\Gamma_0(N)$. The functional equation (1) was first established in [O'S]. In O'Sullivan's paper, the new scattering matrix $\Phi^*(s)$ was given as an infinite sum over double cosets. Using the results developed in Section 4 of this paper, we show

Theorem 1. *Let Φ and Φ^* be as in (1). Then*

$$\phi^*_{\mathfrak{ab}}(s) = T_{\mathfrak{ab}}\ \phi_{\mathfrak{ab}}(s),$$

where

$$T_{\mathfrak{ab}} = 2\pi i \int_{\mathfrak{a}}^{\mathfrak{b}} f(w)dw.$$

This theorem was established by the first author in collaboration with O'Sullivan, and we thank him for allowing us to include it here.

Following Siegel [Si] we will show that it is possible to obtain the hyperbolic Fourier expansion of $E^*_{\mathfrak{a}}(z,s)$ which in turn leads to a new type of zeta function twisted by a modular symbol. We now describe the zeta functions which arise.

Let

$$\rho = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

be a hyperbolic matrix in $\Gamma_0(N)$, i.e., $|\alpha+\delta| > 2$. The two fixed points of ρ,

$$w = \frac{\beta + \sqrt{(\alpha+\delta)^2 - 4}}{2\gamma}, \quad w' = \frac{\beta - \sqrt{(\alpha+\delta)^2 - 4}}{2\gamma}$$

lie in the real quadratic field $K = \mathbb{Q}[\sqrt{D}],\ D = (\alpha+\delta)^2 - 4$. We let ϵ and ϵ^{-1} be the two eigenvalues of ρ. We make the following assumptions:

A1: The level N is squarefree.

A2: The eigenvalue ϵ is a fundamental unit of O_K and $\epsilon > 1$.

A3: The modular symbol $\langle \rho, f \rangle = 0$.

The first two assumptions may be relaxed at the expense of some added complications. The third assumption is essential for the hyperbolic Fourier expansion of section 5.

To state our main result, we introduce some more notation. Since we have assumed N is squarefree, inequivalent cusps of Γ correspond to the divisors of N. For each divisor v of N, with corresponding cusp $\mathfrak{a} \sim 1/v$, we denote by $\mathfrak{J}_{\mathfrak{a}}$ the fractional ideal of K generated by 1 and vw,

$$\mathfrak{J}_{\mathfrak{a}} = \{cvw + d : c, d \in \mathbb{Z}\}.$$

For $j = pw + q \in K$ with $p, q \in \mathbb{Q}$ we define $j' = pw' + q$. For $cw + d$ an integer in K, we define

$$\langle cw + d, f \rangle = \left\langle \begin{pmatrix} * & * \\ c & d \end{pmatrix}, f \right\rangle = -2\pi i \int_{-d/c}^{i\infty} f(w) dw.$$

Let $\chi_0^{(v)}$ denote the trivial Dirichlet character mod v and extend $\chi_0^{(v)}$ to O_K by defining $\chi_0^{(v)}(cw + d) = \chi_0^{(v)}(d)$. Fix an integer n. Associated to $\chi_0^{(v)}$ we have the Grössencharakter ψ defined on principal ideals of O_K by

$$\psi((cw + d)) = \chi_0^{(v)}(d) \left| \frac{cw + d}{cw' + d} \right|^{-\frac{\pi i n}{\log \epsilon}}.$$

The principal object of study in this paper is the L-function $L_{\mathfrak{a}}^*(s, \psi)$ which is defined as a Dirichlet series

$$L_{\mathfrak{a}}^*(s, \psi) = \sum_{0 \neq (j) \subseteq \mathfrak{J}_{\mathfrak{a}}} \langle j, f \rangle \psi((j)) (\mathbb{N} j)^{-s},$$

where the sum is taken over all non-zero principal ideals contained in $\mathfrak{J}_{\mathfrak{a}}$. We view $L_{\mathfrak{a}}^*(s, \psi)$ as a twist, by the modular symbol $\langle \cdot, f \rangle$, of the classical Hecke L-function

$$L_{\mathfrak{a}}(s, \psi) = \sum_{0 \neq (j) \subseteq \mathfrak{J}_{\mathfrak{a}}} (\mathbb{N} j)^{-s} \psi((j)).$$

Let

$$G_n(s) = \frac{\Gamma\left(\frac{1}{2}\left(s - \frac{\pi i n}{\log \epsilon}\right)\right) \Gamma\left(\frac{1}{2}\left(s + \frac{\pi i n}{\log \epsilon}\right)\right)}{\Gamma(s)}.$$

