Approximation of Eigenforms of Infinite Slope by Eigenforms of Finite Slope

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1 Introduction

Fix a prime p. Consider a classical newform

$$F = \sum_{n\geq 1} a_n q^n \in S_k\left(\Gamma_1(Np^t), \overline{\mathbf{Q}}_p\right)$$

where k and N are positive integers and $p \nmid N$ is a prime (by a *newform* we mean a Hecke eigenform that lies in the new subspace and is normalized so that $a_1 = 1$). The *slope* of F is $\mathrm{ord}_p(a_p)$, where $\mathrm{ord}_p(p) = 1$. By [Shi94, Prop. 3.64], the twist

$$F^\chi = \sum \chi(n) a_n q^n$$

of F by any Dirichlet character χ of conductor dividing p is an eigenform on $\Gamma_1(Np^{\max\{t+1,2\}})$. This twist has infinite slope.

In Section 2, we prove that if F has finite slope then it is possible to approximate F^χ arbitrarily closely by (classical) finite slope *eigenforms*. Assuming refinements of standard conjectures, the best estimate we obtain for the smallest weight of an approximating eigenforms is exponential in the approximating modulus p^A. Section 4 contains computations that suggest that the best estimates should have weight that is linear in p^A.

One motivation for the question of approximation of infinite slope eigenforms by finite slope eigenforms is the desire to understand the versal deformation space of a residual modular representation [Maz89] (the deformation space of an irreducible representation is universal [Maz89] as is the deformation space of a residual pseudo-representation [CM98]). In [GM98] (see also [Maz97], and [Böc01] for a generalization), it was shown that the Zariski closure of the locus of finite slope modular deformations of an absolutely irreducible "totally unobstructed" residual modular representation is Zariski dense in the associated representation space but very little is known about the topological closure of this locus. For example, it is not known if it contains any nonempty open sets. Our result implies that it contains tamely ramified twists of modular deformations. We also show in Section 3.1 that a result of Hatada implies that in at least one (albeit not irreducible) case it does not contain all modular deformations.

Our investigation began with with our answer in Section 2 to a question of Jochnowitz. The idea of studying the p-adic variation of modular forms began with Serre [Ser73] and was since developed by Katz [Kat75] and Hida [Hid86] (see also [Gou88] for a sketch of the theory). It follows, in particular, from their work, that one can approximate all forms on $X_0(p^n)$ with forms on the j-line $X_0(1)$, but *not* necessarily with *eigenforms*.

We prove the above result about twists in Section 2, then state some questions about approximation by finite slope forms in Section 2.1. We explain how to reinterpret Hatada's result in Section 3.1, then present the results of our computations in Section 4.

Based on the results and computations discussed in this article, Mazur has suggested that it may be the case that an infinite slope eigenform can be approximated by finite slope eigenforms only if the corresponding representation is what he calls *tamely semistable* (i.e., semistable, in the sense of [CF00], after a tame extension).

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2 Approximating Teichmüller Twists of Finite Slope Eigenforms

This section is the theoretical heart of the paper. We prove that the infinite slope eigenforms obtained as twists of finite slope eigenforms by powers of the Teichmüller character can always be approximated by finite slope eigenforms. We first show that certain overconvergent eigenforms of sufficiently close weight are congruent and have the same slope. Then we use the θ operator on overconvergent forms to deduce the main result (Theorem 2.1) below.

Let p be a prime. All eigenforms in this section will be cusp forms with coefficients in $\overline{\mathbf{Q}}_p$ normalized so that $a_1 = 1$. Suppose $F = \sum_{n\geq 1} a_n q^n$ is an eigenform and $\chi : (\mathbf{Z}/M\mathbf{Z})^* \to \mathbf{C}_p^*$ is a Dirichlet character with modulus M, which we extend to $\mathbf{Z}/M\mathbf{Z}$ by setting $\chi(n) = 0$ if $(n, M) \neq 1$. Then the twist of F by χ is the eigenform

$$F^\chi = \sum_{n\geq 1} \chi(n) a_n q^n.$$

Let $\omega\colon (\mathbf{Z}/p\mathbf{Z})^* \to \mathbf{Z}_p^*$ be the Teichmüller character (so $\omega(n) \equiv n \pmod{p}$). The following theorem concerns finite slope approximations of twists of F by powers of ω. For example, it concerns the twist

$$F^{\omega^0} = \sum_{(n,p)=1} a_n(F) q^n$$

of F by the trivial character mod p, which we call the "p-deprivation" of F and which has infinite slope.

