Possibilities for Shafarevich-Tate Groups of Modular Abelian Varieties

William Stein

Harvard University

August 22, 2003 for Microsoft Research

$$Ш(A_f/\mathbf{Q}) = \mathrm{Ker}\left(H^1(\mathbf{Q}, A_f) \to \bigoplus_{\text{all } v} H^1(\mathbf{Q}_v, A_f)\right)$$

$$Ш(A_f/\mathbf{Q}) = \mathrm{Ker}\left(H^1(\mathbf{Q}, A_f) \to \bigoplus_{\text{all } v} H^1(\mathbf{Q}_v, A_f)\right)$$

$$Ш(A_f/\mathbf{Q})$$

$$Ш(A_f/\mathbf{Q})$$

Overview of Talk

1. Abelian Varieties
2. Shafarevich-Tate Groups
3. Nonsquare Shafarevich-Tate Groups

Abelian Varieties

Abelian Variety: A projective **group** variety (group law is automatically abelian).

Examples:

$$y^2 + y = x^3 - x$$

1. Elliptic curves
2. Jacobians of curves
3. Modular abelian varieties
4. Weil restriction of scalars

Jacobians of Curves

If X is an algebraic curve then

$$\mathrm{Jac}(X) = \{ \text{ divisor classes of degree 0 on } X \}.$$

Example: Let $X_1(N)$ be the modular curve parametrizing pairs

$$(E, \text{ embedding of } \mathbf{Z}/N\mathbf{Z} \text{ into } E).$$

The Jacobian of $X_1(N)$ is $J_1(N)$.

The Modular Jacobian $J_1(N)$

- $J_1(N)$ = Jacobian of $X_1(N)$

- The **Hecke Algebra**:

$$\mathbf{T} = \mathbf{Z}[T_1, T_2, \ldots] \hookrightarrow \mathrm{End}(J_1(N))$$

- **Cuspidal Modular Forms**:

$$S_2(\Gamma_1(N)) = H^0\left(X_1(N), \Omega^1_{X_1(N)}\right)$$

Modular Abelian Varieties

A modular abelian variety is any quotient of $J_1(N)$.

Goro Shimura associated an abelian variety A_f to any newform f:

$$A_f := J_1(N)/I_f J_1(N)$$

where

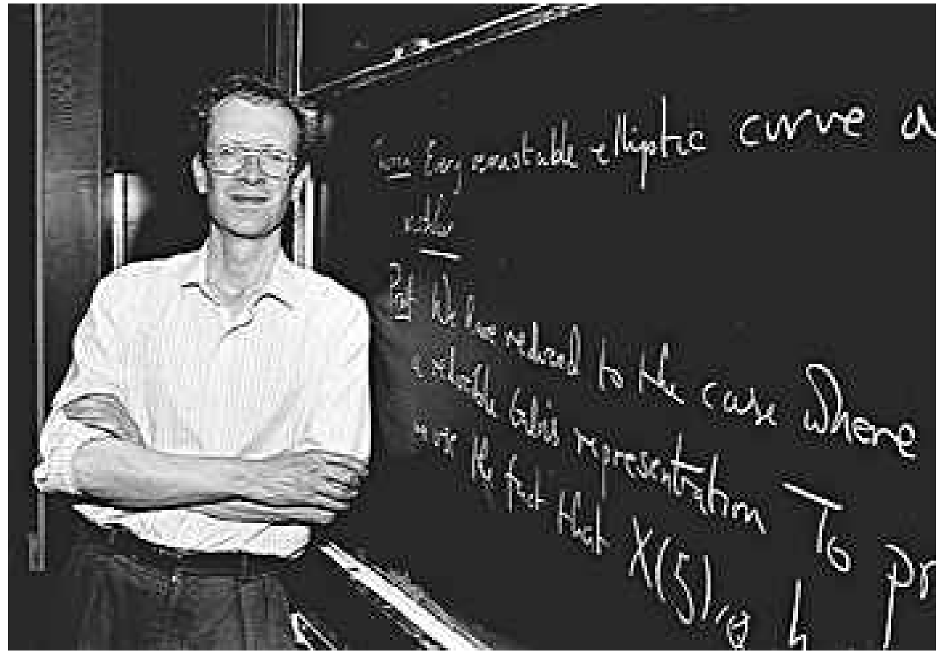
$$f = q + \sum_{n\geq 2} a_n q^n \in S_2(\Gamma_1(N))$$

$$I_f = \mathrm{Ker}(\mathbf{T} \to \mathbf{Z}[a_1, a_2, a_3, \ldots]), \;\; T_n \mapsto a_n$$

Extra structure

- A is an abelian variety over $\mathbf{Q}$
- The ring $\mathbf{Z}[a_1, a_2, \ldots]$ is a subring of $\mathrm{End}(A)$
- The dimension of A equals the degree of the field generated by the a_n

They Are Interesting!