Define

$$\xi_{\mathfrak{a}}(s, \psi) = G_n(s) \frac{(N(w - w')/v)^{-s}}{2 \log \epsilon \; L(2s, \chi_0^{(v)})} L_{\mathfrak{a}}(s, \psi)$$

and

$$\xi_{\mathfrak{a}}^*(s,\psi) = G_n(s)\frac{(N(w-w')/v)^{-s}}{2\log\epsilon\ L(2s,\chi_0^{(v)})}[T_{\mathfrak{a}\infty}L_{\mathfrak{a}}(s,\psi)+L_{\mathfrak{a}}^*(s,\psi)].$$

Let

$$\Lambda^*(s,\psi) = {}^t(\cdots,\xi_{\mathfrak{a}}^*(s,\psi),\cdots)_{\mathfrak{a}}$$

and

$$\Lambda(s,\psi) = {}^t(\cdots,\xi_{\mathfrak{a}}(s,\psi),\cdots)_{\mathfrak{a}}$$

be the associated column vectors of L-functions.

Theorem 2. *Assume* **A1-A3.** *Then the column vector L-functions Λ, Λ^* have an analytic continuation to the complex plane and satisfy the functional equation*

$$\Lambda^*(s,\psi) = \Phi(s)\Lambda^*(1-s,\psi)+\Phi^*(s)\Lambda(1-s,\psi),$$

where $\Phi(s)$ (resp. $\Phi^*(s)$) is the scattering matrix for $\mathcal{E}(z,s)$ (resp. $\mathcal{E}^*(z,s)$). Moreover, for $n\neq 0$, $L_{\mathfrak{a}}^*(s,\psi)$ has a simple pole at $s=1$ with residue given by

$$\frac{N(w-w')L(2,\chi_0^{(v)})}{v\,Vol(\Gamma_0(N)\backslash\mathbb{H})}\int_1^{\epsilon^2} F_{\mathfrak{a}}(\kappa^{-1}(ir))e^{\frac{-\pi i n}{\log\epsilon}}\frac{dr}{r},$$

with

$$\kappa = \begin{pmatrix} 1 & -w \\ 1 & -w' \end{pmatrix}$$

and $F_{\mathfrak{a}}(z) = 2\pi i\int_{\mathfrak{a}}^z f(w)dw$, the antiderivative of f.

§2. Rankin-Selberg L-functions

We repeat and elaborate some of the definitions given in the previous section. Define the Eisenstein series

$$E_{\mathfrak{a}}(z,s) = \sum_{\gamma\in\Gamma_{\mathfrak{a}}\backslash\Gamma}\mathrm{Im}(\sigma_{\mathfrak{a}}^{-1}\gamma z)^s$$

and its derivative,

$$\begin{aligned} E'_{\mathfrak{a}}(z,s) &= y\frac{\partial}{\partial\bar{z}}E_{\mathfrak{a}}(z,s) \\ &= \frac{is}{2}\sum_{\gamma\in\Gamma_{\mathfrak{a}}\backslash\Gamma}\mathrm{Im}(\sigma_{\mathfrak{a}}^{-1}\gamma z)^s\frac{j(\sigma_{\mathfrak{a}}^{-1}\gamma,z)^2}{|j(\sigma_{\mathfrak{a}}^{-1}\gamma,z)|^2} \end{aligned}$$

where $j(\gamma, z) = cz + d$. The Eisenstein series have a Fourier expansion given by

$$E_{\mathfrak{a}}(\sigma_{\mathfrak{b}} z, s) = \delta_{\mathfrak{a}\mathfrak{b}} y^s + \phi_{\mathfrak{a}\mathfrak{b}}(s) y^{1-s} + \sum_{n \neq 0} \phi_{\mathfrak{a}\mathfrak{b}}(n, s) W_s(nz)$$

where $W_s(z)$ is the Whittaker function

$$W_s(z) = \frac{\sqrt{y}}{\Gamma(s)} K_{s-\frac{1}{2}}(2\pi y) e^{2\pi i x},$$

and

$$K_s(y) = \tfrac{1}{2} \int_0^\infty e^{-\frac{y}{2}(u + \frac{1}{u})} u^{-s} \frac{du}{u}$$

is the Bessel function. The matrix

$$\Phi(s) = (\phi_{\mathfrak{a}\mathfrak{b}}(s))$$

is called the scattering matrix of the Eisenstein series; it is the matrix appearing in the functional equation of Section 1.