Theorem 2.1. *Suppose F is a classical eigenform on $X_1(Np^t)$, $t \geq 1$, over $\overline{\mathbf{Q}}_p$ of weight k, character ψ, and finite slope at p. Let $A \in \mathbf{Z}_{>0}$ and $r, s \in \mathbf{Z}_{\geq 0}$ with $r, s < p - 1$. Then there exists a classical finite slope eigenform G on $X_1(Np^t)$ with $G(q) \equiv F^{\omega^r}(q) \pmod{p^A}$ such that G has weight congruent to $k + 2r - s$ modulo $p - 1$ and character $\psi \cdot \omega^s$.*

(The slope of G will be at least A, since the pth Fourier coefficient of F^{ω^r} is 0.)

Let $\mathbf{q} = 4$ if $p = 2$ and p otherwise. Let $\tau : \mathbf{Z}_p^* \to \mathbf{C}_p^*$ be the character of finite order such that $a \equiv \tau(a) \pmod{\mathbf{q}}$. We only need to assume that $F = \sum_{n\geq 1} a_n q^n$ is an overconvergent eigenform of tame level N of finite slope with arithmetic weight-character $\kappa\colon a \to \chi(a)\langle\langle a\rangle\rangle^k$, where χ is a character of finite order whose conductor divides Np^t, k is a possibly negative integer, and $\langle\langle a\rangle\rangle = a/\tau(a)$. (For example, if F is a classical eigenform of weight k and character ψ, then $\chi = \psi\omega^k$.) Recall that the collection of continuous characters on $\mathbf{Z}_p^*$ is a metric space, with

$$d(\rho, \psi) = \max\{|\rho(a) - \psi(a)| : a \in \mathbf{Z}_p^*\},$$

where $|\,|$ is the absolute value on $\mathbf{C}_p$ normalized so that $|p| = 1/p$. We need,

Proposition 2.2. *Suppose $L \in \mathbf{Z}_{\geq 0}$ and H is an overconvergent eigenform of tame level N, finite slope and weight-character κ. Then if γ is a weight-character sufficiently close to κ there exists an overconvergent eigenform R of weight-character γ with the same slope as H such that*

$$H(q) \equiv R(q) \pmod{p^L}.$$

Proof. We will use the notation of the "*R-families*" section (in §B5) of [Col97b]. In particular, B is an affinoid disk in weight space containing κ and X is an affinoid finite over B such that $A(X)$ is generated by the images of the "Hecke operators" $T(n)$. Moreover, if $x \in X$ and $\eta_x\colon A(X) \to \mathbf{C}_p$ is the corresponding homomorphism, then

$$F_x(q) = \sum_{n \geq 1} \eta_x(T(n)) q^n$$

is the q-expansion of an overconvergent finite slope eigenform and finally there is a point $y \in X$ such that $F_y(q) = H(q)$. Note that X is a subdomain of the eigencurve of tame level N (although the eigencurves of level $N > 1$ are not yet defined in the literature).

The ring $A^0(X)$ is finite over $A^0(B)$ by Corollary 6.4.1/5 of [BGR84]. Let $f_1, \ldots, f_n$ be generators. Let f_0 be a uniformizing parameter on B so that $A(B) = \mathbf{C}_p\langle f_0 \rangle$, where $\mathbf{C}_p\langle f_0 \rangle$ is the ring of power series in f_0 whose coefficients tend to 0 with their degree. Let $Z_L(y)$ be the following Weierstrāss subdomain of X:

$$\{x \in X \colon |f_i(x) - f_i(y)| \leq p^{-L}, 0 \leq i \leq n\}.$$

Since the functions $x \to \eta_x(T(n))$ lie in $A^0(X)$, it follows that if $x \in Z_L(y)$, then

$$F_x(q) \equiv H(q) \pmod{p^L}.$$

Finally, since $Z_L(y)$ is a subdomain of X and X is finite over B, the map from $Z_L(y)$ to B is quasi-finite. It follows from Proposition A5.5 of [Col97b] that its image in B is a subdomain. Since κ is the image of y, its image contains a disk around y. $\square$

