Modularity of Shafarevich-Tate Groups

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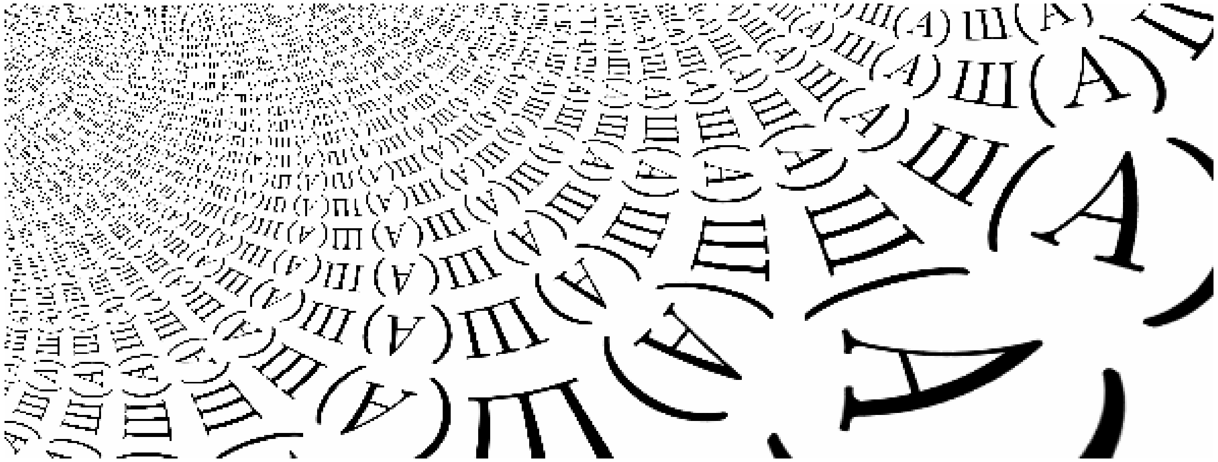


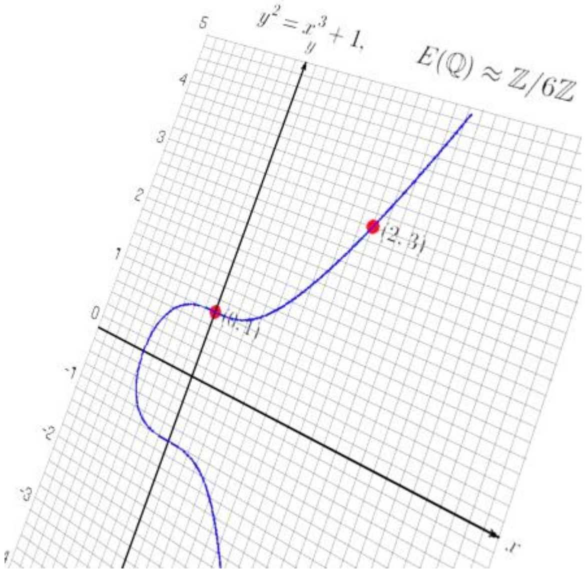
Goal

The goal of this 40-minute talk is to explain the meaning of the following conjecture and give evidence for it.

Conjecture (−). If A is a modular abelian variety, then the Shafarevich-Tate group $Ш(A)$ is modular.

Table of Contents

1. Elliptic Curves and Abelian Varieties



Elliptic curve over $\mathbf{Q}$: $y^2 = x^3 + ax + b$
with $a, b \in \mathbf{Q}$ and $\Delta = -16(4a^3 + 27b^2) \neq 0$

Abelian variety: Any complete group variety.

Examples: Jacobians of curves. Elliptic curves are the abelian varieties of dimension one. **Modular abelian varieties**.

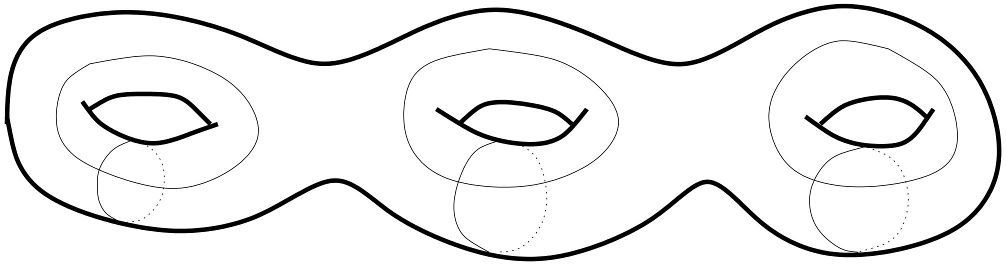
Modular Curves, Modular Forms

Congruence Subgroup:

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbf{Z}) \text{ such that } N \mid c \right\}.$$

Modular Curve: $X_0(N) = \Gamma_0(N) \backslash (\text{upper half plane}) \cup (\text{cusps})$

Example: $X_0(39)$



Algebraic structure over $\mathbf{Q}$:

$$y^2 = (x^4 - 7x^3 + 11x^2 - 7x + 1)(x^4 + x^3 - x^2 + x + 1).$$

Modular Abelian Varieties

Modular Jacobian: $J_0(N) = \mathrm{Jac}(X_0(N))$

Modular Abelian Variety: Any abelian variety quotient of $J_0(N)$ (or of $J_1(N)$, where $J_1(N)$ is defined using $\Gamma_1(N)$).

Theorem (Wiles et al.): All elliptic curves over $\mathbf{Q}$ are modular.

Cusp Forms: $S_2(\Gamma_0(N)) \cong \mathrm{H}^0(X_0(N)_{\mathbf{C}}, \Omega^1)$

Hecke Algebra: $\mathbf{T} = \mathbf{Z}[T_1, T_2, T_3, \ldots] \subset \mathrm{End}(S_2(\Gamma_0(N)))$

Shimura: $\mathbf{T}$-eigenform $f \in S_2(\Gamma_0(N))$ gives $A_f = J_0(N)/I_f J_0(N)$, where $I_f = \mathrm{Ann}_{\mathbf{T}}(f)$. We have $\dim A_f = [\mathbf{Q}(a_2(f), \ldots) : \mathbf{Q}]$.