Fix an integer $k \geq 0$. For $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R})$, we define the slash operator $|_\sigma$ of weight k operating on holomorphic functions $f : \mathbb{H} \to \mathbb{C}$ by

$$f|_\sigma(z) = (ad - bc)^{k/2} (cz + d)^{-k} f\left(\frac{az + b}{cz + d}\right).$$

Let f be a holomorphic weight two cusp form for Γ with Fourier expansion

$$f|_{\sigma_{\mathfrak{a}}}(z) = \sum_1^\infty f_{\mathfrak{a}}(n) e(nz)$$

at the cusp $\mathfrak{a}$. Let

$$F_{\mathfrak{a}}(z) = 2\pi i \int_{\mathfrak{a}}^z f(w) dw.$$

We define the Eisenstein series twisted by a modular symbol

$$E^*_{\mathfrak{a}}(z, s) = \sum_{\gamma \in \Gamma_{\mathfrak{a}} \backslash \Gamma} \langle \gamma, f \rangle \operatorname{Im}(\sigma_{\mathfrak{a}}^{-1} \gamma z)^s$$

and the automorphic function

$$G_{\mathfrak{a}}(z, s) = E^*_{\mathfrak{a}}(z, s) - F_{\mathfrak{a}}(z) E_{\mathfrak{a}}(z, s).$$

It follows that

$$G_{\mathfrak{a}}(z,s) = -\sum_{\gamma\in\Gamma_{\mathfrak{a}}\backslash\Gamma} F_{\mathfrak{a}}(\gamma z)\mathrm{Im}(\sigma_{\mathfrak{a}}^{-1}\gamma z)^s.$$

We compute the Petersson inner product $\langle fE'_{\mathfrak{a}}(\cdot,s)\mathrm{Im}(\cdot),\ \eta_j\rangle$. Here $z = x+iy \in \mathbb{H}$.

$$\begin{aligned}
&\langle fE'_{\mathfrak{a}}(\cdot,s)\mathrm{Im}(\cdot),\ \eta_j\rangle \\
&= \int_{\Gamma\backslash\mathbb{H}} f(z)\bar{\eta}(z)E'_{\mathfrak{a}}(z,s)\mathrm{Im}(z)\frac{dx\ dy}{y^2} \\
&= \frac{is}{2}\sum_{\gamma\in\Gamma_{\mathfrak{a}}\backslash\Gamma}\int_{\Gamma\backslash\mathbb{H}} f(z)\bar{\eta}(z)\mathrm{Im}(\sigma_{\mathfrak{a}}^{-1}\gamma z)^s\frac{j(\sigma_{\mathfrak{a}}^{-1}\gamma,z)^2}{|j(\sigma_{\mathfrak{a}}^{-1}\gamma,z)|^2}\mathrm{Im}(z)\frac{dx\ dy}{y^2} \\
&= \frac{is}{2}\int_0^\infty\int_0^1 f|_{\sigma_{\mathfrak{a}}}(z)\bar{\eta}(\sigma_{\mathfrak{a}}z)(\mathrm{Im}z)^{s+1}dx\frac{dy}{y^2} \\
&= \frac{i}{2}\frac{\Gamma(s+\frac{1}{2}+ir_j)\Gamma(s+\frac{1}{2}-ir_j)}{\pi^s 2^{2s+1}\Gamma(s)}L_{\mathfrak{a}}(s,f\otimes\eta),
\end{aligned}$$

where

$$L_{\mathfrak{a}}(s,f\otimes\eta) := \sum_{n\geq 1}\frac{f_{\mathfrak{a}}(n)\bar{b}_{\mathfrak{a}}(n)}{n^s}.$$

The vector Eisenstein series satisfies the functional equation

$$\mathcal{E}(z,s) = \Phi(s)\mathcal{E}(z,1-s)$$

and after applying $y\frac{\partial}{\partial\bar{z}}$, we also obtain

$$\mathcal{E}'(z,s) = \Phi(s)\mathcal{E}'(z,1-s).$$

Similarly, define the column vector of convolution L-functions $\mathcal{L}(s,f\otimes\eta)$. Then the completed L-function

$$\Lambda(s,f\otimes\eta_j) := \frac{\Gamma(s+\frac{1}{2}+ir_j)\Gamma(s+\frac{1}{2}-ir_j)}{\pi^s 2^{2s+1}\Gamma(s)}\mathcal{L}(s,f\otimes\eta)$$

satisfies the functional equation

$$\Lambda(s,f\otimes\eta_j) = \Phi(s)\Lambda(1-s,f\otimes\eta_j).$$

This follows immediately from the representation

$$\Lambda_{\mathfrak{a}}(s, f \otimes \eta_j) = \frac{2}{i}\langle f E'_{\mathfrak{a}}(\cdot, s)\mathrm{Im}(\cdot),\ \eta_j\rangle$$

and the functional equation for the Eisenstein series.