Proof of Theorem 2.1. Let α be the slope of F. It follows from Proposition 2.2 that if $m \in \mathbf{Z}$ is sufficiently small p-adically there exists an overconvergent eigenform K of tame level N, weight-character $\chi \cdot \langle\langle\ \rangle\rangle^{k-m}$ and slope α such that $K(q) \equiv F(q) \pmod{p^A}$. Suppose $m \geq k$. Then, by Proposition 4.3 of [Col96] (see also [Col97a]) if $F_1 = \theta^{m-k+1} K$, then F_1 is an overconvergent eigenform of weight-character

$$\kappa_1 := \omega^{2(m-k+1)} \cdot \chi \cdot \langle\langle\ \rangle\rangle^{k_1},$$

where $k_1 = m - k + 2$, and F_1 has finite slope $\alpha_1 = \alpha + m - k + 1$. Applying this same process to F_1, for $\ell \in \mathbf{Z}$ sufficiently small p-adically such that $\ell \geq k_1$, we obtain an overconvergent finite slope eigenform F_2 of weight-character κ_2, where $\kappa_2 = \omega^{2\ell} \cdot \chi \cdot \langle\langle\ \rangle\rangle^{k_2}$ and where $k_2 = \ell - k_1 + 2 = k + \ell - m$, such that if $F_2(q) = \sum_{n \geq 1} b_n q^n$, then

$$\begin{aligned} b_n &\equiv n^{\ell - k_1 + 1} n^{m-k+1} a_n \\ &\equiv n^{\ell} a_n \pmod{p^A}. \end{aligned}$$

The latter is congruent to $\omega^r(n) a_n \pmod{p^A}$ if $\ell \equiv r \pmod{\varphi(p^A)}$ and $\ell + v(a_p) \geq A$. It follows from [Col96, §8], [Col97a], and [Col97b] that if c is an integer sufficiently small p-adically, such that $c + k_2 > v(b_p) + 1$ (note that $v(b_p)$ is the slope of F_2 so is finite) there exists a classical eigenform G on $X_1(Np^t)$ of weight $k_2 + c = k + \ell - m + c$, slope $v(b_p)$ and character $\omega^{m+r-c} \cdot \psi$ such that $G(q) \equiv F_2(q) \equiv F^{\omega^r}(q) \pmod{p^A}$. We can choose c so that $m + r - c \equiv s \pmod{p-1}$ and then $k_2 + c \equiv k + 2r - s \mod (p-1)$. □

The following corollary addresses a question of Jochnowitz, which motivated this entire investigation:

Corollary 2.3. *Suppose R is a classical eigenform of weight k on $X_1(N)$, let $A \in \mathbf{Z}_{>0}$, and let $r \in \mathbf{Z}_{\geq 0}$ with $r < p - 1$. Then there exists a classical eigenform S on $X_1(N)$ of weight congruent to $k + 2r$ modulo $p - 1$ such that $S(q) \equiv R^{\omega^r}(q) \pmod{p^A}$.*

Proof. Suppose the F in Theorem 2.1 is one of the old eigenforms associated to R on $X_1(Np)$ and $s = 0$. Let G be a classical eigenform of weight $c + k_2$ as mentioned in the proof of the theorem, but suppose $c + k_2 > 2v(b_p) + 1$. Then G is old of weight congruent to $k \mod (p-1)$ and G is congruent to an eigenform S of the same weight on $X_1(N)$ modulo $p^{v(b_p)}$. Since $b_p \equiv 0 \pmod{p^A}$, we obtain the corollary. □

Remark 2.4. Assuming a natural refinement of the Gouvêa-Mazur conjectures, the best estimate we obtain for the weight of H in the above proof is exponential in p^A. Computational evidence suggests that the best estimates should have weights that are linear in p^A (see Section 4).