- **Wiles et al.:** Every elliptic curve over **Q** is modular, i.e., isogenous to an A_f
  Consequence (Ribet): Fermat's Last Theorem

- **Serre's Conjecture:** Every odd irreducible Galois representation

$$\rho : \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \longrightarrow \mathrm{GL}_2(\overline{\mathbf{F}}_\ell)$$

  occurs up to twist in the torsion points of some A_f

Weil Restriction of Scalars

F/K: finite extension of number fields
A/F: abelian variety over F

$R = \mathrm{Res}_{F/K}(A)$ abelian variety over K with

$$\dim(R) = \dim(A) \cdot [F : K]$$

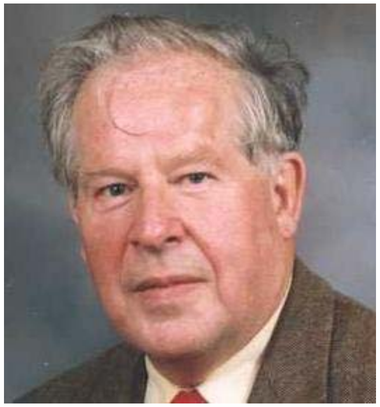
Functorial characterization:

For any K-scheme S,

$$R(S) = A(S \times_K F)$$

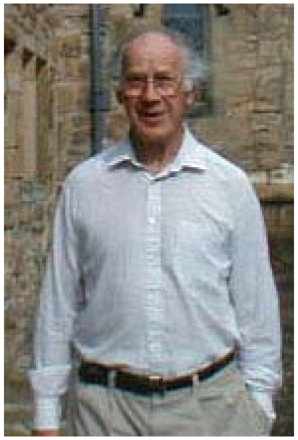
Birch and Swinnerton-Dyer

BSD Conjecture



$$\frac{L^{(r)}(A_f,1)}{r!} \stackrel{\text{conj}}{=} \frac{(\prod c_p)\cdot\Omega_{A_f}\cdot \mathrm{Reg}_{A_f}}{\#A_f(\mathbf{Q})_{\mathrm{tor}}\cdot \#A_f^{\vee}(\mathbf{Q})_{\mathrm{tor}}}\cdot \#\text{Ш}(A_f/\mathbf{Q})$$

$$L(A_f,s) = \prod_{\text{galois orbit}} \left(\sum_{n=1}^{\infty} \frac{a_n^{(i)}}{n^s}\right)$$



$$r = \mathrm{ord}_{s=1} L(A_f,s) \stackrel{\text{conj}}{=} \text{rank of } A_f(\mathbf{Q})$$

$$c_p = \text{order of component group at } p$$

$$\Omega_{A_f} = \text{canonical measure of } A_f(\mathbf{R})$$

The Shafarevich-Tate Group of A_f

Sha is a subgroup of the first Galois cohomology of A_f that measures failure of “local to global”:

$$Ш(A_f/\mathbf{Q}) = \mathrm{Ker}\left(H^1(\mathbf{Q}, A_f) \to \bigoplus_{\text{all } v} H^1(\mathbf{Q}_v, A_f)\right)$$

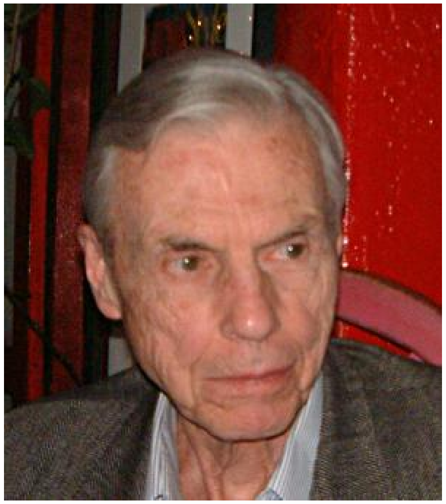
Example:

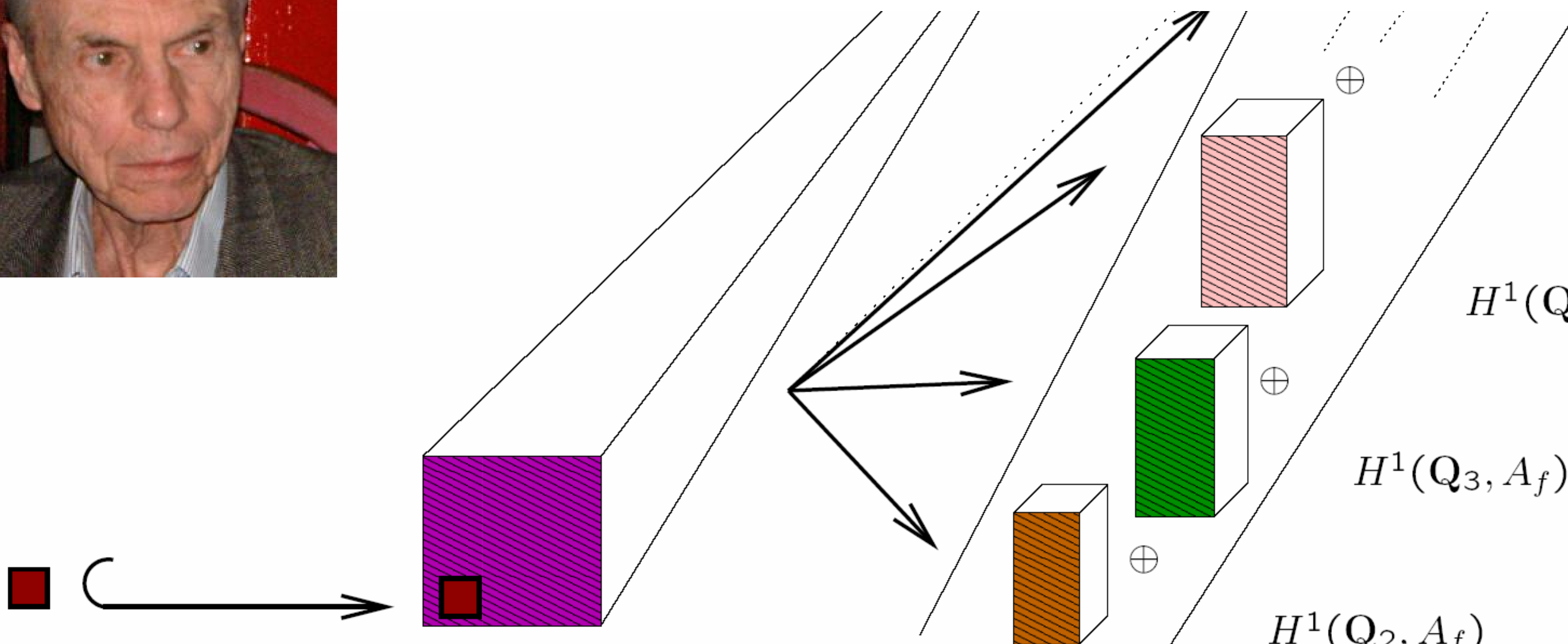
$$[3x^3 + 4y^3 + 5z^3 = 0] \in Ш(x^3 + y^3 + 60z^3 = 0)$$

Conjecture (Shafarevich-Tate):

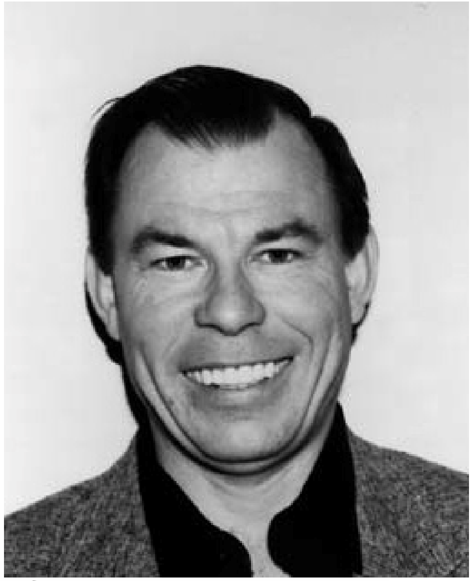
$Ш(A_f/\mathrm{Q})$ *is finite.*

The Shafarevich-Tate Group $Ш(A_f/\mathbf{Q})$





Finiteness Theorems of Kato, Kolyvagin, Logachev, and Rubin



Hypothesis: Suppose dim $A = 1$ and $\mathrm{ord}_{s=1} L(A,s) \leq 1$.