2. The Birch and Swinnerton-Dyer Conjecture

Let A be an abelian variety over $\mathbf{Q}$ (e.g., $A = A_f$ modular).

Conjecture of :

1. $r = \operatorname{rank} A(\mathbf{Q}) \overset{\text{conj.}}{=} \operatorname{ord}_{s=1} L(A, s)$, and

2. $$\frac{L^{(r)}(A,1)}{r!} = \frac{\prod c_p \cdot \Omega_A \cdot \mathrm{Reg}(A) \cdot \#\text{Ш}(A)}{\#A(\mathbf{Q})_{\mathrm{tor}} \cdot \#A^\vee(\mathbf{Q})_{\mathrm{tor}}}.$$

Results: Kolyvagin, Kato, Rubin, etc.

Shafarevich-Tate Group 

Definition:

$$\text{Ш}(A) = \mathrm{Ker}\left(\mathrm{H}^1(K, A) \to \bigoplus_{\text{all } v} \mathrm{H}^1(K_v, A)\right).$$

$\mathrm{H}^1(K, A)$ is **Galois cohomology**. Interpret geometrically as the **Weil-Chatalet group**:

$$\mathrm{WC}(A/K) = \{\,\text{principal homogenous spaces } X \text{ for } A\,\}/\sim .$$

$\text{Ш}(A)$ is the subgroup of locally trivial classes. **Example:**

$$3x^3 + 4y^3 + 5z^3 = 0 \in \text{Ш}(x^3 + y^3 + 60z^3 = 0)[3].$$

3. Visibility of Sha

$$0 \longrightarrow A \xrightarrow{i} B \longrightarrow C \longrightarrow 0$$

1998 – Barry Mazur introduced **Visibility**:

$$\begin{aligned}\mathrm{Vis}_B(\mathrm{H}^1(K,A)) &= \mathrm{Ker}\left(\mathrm{H}^1(K,A) \to \mathrm{H}^1(K,B)\right) \\ &\cong \mathrm{Coker}(B(K) \to C(K))\end{aligned}$$

$$\mathrm{Vis}_B(\text{Ш}(A)) = \mathrm{Ker}(\text{Ш}(A) \to \text{Ш}(B)).$$

Visibility

$$\begin{tikzcd}
 & X = \pi^{-1}(P) \arrow[r] \arrow[d, dash] & P \arrow[r, mapsto] \arrow[d, dash] & c \arrow[d, dash] & \\
0 \arrow[r] & A(K) \arrow[r] & B(K) \arrow[r, "\pi"] & C(K) \arrow[r] & \mathrm{Vis}_B(\mathrm{H}^1(K, A)) \arrow[r] & 0
\end{tikzcd}$$

Why? To write down X using equations is terrifying; to give P is just to give a rational point. Visibility concisely encodes connections between Mordell-Weil and Shafarevich-Tate groups.

Give nonzero $c \in Ш(A)[5]$ with $\dim(A) = 20$ by giving

$$(0, 0) \in [y^2 + y = x^3 + x^2 - 2x]$$

Everything is Visible Somewhere

Theorem (–): If $c \in \mathrm{H}^1(K, A)$ then there exists B such that $i : A \hookrightarrow B$ and $c \in \mathrm{Vis}_B(\mathrm{H}^1(K, A))$.

Proof. Let L be such that $\mathrm{res}_{L/K}(c) = 0$. Then

$$c \longmapsto \mathrm{res}_{L/K}(c) = 0$$

$$\begin{tikzcd}
\mathrm{H}^1(K, A) \arrow[r] \arrow[dr] & \mathrm{H}^1(K, \mathrm{Res}_{\mathbf{L/K}}(\mathbf{A_L})) \arrow[d] \\
& \mathrm{H}^1(L, A)
\end{tikzcd}$$

Note: If $A/\mathbf{Q}$ is modular and L is abelian, then B is modular.

4. Modularity of Sha

Definition (Modular): An element $c \in Ш(A)$ is *modular* if it is visible in a modular abelian variety. I.e., if there is a factor B of some $J_0(N)$ (or $J_1(N)$) and an inclusion $i : A \hookrightarrow B$ such that $i_*(c) = 0$. Torsor corresponding to c is modular in "usual" sense.

Modularity Conjecture (–):

If A is modular, then every element of $Ш(A)$ is modular.

Theorem (Klenke, Mazur, Stein): If $c \in Ш(E)[p]$ with $p = 2, 3$ and E an elliptic curve, then c is modular.

Related questions:

1. Levels N such that such that c is modular of level N?
2. Which genus one curves are modular?

Why Care about Modularity?

• It could yield results about structure of $Ш(A)$, e.g., finiteness.

• It would give a nice "explanation" of where all $Ш(A)$ "comes from" — it all comes from Mordell-Weil groups.

• It motivates proving new results about the arithmetic of modular abelian varieties.

• It provides powerful computational tools for explicitly working with Shafarevich-Tate groups.