Remark 2.5. Jochnowitz and Mazur have independently observed that the above argument can be used to prove the following result: *Suppose F is an overconvergent eigenform of arithmetic weight-character κ, which is a limit of overconvergent eigenforms of finite slope. If $\iota\colon \mathbf{Z}_p^* \to \mathbf{Z}_p^*$ is the identity character, then the twist $F^{\iota/\kappa}(q)$ of F by ι/κ, which is the q-expansion of a convergent eigenform of weight-character ι^2/κ, is the limit of overconvergent eigenforms of finite slope.*

Remark 2.6. One can also approach the p-deprivation (the twist by the 0th power of Teichmüller) of a finite slope eigenform F by using the evil twins of eigenforms approaching F.

2.1 Questions

Some natural questions arise:

1. Is every p-adic convergent eigenform which is the limit of finite slope overconvergent eigenforms an overconvergent eigenform? (We can show the twist of an overconvergent eigenform by a Dirichlet character is overconvergent.)

2. Which infinite slope eigenforms are limits of finite slope eigenforms?

3. If $F(q)$ is the q-expansion of an overconvergent eigenform of weight-character κ, is $F^{\iota/\kappa}(q)$ the q-expansion of an overconvergent eigenform of weight-character

ι^2/κ (recall that ι is the identity character $\mathbf{Z}_p^* \xrightarrow{\sim} \mathbf{Z}_p^*$)? Another closely related question is as follows: Suppose ρ is the representation of the absolute Galois group of $\mathbf{Q}$ attached to an overconvergent eigenform and let χ denote the cyclotomic character. Then is the representation $\rho \otimes \chi \cdot \det(\rho)^{-1}$ attached to an overconvergent eigenform?

3 An Infinite Slope Eigenform that is Not Approximable

In Section 3.1, we prove an extension to higher level of a theorem of Hatada about the possibilities for systems of Hecke eigenvalues modulo 8. We use this result to deduce that the normalized weight 2 cusp form on $X_0(32)$ is not 2-adically approximable by normalized eigenforms of tame level 1 and finite slope. In Section 3.2 we give an example of an infinite slope eigenform of level 27 that computer computations suggest cannot be approximated by finite slope forms. For related investigations, see [CE03].

3.1 An Extension of a Theorem of Hatada

Theorem 3.1. *If $F = \sum a_n q^n$ is a normalized cuspidal newform over $\mathbf{C}_2$ of finite slope on $X_0(2^n)$, then $a_2 \equiv 0 \pmod 8$ and $a_p \equiv p+1 \pmod 8$ for all odd primes p.*

Proof. Suppose F has weight k and finite slope α. The assumption that F has finite slope implies $n \leq 1$. If $n = 0$ the assertion of Theorem 3.1 was proved by Hatada in [Hat79], so we may assume that $n = 1$ and $\alpha = (k-2)/2$ (in general, the slope of a newform on $\Gamma_0(p)$ of weight k is $(k-2)/2$). Note that $\alpha \geq 3$ since there are no newforms on $X_0(2)$ of weight < 8. It follows from Theorems A of [Col97b] (see §B2 of [Col97b] for the extension to $p = 2$) and Theorem B5.7 of [Col97b] that if j is an integer sufficiently close 2-adically to k, then there exists a classical normalized cuspidal eigenform G on $X_0(2)$ of weight j and slope α such that

$$G(q) \equiv F(q) \pmod 8.$$

If in addition we assume that $j > 2(\alpha + 1)$, then G must be old (since the slope of a newform of weight j is $(j-2)/2 \neq \alpha$). Thus there is a cuspidal eigenform $H = \sum b_n q^n$ of level 1 such that G is a linear combination of $H(q)$ and $H(q^2)$. More precisely,

$$G(q) = H(q) - \rho H(q^2)$$

where ρ is a root of $P(X) = X^2 - b_2 X + 2^{j-1}$. By Hatada's theorem $\mathrm{ord}_2(b_2) \geq 3$, and $j \geq 12$, so the slopes of the Newton polygon of $P(X)$ at 2 are both at least 3. Thus $G(q) \equiv H(q) \pmod 8$, which proves the theorem because H has level 1. □

