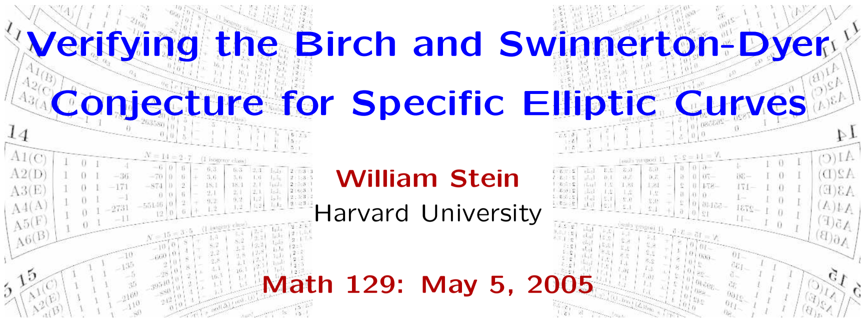

# Verifying the Birch and Swinnerton-Dyer Conjecture for Specific Elliptic Curves

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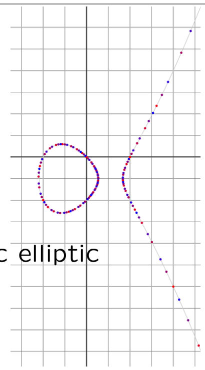


This talk reports on a project to verify the Birch and Swinnerton-Dyer conjecture for many specific elliptic curves over $\mathbb{Q}$.

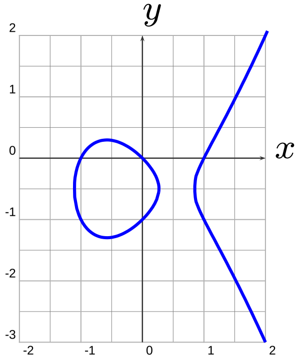
**Joint Work:** Grigor Grigorov, Andrei Jorza, Corina Patrascu, Stefan Patrikis

**Thanks:** John Cremona, Stephen Donnelly, Ralph Greenberg, Grigor Grigorov, Barry Mazur, Robert Pollack, Nick Ramsey, Tony Scholl, Micahel Stoll.



## Elliptic Curves over the Rational Numbers $\mathbb{Q}$

An elliptic curve is a nonsingular plane cubic curve with a rational point (possibly "at infinity").




$y^2 + y = x^3 - x$

EXAMPLES

$$y^2 + y = x^3 - x$$

$$x^3 + y^3 = z^3 \text{ (projective)}$$

$$y^2 = x^3 + ax + b$$

$$\cancel{3x^3 + 4y^3 + 5z^3 = 0}$$



## Mordell's Theorem

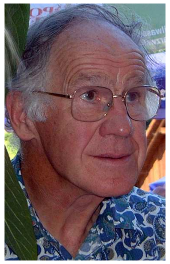
Theorem (Mordell). The group $E(\mathbb{Q})$ of rational points on an elliptic curve is a finitely generated abelian group, so

$$E(\mathbb{Q}) \cong \mathbb{Z}^r \oplus T,$$

with $T = E(\mathbb{Q})_{\text{tor}}$ finite.

Mazur classified the possibilities for $T$.

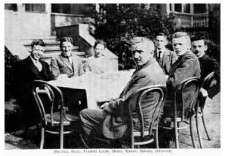
**Folklore conjecture:** $r$ can be arbitrary, but the biggest $r$ ever found is (probably) 24.



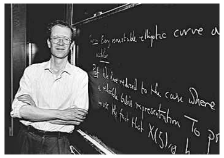
# Conjectures Proliferated

"The subject of this lecture is rather a special one. I want to describe some computations undertaken by myself and Swinnerton-Dyer on EDSAC, by which we have calculated the zeta-functions of certain elliptic curves. As a result of these computations we have found an analogue for an elliptic curve of the Tamagawa number of an algebraic group; and conjectures have proliferated. [...] though the associated theory is both abstract and technically complicated, the objects about which I intend to talk are usually simply defined and often machine computable; experimentally we have detected certain relations between different invariants, but we have been unable to approach proofs of these relations, which must lie very deep." – Birch 1965