In the same way, we may show that

$$\begin{aligned} &\frac{2}{i}\langle f E'_{\mathfrak{a}}(\cdot, s)\mathrm{Im}(\cdot),\ E_{\mathfrak{b}}(\cdot, \tfrac{1}{2} + ir)\rangle \\ &\qquad = \frac{\Gamma(s + \frac{1}{2} + ir)\Gamma(s + \frac{1}{2} - ir)}{\pi^s 2^{2s+1}\Gamma(s)} L_{\mathfrak{a}}(s, f \otimes E_{\mathfrak{b}}(\tfrac{1}{2} + ir)), \end{aligned}$$

where we have defined

$$L_{\mathfrak{a}}(s, f \otimes E_{\mathfrak{b}}(\tfrac{1}{2} + ir)) = \sum_{n \geq 1} \frac{f_{\mathfrak{a}}(n)\bar{\phi}_{\mathfrak{b}\mathfrak{a}}(n, \frac{1}{2} + ir)}{n^s}.$$

As before, define the column vector of L-functions $\mathcal{L}(s, f \otimes E_{\mathfrak{b}}(\frac{1}{2} + ir))$ and the completed L-function by

$$\Lambda(s, f \otimes E_{\mathfrak{b}}(\tfrac{1}{2} + ir)) := \frac{\Gamma(s + \frac{1}{2} + ir)\Gamma(s + \frac{1}{2} - ir)}{\pi^s 2^{2s+1}\Gamma(s)} \mathcal{L}(s, f \otimes E_{\mathfrak{b}}(\tfrac{1}{2} + ir)).$$

This satisfies the functional equation

$$\Lambda(s, f \otimes E_{\mathfrak{b}}(1/2 + ir)) = \Phi(s)\Lambda(1 - s, f \otimes E_{\mathfrak{b}}(1/2 + ir)).$$

§3. A Functional Equation for $\mathcal{G}(z, s)$

Let $\eta_1, \eta_2, \ldots$ be an orthonormal basis of Maass cusp forms with Fourier expansions given by

$$\eta_j(\sigma_{\mathfrak{a}} z) = \sum_{n \neq 0} b_{\mathfrak{a},j}(n)\sqrt{|n|y}K_{ir_j}(2\pi|n|y)e(nx).$$

Here, $\lambda_j = 1/4 + r_j^2$ denotes the eigenvalue of η_j. The Selberg spectral decomposition says that every $g \in \mathcal{L}^2(\Gamma\backslash\mathbb{H})$ which is orthogonal to the constants has the representation

$$g(z) = \sum_{j=1}^{\infty}\langle g, \eta_j\rangle\eta_j(z) + \frac{1}{4\pi}\sum_{\mathfrak{a}}\int_{-\infty}^{+\infty}\langle g, E_{\mathfrak{a}}(\cdot, 1/2 + ir)\rangle E_{\mathfrak{a}}(z, 1/2 + ir)dr.$$

We will use the Selberg spectral decomposition to obtain the meromorphic continuation and functional equation for the Eisenstein series formed with modular symbols.

Recall the definitions

$$F_{\mathfrak{a}}(z) = 2\pi i \int_{\mathfrak{a}}^{z} f(w)dw$$

and

$$\begin{aligned} G_{\mathfrak{a}}(z,s) &= E^*_{\mathfrak{a}}(z,s) - F_{\mathfrak{a}}(z)E_{\mathfrak{a}}(z,s) \\ &= -\sum_{\gamma\in\Gamma_{\mathfrak{a}}\backslash\Gamma} F_{\mathfrak{a}}(\gamma z)\mathrm{Im}(\sigma_{\mathfrak{a}}^{-1}\gamma z)^s. \end{aligned}$$

After a change of variables, we get

$$F_{\mathfrak{a}}(\sigma_{\mathfrak{a}} z) = \sum_{n\geq 1} \frac{f_{\mathfrak{a}}(n)}{n} e^{2\pi i n z}.$$

We define the column vector

$$\mathcal{G}(z,s) = {}^t(G_{\mathfrak{a}}(z,s))_{\mathfrak{a}} = \mathcal{E}^*(z,s) - \mathcal{F}(z)\mathcal{E}(z,s),$$

where $\mathcal{F}$ is the diagonal matrix $diag(\ldots, F_{\mathfrak{a}}(z), \ldots)$ indexed by inequivalent cusps $\mathfrak{a}$. As in [Go2] one may compute the inner products of $G_{\mathfrak{a}}(z,s)$ with the Maass cusp forms and the Eisenstein series on the line $\mathrm{Re}(s) = 1/2$. Doing this, we find

$$\langle G_{\mathfrak{a}}(\cdot,s),\ \eta_j\rangle = \frac{\Gamma(s+\frac{1}{2}-ir_j)\Gamma(s+\frac{1}{2}+ir_j)L_{\mathfrak{a}}(s,f\otimes\eta_j)}{\pi^{s-1}2^{2s-1}\Gamma(s)(s-\frac{1}{2}-ir_j)(s-\frac{1}{2}+ir_j)}$$

and

$$\langle G_{\mathfrak{a}}(\cdot,s),\ E_{\mathfrak{b}}(\cdot,\tfrac{1}{2}+ir)\rangle = \frac{\Gamma(s+\frac{1}{2}-ir)\Gamma(s+\frac{1}{2}+ir)L_{\mathfrak{a}}(s,f\otimes E_{\mathfrak{b}}(\frac{1}{2}+ir))}{\pi^{s-1}2^{2s-1}\Gamma(s)(s-\frac{1}{2}-ir)(s-\frac{1}{2}+ir)}.$$