Corollary 3.2. *Let G be the normalized weight 2 cusp form on $X_0(32)$. Then G is not 2-adically approximable by normalized eigenforms of tame level 1 and finite slope.*

Proof. If F_{32} were approximable there would have to be a normalized eigenform F on $X_0(2)$ such that $F_{32}(q) \equiv F(q) \pmod 8$. However, $F_{32}(q) = \sum_{n=1}^{\infty} a_n q^n$ where,

$$a_p = \begin{cases} 2x & \text{if } p = x^2 + y^2, \quad \text{written so } x + y \equiv x^2 \pmod 4 \\ 0 & \text{otherwise.} \end{cases}$$

As $a_3 = 0 \not\equiv 4 \pmod 8$, we see from Theorem 3.1 that F does not exist. □

Remark 3.3. If $p \equiv 1 \pmod 4$ then the coefficient of a_p in F_{32} agrees modulo 8 with $p+1$. If p is 3 mod 4 it does not because for F_{32} the coefficient vanishes. What is happening is that there is a reducible mod 8 pseudo-representation (namely the trivial one-dimensional representation plus the cyclotomic character) such that any finite slope level 2^n form gives this pseudo-representation mod 8. Conversely the mod 8 representation associated to F_{32} is the direct sum of the quadratic character associated to $\mathbf{Q}(i)$ and the cyclotomic character. Hence the congruence works when $p = 1 \mod 4$ but not otherwise.

3.2 Another Eigenform that Conjecturally Cannot be Approximated

In this section we consider an infinite slope eigenform that is not a Teichmüller twist of a finite slope eigenform. We conjecture that this eigenform cannot be approximated arbitrarily closely by finite slope eigenforms.

Conjecture 3.4. *There are exactly five residue classes in* $(\mathbf{Z}/9\mathbf{Z})[[q]]$ *of normalized eigenforms in* $S_k(\Gamma_0(N))$ *where* $k \geq 1$ *and* $N = 1, 3, 9$. *They are given in the following table, where the indicated weight is the smallest weight where that system of eigenvalues occurs (the level is* 1 *in each case):*

Weight	[$a_2, a_3, \ldots, a_{43}$ mod 9]
12	[3, 0, 6, 5, 3, 8, 0, 2, 6, 3, 8, 2, 6, 5]
16	[0, 0, 0, 2, 0, 2, 0, 2, 0, 0, 2, 2, 0, 2]
20	[6, 0, 3, 8, 6, 5, 0, 2, 3, 6, 5, 2, 3, 8]
24	[6, 0, 3, 5, 6, 8, 0, 2, 3, 6, 8, 2, 3, 5]
32	[3, 0, 6, 8, 3, 5, 0, 2, 6, 3, 5, 2, 6, 8]

The system of eigenvalues mod 9 *associated to the weight* 2 *form* F *on* $X_0(27)$ *is*

[0, 0, 0, 8, 0, 5, 0, 2, 0, 0, 5, 2, 0, 8],

so we conjecture that there is no eigenform f *on* $\Gamma_0(N)$ *with* $N \mid 9$ *such that* $f \equiv F$ $\pmod 9$.

As evidence, we verified that each of the mod 9 reductions of each newform of level 1 and weight $k \leq 74$ has one of the five systems of Hecke eigenvalues listed in the table. We also verified that all newforms of levels 3 and 9 and weight $k \leq 40$ have corresponding system of eigenvalues mod 9 in the above table. We checked using the method described in Section 4 that there is no newform of level 1 with weight $k \leq 300$ that approximates the weight 2 form on $X_0(27)$ modulo 9.