Kolyvagin: $Ш(A/\mathbf{Q})$ is finite.

Kato: If χ is a Dirichlet character corresponding to an abelian extension $K/\mathbf{Q}$ with $L(A,\chi,1) \neq 0$ then the χ-component of $Ш(A/K)$ is finite.

(**Rubin:** Similar results first when A has CM.)

The Dual Abelian Variety

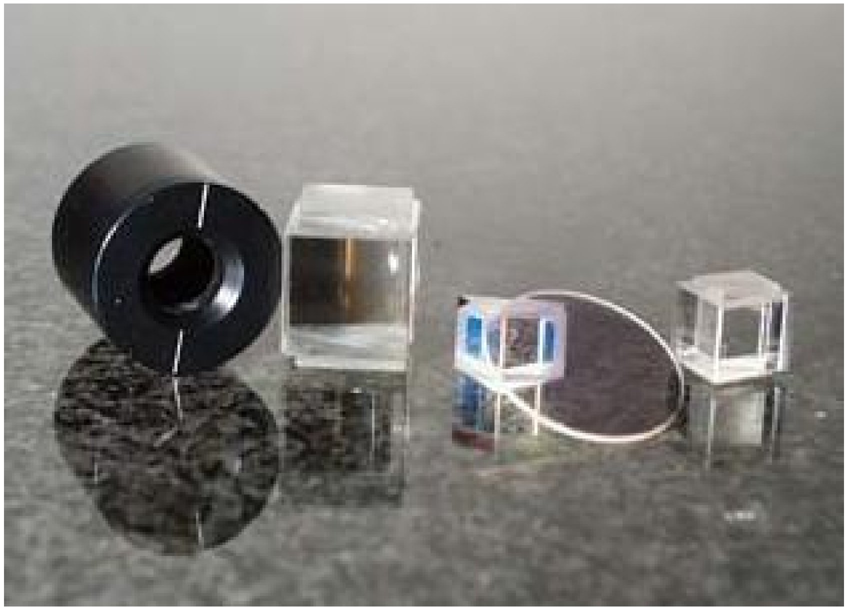
The **dual of *A*** is an abelian variety isogenous to A that parametrizes classes of invertible sheaves on A that are algebraically equivalent to zero.

$$A^\vee = \mathrm{Pic}^0(A)$$

The dual is functorial:

$$\text{If } A \to B \text{ then } B^\vee \to A^\vee.$$

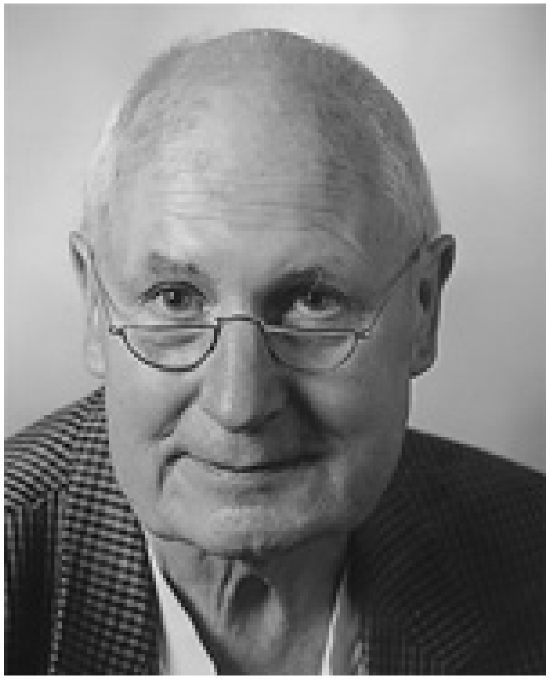
Polarized Abelian Varieties



A *polarization* of A is an isogeny (homomorphism) from A to its dual that is induced by a divisor on A. A polarization of degree 1 is called a *principal polarization*.