Visibility Construction

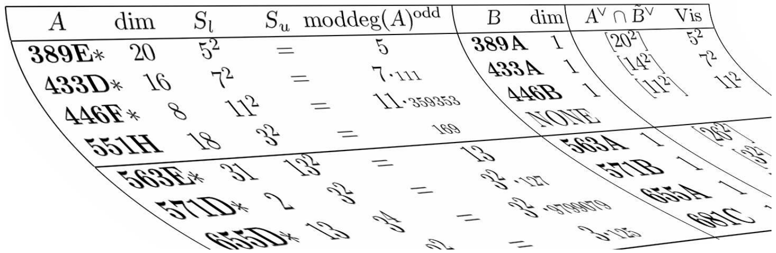
Theorem (Agashe, –): *Suppose* $A, B \subset J_0(N)$, *that* $B[p] \subset A$, *and other technical hypotheses. Then*

$$B(\mathbf{Q})/pB(\mathbf{Q}) \hookrightarrow \mathrm{Vis}_{J_0(N)}(\text{Ш}(A)).$$

Proof. Use the following diagram, chase some exact sequences, and apply subtle properties of Néron models. (Also more general Hecke-equivariant version, proved recently with Jetchev.)

$$\begin{tikzcd} B[p] \arrow[r] \arrow[d] & B \arrow[r, "p"] \arrow[d] & B \arrow[d] \\ A \arrow[r] & J_0(N) \arrow[r] & C. \end{tikzcd}$$

5. Some Data



A	dim	S_l	S_u	moddeg$(A)^{\text{odd}}$	B	dim	$A^\vee \cap B^\vee$	Vis
389E$*$	20	5^2	$=$	5	**389A**	1	$[20^2]$	5^2
433D$*$	16	7^2	$=$	$7 \cdot 111$	**433A**	1	$[14^2]$	7^2
446F$*$	8	11^2	$=$	$11 \cdot 359353$	**446B**	1	$[11^2]$	11^2
551H	18	3^2	$=$	169	NONE			
563E$*$	31	13^2	$=$	13	**563A**	1	$[26^2]$	[illegible]
571D$*$	2	3^2	$=$	$3^2 \cdot 127$	**571B**	1	[illegible]	[illegible]
655D$*$	13	3^4	$=$	$3^2 \cdot 9799079$	**655A**	1	[illegible]	[illegible]
[illegible]	[illegible]	3^2	$=$	$3 \cdot 125$	**681C**	[illegible]	[illegible]	[illegible]

Suppose $A \subset J_0(N)$

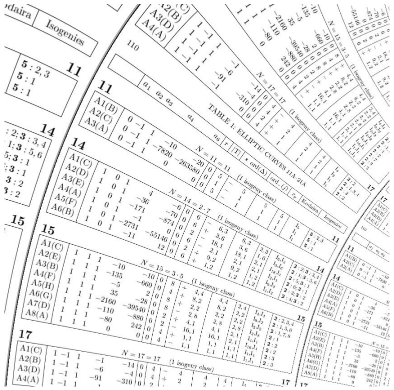
(a) Visibility of $Ш(A)$ in $J_0(N)$, when A is an elliptic curve.

(b) Visibility of $Ш(A)$ in $J_0(N)$, general A.

(c) Visibility of $Ш(A)$ in $J_0(Np)$, general A. (Modularity)

(a) Elliptic Curve tables from Cremona-Mazur

Visibility of $\text{Ш}(A)$ in $J_0(N)$, when $A = E$ is an **elliptic curve**.



The marks (1), (2), (3) and (4) in the last column refer to the notes after the tables.

Table 1. Odd $|\text{Ш}_E| > 1$, all $N \leq 5500$

E	$\sqrt{|\text{Ш}_E|}$	m_E	**F**	m_F	Remarks
681B	3	$3 \cdot 5^3$	**681C**	$2^5 \cdot 3$	
1058D	5	$2^3 \cdot 3 \cdot 5 \cdot 7 \cdot 23$	**1058C**	$2^4 \cdot 5$	
1246B	5	$2^6 \cdot 3^4 \cdot 5$	**1246C**	$2^6 \cdot 5$	
1664K	5	$2^7 \cdot 5 \cdot 7$	**1664N**	$2^6 \cdot 5$	
1913B	3	$3 \cdot 103$	**1913A**	$2^2 \cdot 3 \cdot 5^2$	
2006E	3	$2^6 \cdot 3 \cdot 5 \cdot 7 \cdot 23$	**2006D**	$2^7 \cdot 3$	
2366D	3	$2^4 \cdot 3^2 \cdot 13$	**2366E**	$2^5 \cdot 3^2 \cdot 5$	**E** has rational 3-torsion
2366F	5	$2^4 \cdot 3 \cdot 5 \cdot 13 \cdot 19$	**2366E**	$2^5 \cdot 3^2 \cdot 5$	
2429B	3	$2 \cdot 3 \cdot 73$	**2429D**	$2^3 \cdot 3 \cdot 13$	
2534E	3	$2^2 \cdot 3^2 \cdot 5^3 \cdot 11$	**2534G**	$2^5 \cdot 3^2 \cdot 13$	
2534F	3	$2^2 \cdot 3^2 \cdot 5 \cdot 7$	**2534G**	$2^5 \cdot 3^2 \cdot 13$	
2541D	3	$2^6 \cdot 3^2 \cdot 7 \cdot 11$	**2541C**	$2^5 \cdot 3^2$	
2574D	5	$2^7 \cdot 3^2 \cdot 5 \cdot 7^2$	**2574G**	$2^8 \cdot 5$	
2601H	3	$2^8 \cdot 3 \cdot 17$	**2601L**	$2^8 \cdot 3$	
2674B	3	$2^4 \cdot 3^3 \cdot 13$	**2674A**	$2^4 \cdot 3^2$	
2710C	3	$2^5 \cdot 3^3 \cdot 7$	**2710B**	$2^5 \cdot 3^2$	
2718D	3	$2^6 \cdot 3 \cdot 5 \cdot 7 \cdot 29$	**2718F**	$2^6 \cdot 3 \cdot 5$	
2768C	3	$2^2 \cdot 3 \cdot 41$	**2768B**	$2^5 \cdot 3 \cdot 7$	
2834D	5	$2^2 \cdot 3 \cdot 5 \cdot 109$	**2834C**	$2^6 \cdot 3^2 \cdot 5$	
2849A	3	$2^5 \cdot 5 \cdot 61$	**NONE**	–	
2900D	5	$2^5 \cdot 3^4 \cdot 5$	**2900C**	$2^6 \cdot 3 \cdot 5$	
2932A	3	$3 \cdot 277$	**none**	–	
2955B	3	$2^3 \cdot 3^5 \cdot 5$	**2955C**	$2^6 \cdot 3^3$	
3054A	3	$2 \cdot 3 \cdot 5^2 \cdot 11$	**3054C**	$2^4 \cdot 3 \cdot 5 \cdot 7$	
3185C	5	$2^4 \cdot 3 \cdot 5 \cdot 7 \cdot 11^2$	**3185B**	$2^4 \cdot 3 \cdot 5$	
3306B	3	$2^4 \cdot 3^3 \cdot 5^2$	**1102A**	$2^5 \cdot 3^2$	(1)
3364C	7	$2^6 \cdot 3^2 \cdot 5^2 \cdot 7$	**none**	–	
3384A	5	$2^{10} \cdot 3 \cdot 5 \cdot 11$	**3384C**	$2^8 \cdot 5$	
3536H	3	$2^9 \cdot 3^2 \cdot 5 \cdot 11$	**3536G**	$2^7 \cdot 3^2$	
3555E	3	$2^3 \cdot 3 \cdot 5 \cdot 17$	**3555D**	$2^7 \cdot 3 \cdot 5$	
3712J	3	$2^6 \cdot 3 \cdot 13$	**3712I**	$2^6 \cdot 3$	
3879E	3	$2^6 \cdot 3^4 \cdot 5$	**3879D**	$2^5 \cdot 3^3$	
3933A	3	$2^5 \cdot 3 \cdot 5 \cdot 13$	**3933B**	$2^6 \cdot 3 \cdot 5$	
3952C	5	$2^4 \cdot 3 \cdot 5 \cdot 13 \cdot 17$	**3952E**	$2^5 \cdot 3 \cdot 5$	
3954C	3	$2^4 \cdot 3 \cdot 5^3 \cdot 7^2$	**3954D**	$2^5 \cdot 3 \cdot 5$	
4092A	5	$2^7 \cdot 3 \cdot 5 \cdot 19$	**4092B**	$2^6 \cdot 3 \cdot 5$	
4229A	3	$2^3 \cdot 3 \cdot 7 \cdot 13$	**none**	–	
4343B	3	$2^4 \cdot 1583$	**NONE**	–	
4592D	5	$2^8 \cdot 3^2 \cdot 5 \cdot 17$	**4592G**	$2^6 \cdot 3^2 \cdot 5$	