# The $L$-Function



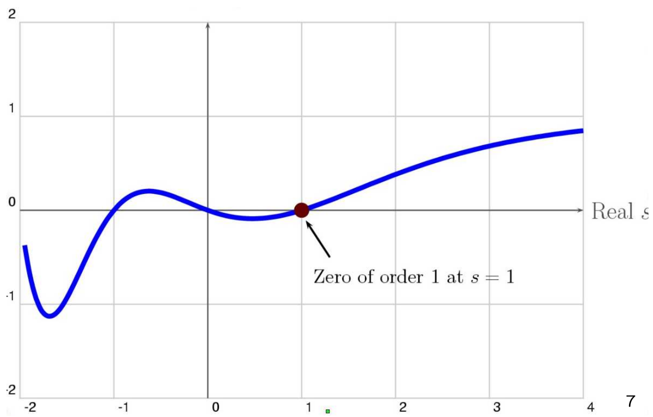
Theorem (Wiles et al., Hecke) The following function extends to a holomorphic function on the whole complex plane:

$$L(E,s) = \prod_{p \nmid \Delta} \left( \frac{1}{1 - a_p \cdot p^{-s} + p \cdot p^{-2s}} \right) \cdot \prod_{p \mid N} L_p(E,s)$$

Here $a_p = p + 1 - \#E(\mathbb{F}_p)$ for all $p \nmid \Delta$, where $\Delta$ is divisible by the primes of bad reduction for $E$. We do not include the factors $L_p(E,s)$ at bad primes here.

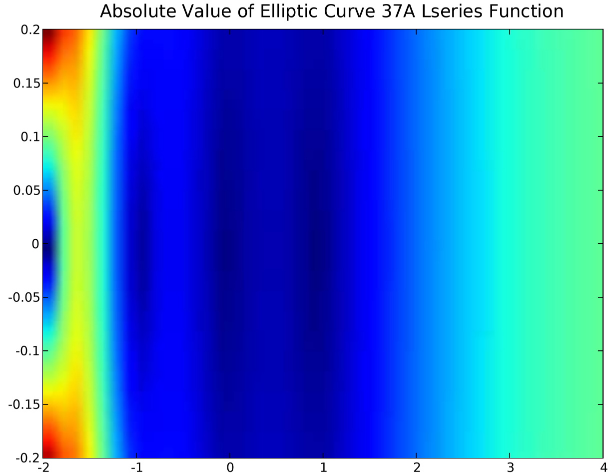


# Real Graph of the $L$-Series of $y^2 + y = x^3 - x$






# Graph of $L$-Series of $y^2 + y = x^3 - x$



## The Birch and Swinnerton-Dyer Conjecture

Conjecture: Let $E$ be any elliptic curve over $\mathbb{Q}$. The order of vanishing of $L(E,s)$ as $s=1$ equals the rank of $E(\mathbb{Q})$.

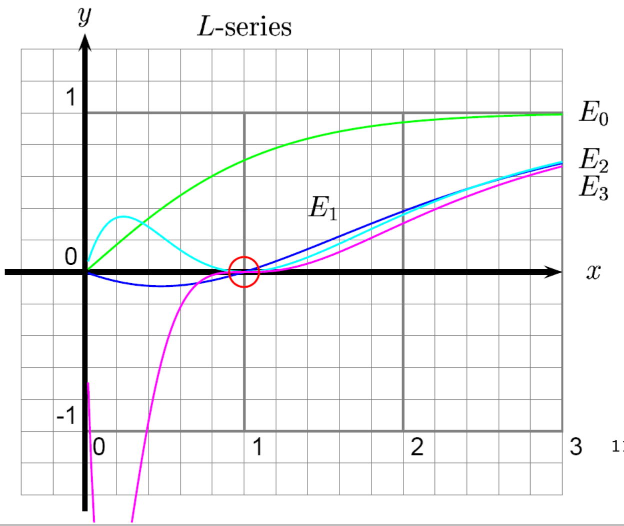


## The Kolyvagin and Gross-Zagier Theorems

Theorem: If the ordering of vanishing $\mathrm{ord}_{s=1} L(E,s)$ is $\leq 1$, then the conjecture is true for $E$.