Theorem. *If A is the Jacobian of a curve, then A is canonically principally polarized. For example, elliptic curves are principally polarized.*

Cassels-Tate Pairing



A/F: abelian variety over number field

Theorem. If A is principally polarized by a polarization arising from an F-rational divisor, then there is a nondegenerate alternating pairing on $\text{Ш}(A/F)_{/\,\mathrm{div}}$, so for all p:

$$\#\text{Ш}(A/F)[p^\infty]_{/\,\mathrm{div}} = \square$$

(Same statement away from minimal degree of polarizations.)

Corollary. If $\dim A = 1$ and $\text{Ш}(A/F)$ finite, then

$$\#\text{Ш}(A/F) = \square$$

What if the abelian variety *A* is not an elliptic curve?

Assume $\#Ш(A/F)$ is finite. Overly optimistic literature:

- Page 306 of [Tate, 1963]: If A is a Jacobian then

$$\#Ш(A/F) = \square.$$

- Page 149 of [Swinnerton-Dyer, 1967]: Tate proved that

$$\#Ш(A/F) = \square.$$

Michael Stoll's Computation

During a grey winter day in 1996, Michael Stoll sat puzzling over a computation in his study on a majestic embassy-peppered hill near Bonn overlooking the Rhine. He had implemented an algorithm for computing 2-torsion in Shafarevich-Tate groups of Jacobians of hyperelliptic curves. He stared at a curve X for which his computations were in direct contradiction to the previous slide!

$$\#\text{Ш}(\mathrm{Jac}(X)/\mathbf{Q})[2] = 2.$$

What was wrong????

Poonen-Stoll

From: Michael Stoll (9 Dec 1996)
Dear Bjorn, Dear Ed:
[...] your results would imply that Sha[2] = Z/2Z
in contradiction to the fact that the order of Sha[2] should be a square (always assuming, as everybody does, that Sha is finite).
So my question is (of course): What is wrong ?

From: Bjorn Pooenen (9 Dec 96)
Dear Michael:
Thanks for your e-mails. I'm glad someone is actually taking the time to think about our paper critically! [...]
I would really like to resolve the apparent contradiction, because I am sure it will end with us learning something!
(And I don't think that it will be that Sha[2] can have odd dimension!)

From: Bjorn Poonen (11 hours later)
Dear Michael:
I think I may have resolved the problem. There is nothing wrong with the paper, or with the calculation. **The thing that is wrong is the claim that Sha must have square order!**

Poonen-Stoll Theorem

Theorem (Annals, 1999): *Suppose J is the Jacobian of a curve and J has finite Shafarevich-Tate group. Then*

$$\#Ш(J/F) = \square \text{ or } 2 \cdot \square$$

Example: The Jacobian of this curve has Sha of order 2

$$y^2 = -3(x^2+1)(x^2-6x+1)(x^2+6x+1)$$

Is Sha Always Square or Twice a Square?

Poonen asked at the Arizona Winter School in 2000, *"Is there an abelian variety A with Shafarevich-Tate group of order* ***three****?"*

In 2002 I finally found Sha of order 3 (times a square):