4592F	3	$2^6 \cdot 3^3 \cdot 7^2$	**4592C**	$2^6 \cdot 3^3$	
4592F	3	$2^6 \cdot 3^3 \cdot 7^2$	**4592G**	$2^6 \cdot 3^2 \cdot 5$	
4606B	3	$2^8 \cdot 3^3 \cdot 5 \cdot 7$	**4606C**	$2^7 \cdot 3^3$	
4675J	3	$2^2 \cdot 3^3 \cdot 5^3$	**4675I**	$2^6 \cdot 3^3$	
4914N	3	$2^4 \cdot 3^5$	**none**	–	**E** has rational 3-torsion
4963C	3	$2^2 \cdot 3 \cdot 71$	**4963D**	$2^9 \cdot 3$	
5046H	3	$2^4 \cdot 3 \cdot 5^2 \cdot 7$	**5046J**	$2^4 \cdot 3 \cdot 5 \cdot 11$	
5054C	3	$2^3 \cdot 3^3 \cdot 11$	**none**	–	(2)
5073D	3	$2^5 \cdot 3 \cdot 5 \cdot 7 \cdot 23$	**none**	–	
5082C	5	$2^4 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 13$	**5082D**	$2^8 \cdot 3 \cdot 5$	
5136B	3	$2^4 \cdot 3 \cdot 59$	**1712D**	$2^5 \cdot 7$	(1)
5389A	3	$2^2 \cdot 2333$	**NONE**	–	
5499E	3	$2^7 \cdot 3^4 \cdot 5$	**5499F**	$2^7 \cdot 3^3$	

Table 2. Even $|Ш_E|$, no rational 2-torsion, all $N \leq 5500$

E	$\sqrt{\lvert Ш_E \rvert}$	m_E	**F**	m_F	Remarks
571A	2	$2^3 \cdot 3 \cdot 5$	**571B**	$2^4 \cdot 3$	
1058B	2	$2^4 \cdot 5 \cdot 23$	**1058C**	$2^4 \cdot 5$	
1309A	4	$2^7 \cdot 3^2 \cdot 17$	**1309B**	2^8	(cong. mod 4)
1325D	2	$2^3 \cdot 3^3 \cdot 5$	**1325E**	$2^3 \cdot 3^3$	
1613B	2	$2^4 \cdot 19$	**1613A**	$2^4 \cdot 5$	
1701I	2	$2^4 \cdot 3^4$	**1701J**	$2^4 \cdot 3^3$	(cong. mod 4)
1717A	2	$2^3 \cdot 41$	**1717B**	$2^3 \cdot 13$	
1738B	2	$2^{11} \cdot 3^3 \cdot 7$	**1738A**	2^8	(cong. mod 4)
1849D	2	$2^4 \cdot 3 \cdot 7 \cdot 11$	**1849A**	$2^3 \cdot 3 \cdot 11$	
1856G	2	$2^8 \cdot 3 \cdot 5$	**1856D**	2^8	(cong. mod 4)
1862C	2	$2^4 \cdot 3^3 \cdot 7$	**1862A**	$2^4 \cdot 3^3$	
1888B	2	$2^8 \cdot 3$	**1888A**	2^7	
1917E	2	$2^3 \cdot 3^4$	**1917C**	$2^3 \cdot 3^3$	
2023A	2	$2^4 \cdot 3^3 \cdot 17$	**2023B**	$2^4 \cdot 3^3$	
2045B	4	$2^3 \cdot 3 \cdot 5 \cdot 7 \cdot 17$	**2045C**	$2^3 \cdot 3^3 \cdot 13$	(cong. mod 2)
2045B	4	$2^3 \cdot 3 \cdot 5 \cdot 7 \cdot 17$	**4090B**	$2^6 \cdot 7$	(1)
2089D	2	$2^5 \cdot 3 \cdot 5$	**2089E**	$2^5 \cdot 11$	
2224E	2	$2^7 \cdot 17$	**2224F**	$2^7 \cdot 3$	(cong. mod 4)
2265A	2	$2^5 \cdot 3^2 \cdot 5^2 \cdot 7$	**2265B**	$2^5 \cdot 5 \cdot 7$	(cong. mod 4)
2409B	2	$2^9 \cdot 5^2$	**2409D**	$2^5 \cdot 7^2$	
2541A	2	$2^5 \cdot 3^4 \cdot 11$	**2541C**	$2^5 \cdot 3^2$	
2554B	2	$2^5 \cdot 13$	**2554C**	$2^4 \cdot 3^2 \cdot 7$	
2563C	2	$2^6 \cdot 3 \cdot 7$	**2563D**	$2^4 \cdot 3 \cdot 5$	
2619C	2	$2^4 \cdot 3^2 \cdot 5$	**2619D**	$2^4 \cdot 3 \cdot 5$	
2678A	4	$2^9 \cdot 3^2 \cdot 23$	**2678B**	$2^7 \cdot 3$	
2678A	4	$2^9 \cdot 3^2 \cdot 23$	**2678I**	$2^5 \cdot 3 \cdot 11$	(cong. mod 2)
2710A	2	$2^5 \cdot 3 \cdot 5^2$	**2710B**	$2^5 \cdot 3^2$	