## What about Taylor series of $L(E,s)$ around $s=1$?






## Taylor Series

For $y^2+y=x^3-x$, the **Taylor series** about 1 is

$$L(E,s) = 0 + (s-1)0.3059997\ldots$$
$$+(s-1)^2 0.18636\ldots + \cdots$$

In general, it's

$$L(E,s) = c_0 + c_1 s + c_2 s^2 + \cdots.$$

Big Mystery: Do these Taylor coefficients $c_n$ have any deep arithmetic meaning?

## BSD Formula Conjecture

Let $r = \mathrm{ord}_{s=1} L(E,s)$. Then Birch and Swinnerton-Dyer made a famous guess for the first nonzero coefficient $c_r$:

$$c_r = \frac{\Omega_E \cdot \mathrm{Reg}_E \cdot \prod_{p|N} t_p}{\#E(\mathbb{Q})_{\mathrm{tor}}^2} \cdot \#Ш(E)$$

- $\#E(\mathbb{Q})_{\mathrm{tor}}$ – **torsion** order
- $t_p$ – **Tamagawa numbers**
- $\Omega_E$ – **real volume** $\int_{E(\mathbb{R})} \omega_E$
- $\mathrm{Reg}_E$ – **regulator** of $E$
- $Ш(E) = \mathrm{Ker}(\mathrm{H}^1(\mathbb{Q}, E) \to \bigoplus_v \mathrm{H}^1(\mathbb{Q}_v, E))$ – **Shafarevich-Tate group**



## What about $c_{r+1}$, $c_{r+2}$, etc?

I think nobody has even a **wild and crazy** guess for an interpretation of these.

They are probably not "periods" like $c_r$ is, so perhaps should not have any nice interpretation...



## Motivating Problem 1

**Motivating Problem 1.** For specific curves, compute every quantity appearing in the BSD formula conjecture **in practice.**

NOTE:

This is **not** meant as a theoretical problem about computability, though by compute we mean "compute with proof."



## Status

1. When $r_{\mathrm{an}} = \mathrm{ord}_{s=1} L(E,s) \leq 3$, then we can compute $r_{\mathrm{an}}$.
   **Open Problem:** Show that $r_{\mathrm{an}} \geq 4$ for some elliptic curve.
2. "Relatively easy" to compute $\#E(\mathbb{Q})_{\mathrm{tor}}$, $c_p$, $\Omega_E$.
3. Computing $\mathrm{Reg}_E$ essentially same as computing $E(\mathbb{Q})$; interesting and sometimes very difficult.
4. Computing $\#Ш(E)$ is currently **very very difficult**.
   **Theorem (Kolyvagin):**
   $r_{\mathrm{an}} \leq 1 \implies Ш(E)$ is finite (with bounds)
   **Open Problem:**
   Prove that $Ш(E)$ is finite for some $E$ with $r_{\mathrm{an}} \geq 2$.

# Victor Kolyvagin

Kolyvagin's work on Euler systems is crucial to our project.




# Motivating Problem 2: Cremona's Book

**Motivating Problem 2.** Prove BSD for every elliptic curve over $\mathbb{Q}$ of conductor at most 1000, i.e., in Cremona's book.

1. By Tate's isogeny invariance of BSD, it suffices to prove BSD for each **optimal** elliptic curve of conductor $N \leq 1000$.
2. **Rank part** of the conjecture has been verified by Cremona for all curves with $N \leq 40000$.
3. All of the quantities in the conjecture, **except** for $\#\text{Ш}(E/\mathbb{Q})$, have been computed by Cremona for conductor $\leq 40000$.
4. **Cremona (Ch. 4, pg. 106):** We have $2 \nmid \#\text{Ш}(E)$ for **all** optimal curves with conductor $\leq 1000$ except 571A, 960D, and 960N. So we can mostly ignore 2 henceforth.



# John Cremona

John Cremona's software and book are crucial to our project.



# The Four Nontrivial Ш's

**Conclusion:** In light of Cremona's book and the above results, the problem is to show that $\text{Ш}(E)$ is *trivial* for all but the following four optimal elliptic curves with conductor at most 1000:

| Curve | $a$-invariants | $\text{Ш}(E)_?$ |
|---|---|---|
| 571A | [0,-1,1,-929,-105954] | 4 |
| 681B | [1,1,0,-1154,-15345] | 9 |
| 960D | [0,-1,0,-900,-10098] | 4 |
| 960N | [0,1,0,-20,-42] | 4 |

We first deal with these four.