In vector notation

$$\langle \mathcal{G}(\cdot,s),\ \eta_j\rangle = \frac{1}{4\pi}\frac{\Lambda(s,f\otimes\eta_j)}{(s-\frac{1}{2}-ir_j)(s-\frac{1}{2}+ir_j)} \tag{2}$$

and

$$\langle \mathcal{G}(\cdot, s),\ E_{\mathfrak{b}}(\cdot, \tfrac{1}{2} + ir)\rangle = \frac{1}{4\pi}\frac{\Lambda(s, f \otimes E_{\mathfrak{b}}(\frac{1}{2} + ir)}{(s - \frac{1}{2} - ir)(s - \frac{1}{2} + ir)}. \tag{3}$$

Now, use the Selberg spectral decomposition to write $\mathcal{G}(\cdot, s)$ as a series expansion with coefficients given by the above inner products. Then from the functional equation for the Rankin-Selberg L-functions together with the fact that the denominators of (2) and (3) are invariant under $s \mapsto 1-s$, we deduce that

$$\mathcal{G}(z, s) = \Phi(s)\mathcal{G}(z, 1 - s).$$

Note that all of the formal manipulations of this section are justified because $G_{\mathfrak{a}}(z, s)$ is square integrable for all s.

§4. Proof of Theorem 1

The functional equation for $\mathcal{G}(z, s)$ given in section 4 may be combined with the functional equation given in [O'S] to give a very simple formula for the entries of Φ^*. The equation in [O'S] is

$$\mathcal{E}^*(z, s) = \Phi(s)\mathcal{E}^*(z, 1 - s) + \Phi^*(s)\mathcal{E}(z, 1 - s).$$

Writing

$$\mathcal{E}^*(z, s) = \mathcal{G}(z, s) + \mathcal{F}(z)\mathcal{E}(z, s)$$

and using the functional equation for $\mathcal{G}(z, s)$ we get

$$\Phi^*(1 - s)\mathcal{E}(z, s) = \mathcal{F}(z)\mathcal{E}(z, 1 - s) - \Phi(1 - s)\mathcal{F}(z)\mathcal{E}(z, s). \tag{4}$$

Now replace z by $\sigma_{\mathfrak{b}} z$ and compare the constant term in the Fourier coefficients of both sides. For this we need,

$$\begin{aligned} \text{constant term of } E_{\mathfrak{a}}(\sigma_{\mathfrak{b}} z, s) &= \delta_{\mathfrak{a}\mathfrak{b}} y^s + \phi_{\mathfrak{a}\mathfrak{b}}(s) y^{1-s} \\ \text{constant term of } F_{\mathfrak{a}}(\sigma_{\mathfrak{b}} z) &= T_{\mathfrak{a}\mathfrak{b}} \end{aligned}$$

The constant term of $F_{\mathfrak{a}}(\sigma_{\mathfrak{b}} z)$ is computed as follows:

$$\begin{aligned} F_{\mathfrak{a}}(\sigma_{\mathfrak{b}} z) &= 2\pi i \int_{\mathfrak{a}}^{\sigma_{\mathfrak{b}} z} f(w)dw \\ &= 2\pi i \int_{\mathfrak{a}}^{\mathfrak{b}} f(w)dw + 2\pi i \int_{\mathfrak{b}}^{\sigma_{\mathfrak{b}} z} f(w)dw \\ &= T_{\mathfrak{a}\mathfrak{b}} + \sum_{n \geq 1} \frac{f_{\mathfrak{b}}(n)}{n} e^{2\pi i n z} \end{aligned}$$

Let $\mathfrak{a}_1, \mathfrak{a}_2, \ldots$ denote the inequivalent cusps of $\Gamma_0(N)$. Then the constant term of the j^{th} column on the left side of (4) is

$$\sum_i \phi^*_{\mathfrak{a}_j\mathfrak{a}_i}(1-s)[\delta_{\mathfrak{a}_i\mathfrak{b}}y^s + \phi_{\mathfrak{a}_i\mathfrak{b}}(s)y^{1-s}],$$

and the j^{th} column on the right side of (4) is

$$T_{\mathfrak{a}_j\mathfrak{b}}\phi_{\mathfrak{a}_j\mathfrak{b}}(1-s)y^s - \sum_i \phi_{\mathfrak{a}_j\mathfrak{a}_i}(s)T_{\mathfrak{a}_i\mathfrak{b}}\phi_{\mathfrak{a}_i\mathfrak{b}}(s)y^{1-s}.$$