(b) Abelian Variety Tables from Agashe-Stein

Visibility of $\text{Ш}(A)$ in $J_0(N)$, general A.

Table 1. Visibility of Nontrivial Odd Parts of Shafarevich-Tate Groups

A	dim	S_l	S_u	$\mathrm{moddeg}(A)^{\mathrm{odd}}$	B	dim	$A^\vee \cap \tilde{B}^\vee$	Vis
389E*	20	5^2	$=$	5	**389A**	1	$[20^2]$	5^2
433D*	16	7^2	$=$	$7\cdot 111$	**433A**	1	$[14^2]$	7^2
446F*	8	11^2	$=$	$11\cdot 359353$	**446B**	1	$[11^2]$	11^2
551H	18	3^2	$=$	169	NONE			
563E*	31	13^2	$=$	13	**563A**	1	$[26^2]$	13^2
571D*	2	3^2	$=$	$3^2\cdot 127$	**571B**	1	$[3^2]$	3^2
655D*	13	3^4	$=$	$3^2\cdot 9799079$	**655A**	1	$[36^2]$	3^4
681B	1	3^2	$=$	$3\cdot 125$	**681C**	1	$[3^2]$	–
707G*	15	13^2	$=$	$13\cdot 800077$	**707A**	1	$[13^2]$	13^2
709C*	30	11^2	$=$	11	**709A**	1	$[22^2]$	11^2
718F*	7	7^2	$=$	$7\cdot 5371523$	**718B**	1	$[7^2]$	7^2
767F	23	3^2	$=$	1	NONE			
794G	12	11^2	$=$	$11\cdot 34986189$	**794A**	1	$[11^2]$	–
817E	15	7^2	$=$	$7\cdot 79$	**817A**	1	$[7^2]$	–
959D	24	3^2	$=$	583673	NONE			
997H*	42	3^4	$=$	3^2	**997B**	1	$[12^2]$	3^2
					997C	1	$[24^2]$	3^2
1001F	3	3^2	$=$	$3^2\cdot 1269$	**1001C**	1	$[3^2]$	–
					91A	1	$[3^2]$	–
1001L	7	7^2	$=$	$7\cdot 2029789$	**1001C**	1	$[7^2]$	–
1041E	4	5^2	$=$	$5^2\cdot 13589$	**1041B**	2	$[5^2]$	–
1041J	13	5^4	$=$	$5^3\cdot 21120929983$	**1041B**	2	$[5^4]$	–
1058D	1	5^2	$=$	$5\cdot 483$	**1058C**	1	$[5^2]$	–
1061D	46	151^2	$=$	$151\cdot 10919$	**1061B**	2	$[2^2 302^2]$	–
1070M	7	$3\cdot 5^2$	$3^2\cdot 5^2$	$3\cdot 5\cdot 1720261$	**1070A**	1	$[15^2]$	–
1077J	15	3^4	$=$	$3^2\cdot 1227767047943$	**1077A**	1	$[9^2]$	–
1091C	62	7^2	$=$	1	NONE			
1094F*	13	11^2	$=$	$11^2\cdot 172446773$	**1094A**	1	$[11^2]$	11^2
1102K	4	3^2	$=$	$3^2\cdot 31009$	**1102A**	1	$[3^2]$	–
1126F*	11	11^2	$=$	$11\cdot 13990352759$	**1126A**	1	$[11^2]$	11^2
1137C	14	3^4	$=$	$3^2\cdot 64082807$	**1137A**	1	$[9^2]$	–
1141I	22	7^2	$=$	$7\cdot 528921$	**1141A**	1	$[14^2]$	–
1147H	23	5^2	$=$	$5\cdot 729$	**1147A**	1	$[10^2]$	–
1171D*	53	11^2	$=$	$11\cdot 81$	**1171A**	1	$[44^2]$	11^2
1246B	1	5^2	$=$	$5\cdot 81$	**1246C**	1	$[5^2]$	–
1247D	32	3^2	$=$	$3^2\cdot 2399$	**43A**	1	$[36^2]$	–
1283C	62	5^2	$=$	$5\cdot 2419$	NONE			
1337E	33	3^2	$=$	71	NONE			
1339G	30	3^2	$=$	5776049	NONE			
1355E	28	3	3^2	$3^2\cdot 2224523985405$	NONE			
1363F	25	31^2	$=$	$31\cdot 34889$	**1363B**	2	$[2^2 62^2]$	–
1429B	64	5^2	$=$	1	NONE			
1443G	5	7^2	$=$	$7^2\cdot 18525$	**1443C**	1	$[7^1 14^1]$	–
1446N	7	3^2	$=$	$3\cdot 17459029$	**1446A**	1	$[12^2]$	–