## Divisor of Order:

1. Using a 2-descent we see that $4 \mid \#Ш(E)$ for 571A, 960D, 960N.

2. For $E = 681B$: Using visibility (or a 3-descent) we see that $9 \mid \#Ш(E)$.



## Multiple of Order:

1. For $E = 681B$, the mod 3 representation is surjective, and $3 \,\|\, [E(K) : y_K]$ for $K = \mathbb{Q}(\sqrt{-8})$, so Kolyvagin's theorem implies that $\#Ш(E) = 9$, as required.

2. Kolyvagin's theorem and computation $\implies$ $\#Ш(E) = 4^?$ for 571A, 960D, 960N.

3. Using MAGMA's `FourDescent` command, we compute $\mathrm{Sel}^{(4)}(E/\mathbb{Q})$ for 571A, 960D, 960N and deduce that $\#Ш(E) = 4$. (Note: MAGMA Documentation currently misleading.)



## The Eighteen Optimal Curves of Rank $> 1$

There are 18 curves with conductor $\leq 1000$ and rank $> 1$ (all have rank 2):

389A, 433A, 446D, 563A, 571B, 643A, 655A, 664A, 681C, 707A, 709A, 718B, 794A, 817A, 916C, 944E, 997B, 997C

For these $E$ **nobody** currently knows how to show that $Ш(E)$ is finite, let alone trivial. (But mention, e.g., $p$-adic $L$-functions.)

**Motivating Problem 3:** Prove the BSD Conjecture for all elliptic curve over $\mathbb{Q}$ of conductor at most 1000 and rank $\leq 1$.



**SECRET MOTIVATION:** Our actual motivation is to unify and extend results about BSD and algorithms for elliptic curves. Also, the computations give rise to many surprising and interesting examples.

## Kolyvagin Bound on $\#Ш(E)$

**INPUT:** An elliptic curve $E$ over $\mathbb{Q}$ with $r_{an} \leq 1$.
**OUTPUT:** Odd $B \geq 1$ such that if $p \nmid 2B$, then $p \nmid \#Ш(E/\mathbb{Q})$.

1. [**Choose $K$**] Choose a quadratic imaginary field $K = \mathbb{Q}(\sqrt{D})$ with certain properties, such that $E/K$ has analytic rank 1. Assume $\mathbb{Q}(E[p])$ has degree $\#\,\mathrm{GL}_2(\mathbb{F}_p)$.

2. [**Compute Mordell-Weil**]

   (a) If $r = 0$, compute generator $z$ for $E^D(\mathbb{Q})$ mod torsion.

   (b) If $r = 1$, compute generator $z$ for $E(\mathbb{Q})$ mod torsion.



3. [**Index of Heegner point**] Compute the "Heegner point" $y_K \in E(K)$ associated to $K$. This is a point that comes from the "modularity" map $X_0(N) \to E$.

4. [**Finished**] Output $B = I \cdot A$, where $A$ is the product of primes such that $\mathbb{Q}(E[p])$ has degree less than $\#\,\mathrm{GL}_2(\mathbb{F}_p)$.

**Theorem (Kolyvagin):** $p \nmid 2B \implies p \nmid \#Ш(E/\mathbb{Q})$.

## Our Goal

- There are 2463 optimal curves of conductor at most 1000.
- Of these, 18 have rank 2, which leaves 2445 curves.
- Of these, 2441 are conjectured to have trivial Ш.

Thus our **goal** is to prove that

$$\#Ш(E) = 1$$

for these 2441 elliptic curves.



## Our Strategy

1. [**Find an Algorithm**] Based on deep work of Kolyvagin, Kato, et al.
   **Input:** An elliptic curve over $\mathbb{Q}$ with $r_{an} \leq 1$.
   **Output:** $B \geq 1$ such that if $p \nmid B$, then $p \nmid \#Ш(E)$.

2. [**Compute**] Run the algorithm on our 2441 curves.

3. [**Reducible**] If $E[p]$ is reducible say nothing.