Equating the terms involving y^s, we get

$$\phi^*_{\mathfrak{a}_j\mathfrak{b}}(1-s)y^s = T_{\mathfrak{a}_j\mathfrak{b}}\phi_{\mathfrak{a}_j\mathfrak{b}}(1-s)y^s.$$

Hence, for any two cusps $\mathfrak{a}, \mathfrak{b}$,

$$\phi^*_{\mathfrak{a}\mathfrak{b}}(s) = T_{\mathfrak{a}\mathfrak{b}}\phi_{\mathfrak{a}\mathfrak{b}}(s),$$

as was to be shown.

§5. The Hyperbolic Fourier Expansion for $\mathcal{E}^*(z,s)$

Let ρ be a fixed hyperbolic matrix in $\Gamma_0(N)$. We recall the assumptions made in the introduction:

A1: The level N is squarefree.

A2: The eigenvalues ϵ, ϵ^{-1} are fundamental units in O_K and $\epsilon > 1$.

A3: The modular symbol $\langle \rho, f \rangle = 0$.

We will compute the hyperbolic Fourier expansion of $E^*_{\mathfrak{a}}(z,s)$ with respect to ρ. By **A3**, $E^*_{\mathfrak{a}}(\rho z, s) = E^*_{\mathfrak{a}}(z,s)$.

Let w, w' be the two real fixed points of ρ. Define

$$\kappa = \begin{pmatrix} 1 & -w \\ 1 & -w' \end{pmatrix}.$$

Then

$$\kappa\rho\kappa^{-1} = \begin{pmatrix} \epsilon & \\ & \epsilon^{-1} \end{pmatrix}.$$

The function $E^*_{\mathfrak{a}}(\kappa^{-1}z, s)$ is invariant under $z \mapsto \epsilon^2 z$. Therefore, on the positive imaginary axis (i.e. choosing $z = ir$), $E^*_{\mathfrak{a}}(\kappa^{-1}z, s)$ has the Fourier expansion

$$E^*_{\mathfrak{a}}(\kappa^{-1}(ir), s) = \sum g^*_{\mathfrak{a}}(n, s) e^{\pi i \frac{n \log r}{\log \epsilon}}.$$

The Fourier coefficients are given by

$$g^*_{\mathfrak{a}}(n, s) = \frac{1}{2\log\epsilon} \int_1^{\epsilon^2} E^*_{\mathfrak{a}}(\kappa^{-1}(ir), s) e^{-\pi i \frac{n \log r}{\log \epsilon}} \frac{dr}{r}.$$

A set of inequivalent cusps for $\Gamma_0(N)$ is given by $\{1/v : v|N\}$. The scaling matrix $\sigma_{\mathfrak{a}}$ for the cusp $\mathfrak{a} \sim 1/v$ is given by

$$\sigma_{\mathfrak{a}} = \begin{pmatrix} \sqrt{N/v} & * \\ \sqrt{Nv} & * \end{pmatrix} \in SL_2(\mathbb{R}).$$

A direct computation shows that

$$\mathrm{Im}\left(\sigma_0^{-1}\gamma\kappa^{-1}(ir)\right) = \frac{(rv/N)(w-w')^{-1}}{[(av-c)w' + (bv-d)]^2 r^2 + [(av-c)w + (bv-d)]^2}.$$

As $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ ranges over elements in $\Gamma_{\mathfrak{a}}\backslash\Gamma$, the pairs $(av-c, bv-d)$ range over distinct pairs of integers (c, d) such that $c \equiv 0(v)$ and $(c, d) = 1$.

Furthemore, we observe that for

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N),$$

the modular symbol

$$\begin{aligned}
\left\langle \begin{pmatrix} a & b \\ c & d \end{pmatrix}, f \right\rangle &= -\left\langle \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}, f \right\rangle \\
&= 2\pi i \int_{1/v}^{-\frac{bv-d}{av-c}} f(z)dz \\
&= 2\pi i \int_{1/v}^{i\infty} f(z)dz + 2\pi i \int_{i\infty}^{-\frac{bv-d}{av-c}} f(z)dz \\
&= T_{\mathfrak{a}\infty} + \left\langle \begin{pmatrix} * & * \\ av-c & bv-d \end{pmatrix}, f \right\rangle.
\end{aligned}$$

(Recall our convention from the introduction for defining the modular symbol $\langle\gamma, f\rangle$ when γ is not in $\Gamma_0(N)$.)