Table 2. Visibility of Nontrivial Odd Parts of Shafarevich-Tate Groups

A	dim	S_l	S_u	$\mathrm{moddeg}(A)^{\mathrm{odd}}$	B	dim	$A^\vee \cap \tilde{B}^\vee$	Vis
1466H$*$	23	13^2	$=$	$13\cdot 25631993723$	**1466B**	1	$[26^2]$	13^2
1477C$*$	24	13^2	$=$	$13\cdot 57037637$	**1477A**	1	$[13^2]$	13^2
1481C	71	13^2	$=$	70825	NONE			
1483D$*$	67	$3^2\cdot 5^2$	$=$	$3\cdot 5$	**1483A**	1	$[60^2]$	$3^2\cdot 5^2$
1513F	31	3	3^4	$3\cdot 759709$	NONE			
1529D	36	5^2	$=$	535641763	NONE			
1531D	73	3	3^2	3	**1531A**	1	$[48^2]$	$-$
1534J	6	3	3^2	$3^2\cdot 635931$	**1534B**	1	$[6^2]$	$-$
1551G	13	3^2	$=$	$3\cdot 110659885$	**141A**	1	$[15^2]$	$-$
1559B	90	11^2	$=$	1	NONE			
1567D	69	$7^2\cdot 41^2$	$=$	$7\cdot 41$	**1567B**	3	$[4^4 1148^2]$	$-$
1570J$*$	6	11^2	$=$	$11\cdot 228651397$	**1570B**	1	$[11^2]$	11^2
1577E	36	3	3^2	$3^2\cdot 15$	**83A**	1	$[6^2]$	$-$
1589D	35	3^2	$=$	6005292627343	NONE			
1591F$*$	35	31^2	$=$	$31\cdot 2401$	**1591A**	1	$[31^2]$	31^2
1594J	17	3^2	$=$	$3\cdot 259338050025131$	**1594A**	1	$[12^2]$	$-$
1613D$*$	75	5^2	$=$	$5\cdot 19$	**1613A**	1	$[20^2]$	5^2
1615J	13	3^4	$=$	$3^2\cdot 13317421$	**1615A**	1	$[9^1 18^1]$	$-$
1621C$*$	70	17^2	$=$	17	**1621A**	1	$[34^2]$	17^2
1627C$*$	73	3^4	$=$	3^2	**1627A**	1	$[36^2]$	3^4
1631C	37	5^2	$=$	6354841131	NONE			
1633D	27	$3^6\cdot 7^2$	$=$	$3^5\cdot 7\cdot 31375$	**1633A**	3	$[6^4 42^2]$	$-$
1634K	12	3^2	$=$	$3\cdot 3311565989$	**817A**	1	$[3^2]$	$-$
1639G$*$	34	17^2	$=$	$17\cdot 82355$	**1639B**	1	$[34^2]$	17^2
1641J$*$	24	23^2	$=$	$23\cdot 1491344147471$	**1641B**	1	$[23^2]$	23^2
1642D$*$	14	7^2	$=$	$7\cdot 123398360851$	**1642A**	1	$[7^2]$	7^2
1662K	7	11^2	$=$	$11\cdot 16610917393$	**1662A**	1	$[11^2]$	$-$
1664K	1	5^2	$=$	$5\cdot 7$	**1664N**	1	$[5^2]$	$-$
1679C	45	11^2	$=$	6489	NONE			
1689E	28	3^2	$=$	$3\cdot 172707180029157365$	**563A**	1	$[3^2]$	$-$
1693C	72	1301^2	$=$	1301	**1693A**	3	$[2^4 2602^2]$	$-$
1717H$*$	34	13^2	$=$	$13\cdot 345$	**1717B**	1	$[26^2]$	13^2
1727E	39	3^2	$=$	118242943	NONE			
1739F	43	659^2	$=$	$659\cdot 151291281$	**1739C**	2	$[2^2 1318^2]$	$-$
1745K	33	5^2	$=$	$5\cdot 1971380677489$	**1745D**	1	$[20^2]$	$-$
1751C	45	5^2	$=$	$5\cdot 707$	**103A**	2	$[505^2]$	$-$
1781D	44	3^2	$=$	61541	NONE			
1793G$*$	36	23^2	$=$	$23\cdot 8846589$	**1793B**	1	$[23^2]$	23^2
1799D	44	5^2	$=$	201449	NONE			
1811D	98	31^2	$=$	1	NONE			
1829E	44	13^2	$=$	3595	NONE			
1843F	40	3^2	$=$	8389	NONE			
1847B	98	3^6	$=$	1	NONE			
1871C	98	19^2	$=$	14699	NONE			