We now make some remarks about pseudo-representations when $p = 3$. Let

$$\chi : \mathbf{Z}/27\mathbf{Z} \to \mathbf{Z}/9\mathbf{Z}$$

be the mod 9 cyclotomic character, so χ has order 6 and if $\gcd(n, 3) = 1$ then $\chi(n) = n \in \mathbf{Z}/9\mathbf{Z}$. The pseudo-representation corresponding to a form of weight k giving the system of eigenvalues in the table in Conjecture 3.4 are

Weight	Pseudo-representation
12	$\chi^2 \oplus \chi^3$
16	$1 \oplus \chi^3$
20	$\chi^3 \oplus \chi^4$
24	$1 \oplus \chi^5$
32	$1 \oplus \chi$
$S_2(\Gamma_0(27))$	$\chi^2 \oplus \chi^5$

Note that the square of any pseudo-representation of level 1 in the above table has 1 as an eigenvalue, but the square of the pseudo-representation attached to $S_2(\Gamma_0(27))$ does not have 1 as an eigenvalue. Also,

$$F \equiv f_{16} \otimes \chi^2 \pmod{9},$$

where f_{16} is of weight 16. The order of χ^2 is 3, so χ^2 is not a power of the Teichmüller character (which has order 2) and Theorem 2.1 does not apply.

Further computations *suggest* that the pseudo-representations attached to forms of level 1 with coefficients in $\mathbf{Z}_9$ are

Weight	Pseudo-representations
$k \equiv 0 \pmod{6}$	$1 \oplus \chi^5, \quad \chi^2 \oplus \chi^3$
$k \equiv 2 \pmod{6}$	$1 \oplus \chi, \quad \chi^3 \oplus \chi^4$
$k \equiv 4 \pmod{6}$	$1 \oplus \chi^3$

The pseudo-representations attached to forms of level 27 with coefficients in $\mathbf{Z}_9$ seem to be

Weight	Pseudo-representations
$k \equiv 0 \pmod{6}$	$\chi \oplus \chi^4$
$k \equiv 2 \pmod{6}$	$\chi^2 \oplus \chi^5$
$k \equiv 4 \pmod{6}$	$\chi \oplus \chi^2, \quad \chi^4 \oplus \chi^5$

Also note that if $\chi^i \oplus \chi^j$ is one of the pseudo-representations of level 27 in the table, then the sum of the orders of χ^i and χ^j is 9, whereas at level 1 the sum of the orders is at most 7.

4 Computations About Approximating Infinite Slope Eigenforms

In this section, we investigate computationally how well certain infinite slope form can be approximated by finite slope eigenforms.

4.1 A Question About Families

The following question is an analogue of [GM92, §8] but for eigenforms of infinite slope. Fix a prime p and an integer N with $(N, p) = 1$.

Question 4.1. Suppose $f \in S_{k_0}(\Gamma_0(Np^r))$ is an eigenform having infinite slope (note that f need not be a newform). Is there a "family" of eigenforms $\{f_k\}$, with $f_k \in S_k(\Gamma_0(Np))$, where the weights k run through an arithmetic progression

$$k \in \mathcal{K} = \{k_0 + mp^\nu(p-1) \text{ for } m = 1, 2, \ldots\}$$

for some integer ν, such that

$$f_k \equiv f \pmod{p^n},$$

where $n = \mathrm{ord}_p(k - k_0) + 1$? (When $p = 2$ set $n = \mathrm{ord}_2(k - k_0) + 2$.)

Our question differs from the one in [GM92, §8] because there the form being approximated has finite slope, whereas our form f does not. We know, as discussed in the previous section, that our question sometimes has a negative answer since it might not be possible to approximate f at all.