Therefore,

$$\begin{aligned}
&E^*_{\mathfrak{a}}(\kappa^{-1}(ir), s) \\
&\quad= \sum_{\gamma\in\Gamma_{\mathfrak{a}}\backslash\Gamma_0(N)} \langle\gamma, f\rangle \,\mathrm{Im}\left(\sigma_{\mathfrak{a}}^{-1}\gamma\kappa^{-1}(ir)\right)^s \\
&\quad= \sum_{\substack{(c,d)=1\\ c\equiv 0(v)}} \left[T_{\mathfrak{a}\infty} + \left\langle \begin{pmatrix} * & * \\ c & d \end{pmatrix}, f \right\rangle\right] \left(\frac{rv/N(w-w')}{(cw'+d)^2r^2+(cw+d)^2}\right)^s.
\end{aligned}$$

We introduce the Möbius function μ which satisfies

$$\sum_{e|(c,d)} \mu(e) = \begin{cases} 1 & (c,d)=1 \\ 0 & \text{otherwise} \end{cases}$$

to relax the condition $(c,d)=1$, and conclude that

$$\begin{aligned}
&E^*_{\mathfrak{a}}(\kappa^{-1}(ir), s) = \\
&\qquad \frac{(rv/N(w-w'))^s}{L(2s,\chi)} \sum_{\substack{(c,d)\neq 0\\ c\equiv 0(N)}} \left[T_{\mathfrak{a}\infty} + \left\langle \begin{pmatrix} * & * \\ c & d \end{pmatrix}, f \right\rangle\right] \chi(d) \\
&\qquad\qquad \times \left(\frac{r}{(cw'+d)^2r^2+(cw+d)^2}\right)^s,
\end{aligned}$$

where $\chi = \chi_0^{(v)}$ is the trivial character mod v. Therefore

$$g^*_{\mathfrak{a}}(n,s) = \frac{(v/N(w-w'))^s}{2L(2s,\chi)\log\epsilon} \sum_{\substack{(c,d)\neq 0\\ c\equiv 0(N)}} \left[T_{\mathfrak{a}\infty} + \left\langle \begin{pmatrix} * & * \\ c & d \end{pmatrix}, f \right\rangle\right] \chi(d) I_{c,d},$$

where

$$\begin{aligned}
I_{c,d} &= \int_1^{\epsilon^2} \left(\frac{r}{(cw'+d)^2r^2+(cw+d)^2}\right)^s e^{-\pi i \frac{n\log r}{\log\epsilon}} \frac{dr}{r} \\
&= \mathbb{N}(cw+d)^{-s} \left|\frac{cw+d}{cw'+d}\right|^{\frac{-\pi i n}{\log\epsilon}} \int_{\frac{cw'+d}{cw+d}}^{\epsilon^2\frac{cw'+d}{cw+d}} \left(\frac{r}{r^2+1}\right)^s e^{\pi i \frac{n\log r}{\log\epsilon}} \frac{dr}{r}.
\end{aligned}$$

In the previous expression, $\mathbb{N}(cw+d) := (cw+d)(cw'+d)$.

In the notation of the introduction,

$$g_{\mathfrak{a}}^{*}(n,s) = \frac{(v/N(w-w'))^s}{2L(2s,\chi)\log\epsilon} \sum_{j\in\mathfrak{J}_{\mathfrak{a}}, j\neq 0} \langle j,f\rangle\, \chi(j)(\mathbb{N}j)^{-s} \left|\frac{j}{j'}\right|^{-\frac{\pi i n}{\log\epsilon}} \int_{\frac{j'}{j}}^{\epsilon^2\frac{j'}{j}},$$

where

$$\int_{\frac{j'}{j}}^{\epsilon^2\frac{j'}{j}} = \int_{\frac{j'}{j}}^{\epsilon^2\frac{j'}{j}} \left(\frac{r}{r^2+1}\right)^s e^{-\pi i \frac{n\log r}{\log\epsilon}} \frac{dr}{r}.$$

We write the sum over non-zero integers in the ideal $\mathfrak{J}_{\mathfrak{a}}$ as a double sum over principal ideals (j) contained in $\mathfrak{J}_\infty$ and generators of (j). Since ϵ generates the unit group, we have

$$\begin{aligned}
&\sum_{\substack{j\in\mathfrak{J}_{\mathfrak{a}}\\ j\neq 0}} \langle j,f\rangle\, \chi(j)(\mathbb{N}j)^{-s} \left|\frac{j}{j'}\right|^{-\frac{\pi i n}{\log\epsilon}} \int_{\frac{j'}{j}}^{\epsilon^2\frac{j'}{j}} \\
&= \sum_{(j)\subseteq\mathfrak{J}_{\mathfrak{a}}, j\neq 0} (\mathbb{N}j)^{-s}\langle j,f\rangle \left|\frac{j'}{j}\right|^{\frac{\pi i n}{\log\epsilon}} \sum_{m\in\mathbb{Z}} \int_{\epsilon^{-2m}\frac{j'}{j}}^{\epsilon^{-2(m+1)}\frac{j'}{j}} \left(\frac{r}{r^2+1}\right)^s e^{-\pi i \frac{n\log r}{2\log\epsilon}} \frac{dr}{r}
\end{aligned}$$