TABLE 3. Visibility of Nontrivial Odd Parts of Shafarevich-Tate Groups

A	dim	S_l	S_u	$\mathrm{moddeg}(A)^{\mathrm{odd}}$	B	dim	$A^\vee \cap \tilde{B}^\vee$	Vis
1877B	86	7^2	$=$	1	NONE			
1887J	12	5^2	$=$	$5\cdot 10825598693$	**1887A**	1	$[20^2]$	–
1891H	40	7^4	$=$	$7^2\cdot 44082137$	**1891C**	2	$[4^2 196^2]$	–
1907D∗	90	7^2	$=$	$7\cdot 165$	**1907A**	1	$[56^2]$	7^2
1909D∗	38	3^4	$=$	$3^2\cdot 9317$	**1909A**	1	$[18^2]$	3^4
1913B∗	1	3^2	$=$	$3\cdot 103$	**1913A**	1	$[3^2]$	3^2
1913E	84	$5^4\cdot 61^2$	$=$	$5^2\cdot 61\cdot 103$	**1913A**	1	$[10^2]$	–
					1913C	2	$[2^2 610^2]$	–
1919D	52	23^2	$=$	675	NONE			
1927E	45	3^2	3^4	52667	NONE			
1933C	83	$3^2\cdot 7$	$3^2\cdot 7^2$	$3\cdot 7$	**1933A**	1	$[42^2]$	3^2
1943E	46	13^2	$=$	62931125	NONE			
1945E∗	34	3^2	$=$	$3\cdot 571255479184807$	**389A**	1	$[3^2]$	3^2
1957E∗	37	$7^2\cdot 11^2$	$=$	$7\cdot 11\cdot 3481$	**1957A**	1	$[22^2]$	11^2
					1957B	1	$[14^2]$	7^2
1979C	104	19^2	$=$	55	NONE			
1991C	49	7^2	$=$	1634403663	NONE			
1994D	26	3	3^2	$3^2\cdot 46197281414642501$	**997B**	1	$[3^2]$	–
1997C	93	17^2	$=$	1	NONE			
2001L	11	3^2	$=$	$3^2\cdot 44513447$	NONE			
2006E	1	3^2	$=$	$3\cdot 805$	**2006D**	1	$[3^2]$	–
2014L	12	3^2	$=$	$3^2\cdot 126381129003$	**106A**	1	$[9^2]$	–
2021E	50	5^6	$=$	$5^2\cdot 729$	**2021A**	1	$[100^2]$	5^4
2027C∗	94	29^2	$=$	29	**2027A**	1	$[58^2]$	29^2
2029C	90	$5^2\cdot 269^2$	$=$	$5\cdot 269$	**2029A**	2	$[2^2 2690^2]$	–
2031H∗	36	11^2	$=$	$11\cdot 1014875952355$	**2031C**	1	$[44^2]$	11^2
2035K	16	11^2	$=$	$11\cdot 218702421$	**2035C**	1	$[11^1 22^1]$	–
2038F	25	5	5^2	$5^2\cdot 92198576587$	**2038A**	1	$[20^2]$	–
					1019B	1	$[5^2]$	–
2039F	99	$3^4\cdot 5^2$	$=$	13741381043009	NONE			
2041C	43	3^4	$=$	61889617	NONE			
2045I	39	3^4	$=$	$3^3\cdot 3123399893$	**2045C**	1	$[18^2]$	–
					409A	13	$[9370199679^2]$	–
2049D	31	3^2	$=$	29174705448000469937	NONE			
2051D	45	7^2	$=$	$7\cdot 674652424406369$	**2051A**	1	$[56^2]$	–
2059E	45	$5\cdot 7^2$	$5^2\cdot 7^2$	$5^2\cdot 7\cdot 167359757$	**2059A**	1	$[70^2]$	–
2063C	106	13^2	$=$	8479	NONE			
2071F	48	13^2	$=$	36348745	NONE			
2099B	106	3^2	$=$	1	NONE			
2101F	46	5^2	$=$	$5\cdot 11521429$	**191A**	2	$[155^2]$	–
2103E	37	$3^2\cdot 11^2$	$=$	$3^2\cdot 11\cdot 87441292307157179261$	**2103B**	1	$[33^2]$	11^2
2111B	112	211^2	$=$	1	NONE			
2113B	91	7^2	$=$	1	NONE			
2117E∗	45	19^2	$=$	$19\cdot 1078389$	**2117A**	1	$[38^2]$	19^2

Table 4. Visibility of Nontrivial Odd Parts of Shafarevich-Tate Groups

A	dim	S_l	S_u	$\mathrm{moddeg}(A)^{\mathrm{odd}}$	B	dim	$A^\vee \cap \tilde{B}^\vee$	Vis
2119C	48	7^2	=	89746579	NONE			
2127D	34	3^2	=	$3\cdot 18740561792121901$	**709A**	1	$[3^2]$	–
2129B	102	3^2	=	1	NONE			
2130Y	4	7^2	=	$7\cdot 83927$	**2130B**	1	$[14^2]$	–
2131B	101	17^2	=	1	NONE			
2134J	11	3^2	=	1710248025389	NONE			
2146J	10	7^2	=	$7\cdot 1672443$	**2146A**	1	$[7^2]$	–
2159E	57	13^2	=	31154538351	NONE			
2159D	56	3^4	=	233801	NONE			
2161C	98	23^2	=	1	NONE			
2162H	14	3	3^2	$3\cdot 6578391763$	NONE			
2171E	54	13^2	=	271	NONE			
2173H	44	199^2	=	$199\cdot 3581$	**2173D**	2	$[398^2]$	–
2173F	43	19^2	$3^2\cdot 19^2$	$3^2\cdot 19\cdot 229341$	**2173A**	1	$[38^2]$	19^2
2174F	31	5^2	=	$5\cdot 21555702093188316107$	NONE			
2181E	27	7^2	=	$7\cdot 7217996450474835$	**2181A**	1	$[28^2]$	–
2193K	17	3^2	=	$3\cdot 15096035814223$	**129A**	1	$[21^2]$	–
2199C	36	7^2	=	$7^2\cdot 13033437060276603$	NONE			
2213C	101	3^4	=	19	NONE			
2215F	46	13^2	=	$13\cdot 1182141633$	**2215A**	1	$[52^2]$	–
2224R	11	79^2	=	79	**2224G**	2	$[79^2]$	–
2227E	51	11^2	=	259	NONE			
2231D	60	47^2	=	91109	NONE			
2239B	110	11^4	=	1	NONE			
2251E$*$	99	37^2	=	37	**2251A**	1	$[74^2]$	37^2
2253C$*$	27	13^2	=	$13\cdot 14987929400988647$	**2253A**	1	$[26^2]$	13^2
2255J	23	7^2	=	15666366543129	NONE			
2257H	46	$3^6\cdot 29^2$	=	$3^3\cdot 29\cdot 175$	**2257A**	1	$[9^2]$	–
					2257D	2	$[2^2 174^2]$	–
2264J	22	73^2	=	73	**2264B**	2	$[146^2]$	–
2265U	14	7^2	=	$7^2\cdot 73023816368925$	**2265B**	1	$[7^2]$	–
2271I$*$	43	23^2	=	$23\cdot 392918345997771783$	**2271C**	1	$[46^2]$	23^2
2273C	105	7^2	=	7^2	NONE			
2279D	61	13^2	=	96991	NONE			
2279C	58	5^2	=	1777847	NONE			
2285E	45	151^2	=	$151\cdot 138908751161$	**2285A**	2	$[2^2 302^2]$	–
2287B	109	71^2	=	1	NONE			
2291C	52	3^2	=	427943	NONE			
2293C	96	479^2	=	479	**2293A**	2	$[2^2 958^2]$	–
2294F	15	3^2	=	$3\cdot 6289390462793$	**1147A**	1	$[3^2]$	–
2311B	110	5^2	=	1	NONE			
2315I	51	3^2	=	$3\cdot 4475437589723$	**463A**	16	$[13426312769169^2]$	–
2333C	101	83341^2	=	83341	**2333A**	4	$[2^6 166682^2]$	–