4.2 An Approximation Bound

Let

$$f = \sum_{n\geq 1} a_n q^n \in K[[q]]$$

be a q-expansion with coefficients that generate a number field K. Fix a prime p and an even integer $k \geq 2$. In order to gather some data about Question 4.1, we now define a reasonably easy to compute upper bound on how well f can be approximated by an eigenform in $S_k(\Gamma_0(p))$. Suppose $\ell \geq 1$, let F be the characteristic polynomial of T_ℓ acting on the space $S_k(\Gamma_0(p))$ of classical cusp forms of weight k and tame level 1, and let H be the characteristic polynomial of $a_\ell \in K$. Let G be the resultant of $F(Y)$ and $H(X + Y)$ with respect to the variable Y, normalized so that G is monic. Thus the roots of G are the differences $\alpha - \beta$ where α runs through the roots of F and β runs through the $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$-conjugates of a_ℓ. We can easily compute the p-valuations of the roots of G without finding the roots, because the p-valuations of the roots are the slopes of the newton polygon of G. Let $m_\ell \in \mathbf{Q} \cup \{\infty\}$ be the *maximum* of the slopes of the Newton polygon of G. Let

$$c_k(f, r) = \min\{m_\ell : \ \ell \leq r \text{ is prime}\}.$$

We note that computing $c_k(f,r)$ requires knowing only the characteristic polynomials of Hecke operators T_ℓ on $S_k(\Gamma_0(p))$ and of a_ℓ for primes $\ell \leq r$.

Proposition 4.2. *If there is a normalized eigenform $g \in S_k(\Gamma_0(p))$ such that $f \equiv g \pmod{p^A}$, then $A \leq c_k(f,r)$ for any r.*

Proof. To see this observe that $c_k(f,r)$ is the minimum of the

$$\mathrm{ord}_p(a_n(f) - a_n(g))$$

where $1 \leq n \leq r$ and g runs through all normalized eigenforms in $S_k(\Gamma_0(p))$, and we run through all possible embeddings of f and g into $\overline{\mathbf{Z}}_p[[q]]$. □

The motivation for our definition of $c_k(f,r)$ is that it is straightforward to compute from characteristic polynomials of Hecke operators, even when the coefficients of f lie in a complicated number field. The number $c_k(f,r)$ could overestimate the true extent to which f is approximated by an eigenform in $S_k(\Gamma_0(p))$ in at least two ways:

1. There is an $r' > r$ such that $c_k(f,r') < c_k(f,r)$.

2. No *single* eigenform g is congruent to f, but each coefficient of f is congruent to some coefficient of some eigenform g.

4.3 Some Data About Approximations

Let p be a prime and $f \in S_{k_0}(\Gamma_0(p^r))$ be a newform of infinite slope. Suppose that the answer to Question 4.1 for f is yes. If k is a weight (in the arithmetic progression) then there should be an eigenform $f_k \in S_k(\Gamma_0(p))$ such that $f_k \equiv f \pmod{p^{n+1}}$ where $n = \mathrm{ord}_p(k - k_0)$. Thus we should have

$$\mathrm{ord}_p(k - k_0) + 1 \leq c_k(f,r)$$

for all $r > 1$ and all k in an arithmetic progression $\mathcal{K} = \{k_0 + mp^\nu(p-1) \text{ for } m = 0, 1, 2, \ldots\}$. (When $p = 2$ we should have $\mathrm{ord}_2(k - k_0) + 2 \leq c_k(f,r)$.)

The following or the results of some computations of $c_k(f,r)$.

$\mathbf{p = 2}$:

1. For $k_0 = 6, 10, 12, 14, 16, 20$ let $f \in S_{k_0}(\Gamma_0(4))$ be the unique newform. Then for all k with $k_0 < k \leq 100$ we have $c_k(f, 47) = \mathrm{ord}_2(k - k_0) + 2$.

2. For $k_0 = 18, 22$ let $f \in S_{k_0}(\Gamma_0(4))$ be the unique, up to Galois conjugacy, newform. Then for all k with $k_0 < k \leq 100$ we have $c_k(f, 7) = \mathrm{ord}_2(k - k_0) + 2$.

3. Let $f \in S_4(\Gamma_0(8))$ be the unique newform. For most $4 < k \leq 100$ we have $c_k(f, 47) = \mathrm{ord}_2(k - k_0) + 2$. However, in this range if $\mathrm{ord}_2(k - k_0) \geq 4$ then $c_k(f, 47) = 5$ Since $\mathrm{ord}_2(68 - 4) + 2 = 8$, this is a problem; perhaps this form is not approximated. Very similar behavior occurs for the newforms in $S_6(\Gamma_0(8))$, $S_8(\Gamma_0(8))$, and $S_4(\Gamma_0(16))$.