The inner sum over m divides the positive real axis into non-overlapping intervals, thus the integral evaluates to

$$G_n(s) = \frac{\Gamma\left(\frac{1}{2}\left(s-\frac{\pi i n}{\log\epsilon}\right)\right)\Gamma\left(\frac{1}{2}\left(s+\frac{\pi i n}{\log\epsilon}\right)\right)}{\Gamma(s)}.$$

We conclude that

$$g_{\mathfrak{a}}^{*}(n,s) = \frac{(v/N(w-w'))^s}{2L(2s,\chi)\log\epsilon} G_n(s) \sum_{0\neq(j)\subseteq\mathfrak{J}_{\mathfrak{a}}} [T_{\mathfrak{a}\infty} + \langle j,f\rangle]\chi(j)(\mathbb{N}j)^{-s} \left|\frac{j}{j'}\right|^{-\frac{\pi i n}{\log\epsilon}}.$$

A similar but simpler computation gives the hyperbolic Fourier coefficients of the ordinary Eisenstein series:

$$E_{\mathfrak{a}}(\kappa^{-1}(ir),s) = \sum g_{\mathfrak{a}}(n,s) e^{\pi i \frac{n\log r}{\log\epsilon}},$$

with

$$g_{\mathfrak{a}}(n,s) = \frac{(v/N(w-w'))^s}{2L(2s,\chi)\log\epsilon} G_n(s) \sum_{0\neq(j)\subseteq\mathfrak{J}_{\mathfrak{a}}} \chi(j)(\mathbb{N}j)^{-s} \left|\frac{j}{j'}\right|^{-\frac{\pi i n}{\log\epsilon}}.$$

§6. Proof of Theorem 2

The proof of the first part of Theorem 2 now follows immediately from the functional equation (1) and the results of the previous section. The hyperbolic Fourier coefficients $g_{\mathfrak{a}}^*(n,s)$ and $g_{\mathfrak{a}}(n,s)$ must satisfy (1) as well. But these Fourier coefficients are precisely the L-functions appearing in the Theorem.

We now compute the residue of $L_{\mathfrak{a}}^*(s,\psi)$ at $s=1$. It is known [O'S] that $E_{\mathfrak{a}}^*(z,s)$ has a simple pole at $s=1$ with residue given by

$$\frac{F_{\mathfrak{a}}(z)}{\mathrm{Vol}(\Gamma_0(N)\backslash\mathbb{H})}.$$

Consequently,

$$\mathrm{Res}_{s=1} g_{\mathfrak{a}}^*(n,s) = \frac{1}{2\ \log\epsilon\ \mathrm{Vol}(\Gamma_0(N)\backslash\mathbb{H})}\int_1^{\epsilon^2} F_{\mathfrak{a}}(\kappa^{-1}(ir))e^{-\frac{\pi i n}{\log\epsilon}}\frac{dr}{r}.$$

But

$$g_{\mathfrak{a}}^*(n,s) = \frac{(v/N(w-w'))^s}{2\log\epsilon\ L(2s,\chi)}\left(T_{\mathfrak{a}\infty}L_{\mathfrak{a}}(s,\psi)+L_{\mathfrak{a}}^*(s,\psi)\right).$$

Assume $n\neq 0$. In this case, $L_{\mathfrak{a}}(s,\psi)$ is entire [He]. Therefore,

$$\mathrm{Res}_{s=1} g_{\mathfrak{a}}^*(n,s) = \frac{v}{2N(w-w')\log\epsilon\ L(2,\chi)}\cdot\mathrm{Res}_{s=1}L_{\mathfrak{a}}^*(s,\psi).$$

Solving for the residue of the twisted Grössencharakter L-function,

$$\mathrm{Res}_{s=1}L_{\mathfrak{a}}^*(s,\psi) = \frac{N(w-w')\ L(2,\chi)}{v\cdot\mathrm{Vol}(\Gamma_0(N)\backslash\mathbb{H})}\int_1^{\epsilon^2} F_{\mathfrak{a}}(\kappa^{-1}(ir))e^{\frac{-\pi i n}{\log\epsilon}}\frac{dr}{r}.$$

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