(c) Visibility at Higher Level – Evidence for Modularity Conjecture

Visibility of $Ш(A)$ in $J_0(Np)$, general A.

Recall Ribet's theorem...

Ribet Level Raising

Suppose

- $f = \sum a_n q^n \in S_2(\Gamma_0(N))$ a newform
- $\lambda \subset \mathbf{Z}[a_1, a_2, \ldots]$ a nonzero prime ideal s.t. $A_f[\lambda]$ irreducible.

Theorem: $a_p + p + 1 \equiv 0 \pmod{\lambda} \implies$ there exists a p-newform $g \in S_2(\Gamma_0(Np))$ such that

- $i(A_f[\lambda]) = A_g[\lambda]$ some $i : J_0(N) \to J_0(Np)$, and
- sign of functional equations for $L(f,s)$ and $L(g,s)$ same.

Big Computation: For every level N up to 5000 (and more), use my modular forms package in MAGMA to provably compute:

1. Each newform $f = \sum a_n q^n \in S_2(\Gamma_0(N))$.

2. Whether or not $L(f, 1) = 0$.

3. Whether or not $\mathrm{ord}_{s=1} L(f, s)$ is even.

4. Characteristic polynomials of $a_2, a_3, a_5, \ldots, a_{19}$.

(I hope to redo this computation using only open-source software that I'm currently writing.)

Probable Modularity

• Two forms $f = \sum a_n q^n$ and $g = \sum b_n q^n$ are *probably congruent mod ℓ* (away from level) if for $p < 20$ with $p \nmid N_f N_g$ we have

$$\ell \mid \text{resultant}(\text{charpoly}(a_p), \text{charpoly}(b_p)).$$

• If $A = A_g \subset J_0(N)$, then there is *probably a nonzero element in* $Ш(A)[\ell]$ *modular of level Np* if there is f of level Np such that:

1. f and g are probably congruent modulo ℓ, and
2. $\text{ord}_{s=1} L(f, s)$ is positive and even.

Visibility at Higher Level

A_f with odd invisible $Ш_{\mathrm{an}}[\ell]$	All ℓ-congruent $A_g \subset J_0(Np)_{\mathrm{new}}$ with $Np \leq 5000$ and $\mathrm{ord}_{s=1} L(g,s) \geq 0$ (and higher Np if known)
551, dim 18, $\ell = 3$	$\mathbf{p}=\mathbf{2}$: dim 1, rank 2 $\mathbf{p}=\mathbf{3}$: dim 1, rank 2 $\mathbf{p}=\mathbf{5}$: dim 25, rank 0
767, dim 23, $\ell = 3$	$\mathbf{p}=\mathbf{2}$: dim 1, rank 2 $\mathbf{p}=\mathbf{7}$: dim 1, rank 2 $\mathbf{p}=\mathbf{7}$: dim 52, rank 0
959, dim 24, $\ell = 3$	$\mathbf{p}=\mathbf{2}$: dim 1, rank 2
1091, dim 62, $\ell = 7$	$\mathbf{p}=\mathbf{7}$: dim 2, rank 2
1283, dim 62, $\ell = 5$	$\mathbf{p}=\mathbf{3}$: dim 2, rank 2
1337, dim 33, $\ell = 3$	$\mathbf{p}=\mathbf{2}$: dim 1, rank 2
1339, dim 30, $\ell = 3$	$\mathbf{p}=\mathbf{2}$: dim 1, rank 2
1355, dim 28, $\ell = 3$	$\mathbf{p}=\mathbf{2}$: dim 1, rank 2
1429, dim 64, $\ell = 5$	$\mathbf{p}=\mathbf{2}$: dim 2, rank 2 $\mathbf{p}=\mathbf{3}$: dim 66, rank 0
1481, dim 71, $\ell = 13$	Nothing in range
1513, dim 31, $\ell = 3$	$\mathbf{p}=\mathbf{2}$: dim 1, rank 2
1529, dim 36, $\ell = 5$	$\mathbf{p}=\mathbf{7}$: dim 1, rank 2
1559, dim 90, $\ell = 11$	Nothing in range
1589, dim 35, $\ell = 3$	Nothing in range
1631, dim 37, $\ell = 5$	$\mathbf{p}=\mathbf{2}$: dim 1, rank 2
1679, dim 45, $\ell = 11$	$\mathbf{p}=\mathbf{2}$: dim 2, rank 2
1727, dim 39, $\ell = 3$	$\mathbf{p}=\mathbf{2}$: dim 1, rank 2
2849, dim 1, $\ell = 3$	$\mathbf{p}=\mathbf{3}$: dim 1, rank 2
4343, dim 1, $\ell = 3$	Nothing in range
5389, dim 1, $\ell = 3$	$\mathbf{p}=\mathbf{7}$: dim 1, rank 2

When the second column contains an A_g of rank 2, then $Ш(A_f)[\ell]$ is "very likely" to be visible of level $M = Np$. This is the case for most examples. The "Nothing in range" note means that the smallest p for which there exists g of even analytic rank congruent to f is beyond the range of my current tables. The examples of level 2849, 4343, and 5389 are the odd and definitely invisible examples in Cremona and Mazur's original paper on visibility.

Questions