4. For the two newforms $f \in S_6(\Gamma_0(16))$, we have $c_k(f,47) \leq 3$ for all $k < 100$, so these f probably can not be approximated by finite slope forms.

5. Let f be the 2-deprivation of the unique normalized eigenform in $S_{k_0}(\Gamma_0(1))$ for $k_0 = 12, 16, 18, 20, 22, 26$. Then $c_k(f,47) = \mathrm{ord}_2(k-k_0)+2$ for $12 < k \leq 100$. Same statement for $k_0 = 24, 28$ for the 2-deprivation of one of the Galois conjugates and $c_k(f,47)$ replaced by $c_k(f,7)$.

$\mathbf{p = 3}$:

1. Suppose f is a newform in $S_{k_0}(\Gamma_0(9))$ for $k_0 \leq 12$. Then for $k_0 < k \leq 100$ we have $c_k(f,47) = \mathrm{ord}_3(k-k_0)+1$, except possibly for the nonrational form of weight 8, where we have only checked that $c_k(f,7) \geq \mathrm{ord}_3(k-k_0)+1$.

2. Let f be the twist of a newform in $S_{k_0}(\Gamma_0(1))$ by ω_3 for $k_0 \leq 32$. Then $c_k(f,7) \geq \mathrm{ord}_3(k-k_0)+1$ for $k_0 < k \leq 100$, with equality usually.

3. Let f be the newform in $S_2(\Gamma_0(45))$ of tame level 5. Then $c_{2+(3-1)3^n}(f,7) = n+1$ for $n = 0,1,2,3$ (here we are testing congruences with forms in $S_k(\Gamma_0(15))$).

$\mathbf{p = 5}$:

1. Let $f = q + q^2 + \cdots \in S_4(\Gamma_0(25))$ be a newform. Then $c_{4+4}(f,7) = 1$, $c_{4+4\cdot 5}(f,7) = 2$, and $c_{4+4\cdot 5^2}(f,7) = 3$. Same result for the newform $f = q + 4q^2 + \cdots \in S_4(\Gamma_0(25))$.

2. Let $f = q - q^2 + \cdots \in S_2(\Gamma_0(2\cdot 25))$. Then $c_{2+4}(f,7) = 1$ and $c_{2+4\cdot 5}(f,7) = 2$, where we are testing congruences with forms in $S_k(\Gamma_0(10))$.

3. Let f be one of the newforms in $S_2(\Gamma_0(5^3))$ defined over a quadratic extension of $\mathbf{Q}$. Then $c_{2+4}(f,7) = c_{2+4\cdot 5}(f,7) = c_{2+4\cdot 5^2}(f,2) = 1/2$. Thus it seems unlikely that f can be approximated by forms of finite slope.

$\mathbf{p = 7}$:

1. Let $f \in S_2(\Gamma_0(49))$ be the newform. Then $c_{2+6}(f,7) = 1$ and $c_{2+6\cdot 7}(f,7) = 2$. Same statement for the form $f = q - q^2 \in S_4(\Gamma_0(49))$ at weights $4+6$ and $4+6\cdot 7$.

The data and results of this paper suggests the following:

Guess 4.3. Let p be a prime and N an integer coprime to p. Then the eigenforms on $X_0(Np^t)$ that can be approximated by finite-slope eigenforms are exactly the eigenforms on $X_0(Np^2)$. Suppose f is an infinite slope eigenform that can be approximated by finite slope eigenforms and f has weight k_0. Then for any $k > k_0$ with $k \equiv k_0 \pmod{p-1}$, there is an eigenform f_k on $X_0(Np)$ of weight k such that $f \equiv f_k \pmod{p^n}$ where $n = \mathrm{ord}_p(k-k_0)+1$ (or $+2$ if $p = 2$). (In general one might have to restrict to n sufficiently large.)

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