$$
\begin{aligned}
0 &= -x_1^3 - x_1^2 + (-6x_3x_2 + 3x_3^2)x_1 + (-x_2^3 + 3x_3x_2^2 + (-9x_3^2 - 2x_3)x_2 \\
&\quad + (4x_3^3 + x_3^2 + (y_1^2 + y_1 + (2y_3y_2 - y_3^2)))) \\
0 &= -3x_2x_1^2 + ((-12x_3 - 2)x_2 + 3x_3^2)x_1 + (-2x_2^3 + 3x_3x_2^2 + \\
&\quad (-15x_3^2 - 4x_3)x_2 + (5x_3^3 + x_3^2 + (2y_2y_1 + ((4y_3 + 1)y_2 - y_3^2)))) \\
0 &= -3x_3x_1^2 + (-3x_2^2 + 6x_3x_2 + (-9x_3^2 - 2x_3))x_1 + (x_2^3 + (-9x_3 - 1)x_2^2 \\
&\quad + (12x_3^2 + 2x_3)x_2 + (-9x_3^3 - 3x_3^2 + (2y_3y_1 + (y_2^2 - 2y_3y_2 + (3y_3^2 + y_3))))) \\
0 &= x_1^2x_2^4 - 8x_1^2x_2^3x_3 + 30x_1^2x_2^2x_3^2 - 44x_1^2x_2x_3^3 + 25x_1^2x_3^4 - 2/3x_1x_2^5 + 26/3x_1x_2^4x_3 + 2/3x_1x_2^4 \\
&\quad - 140/3x_1x_2^3x_3^2 - 16/3x_1x_2^3x_3 + 388/3x_1x_2^2x_3^3 + 20x_1x_2^2x_3^2 - 2/3x_1x_2^2y_2^2 + 8/3x_1x_2^2y_2y_3 \\
&\quad - 10/3x_1x_2^2y_3^2 - 490/3x_1x_2x_3^4 - 88/3x_1x_2x_3^3 + 8/3x_1x_2x_3y_2^2 - 40/3x_1x_2x_3y_2y_3 \\
&\quad + 44/3x_1x_2x_3y_3^2 + 250/3x_1x_3^5 + 50/3x_1x_3^4 - 10/3x_1x_3^2y_2^2 + 44/3x_1x_3^2y_2y_3 - 50/3x_1x_3^2y_3^2 \\
&\quad + 1/9x_2^6 - 2x_2^5x_3 - 2/9x_2^5 + 15x_2^4x_3^2 + 26/9x_2^4x_3 + 1/9x_2^4 - 544/9x_2^3x_3^3 - 140/9x_2^3x_3^2 \\
&\quad - 8/9x_2^3x_3 + 2/9x_2^3y_2^2 - 8/9x_2^3y_2y_3 + 10/9x_2^3y_3^2 + 135x_2^2x_3^4 + 388/9x_2^2x_3^3 + 10/3x_2^2x_3^2 \\
&\quad - 2x_2^2x_3y_2^2 + 80/9x_2^2x_3y_2y_3 - 94/9x_2^2x_3y_3^2 - 2/9x_2^2y_2^2 + 8/9x_2^2y_2y_3 - 10/9x_2^2y_3^2 \\
&\quad - 150x_2x_3^5 - 490/9x_2x_3^4 - 44/9x_2x_3^3 + 50/9x_2x_3^2y_2^2 - 244/9x_2x_3^2y_2y_3 + 30x_2x_3^2y_3^2 \\
&\quad + 8/9x_2x_3y_2^2 - 40/9x_2x_3y_2y_3 + 44/9x_2x_3y_3^2 + 625/9x_3^6 + 250/9x_3^5 + 25/9x_3^4 - 50/9x_3^3y_2^2 \\
&\quad + 220/9x_3^3y_2y_3 - 250/9x_3^3y_3^2 - 10/9x_3^2y_2^2 + 44/9x_3^2y_2y_3 - 50/9x_3^2y_3^2 + 1/9y_2^4 \\
&\quad - 8/9y_2^3y_3 + 10/3y_2^2y_3^2 - 44/9y_2y_3^3 + 25/9y_3^4
\end{aligned}
$$

Plenty of Nonsquare Sha!

- **Theorem (Stein):** *For every prime* $p < 25000$ *there is an abelian variety* A *over* **Q** *such that*

$$\#\text{Ш}(A/\mathrm{Q}) = p \cdot \square$$

- **Conjecture (Stein):** *Same statement for all p.*

Constructing Nonsquare Sha

While attempting to connect groups of points on elliptic curves of high rank to Shafarevich-Tate groups of abelian varieties of rank 0, I found a construction of nonsquare Shafarevich-Tate groups.

The Main Theorem

Theorem (Stein). *Suppose E is an elliptic curve and p an odd prime that satisfies various technical hypothesis. Suppose ℓ is a prime congruent to 1 mod p (and not dividing N_E) such that*

$$L(E, \chi_{p,\ell}, 1) \neq 0 \text{ and } a_\ell(E) \not\equiv \ell + 1 \pmod{p}$$

Here $\chi_{p,\ell} : (\mathbf{Z}/\ell)^* \twoheadrightarrow \mu_p$ *is a Dirichlet character of order p and conductor ℓ corresponding to an abelian extension K. Then there is a twist A of a product of p - 1 copies of E and an exact sequence*

$$0 \to E(\mathbf{Q})/pE(\mathbf{Q}) \to Ш(A/\mathbf{Q})[p^\infty] \to Ш(E/K)[p^\infty] \to Ш(E/\mathbf{Q})[p^\infty] \to 0.$$

If E has odd rank and $Ш(E/\mathbf{Q})[p^\infty]$ *is finite then* $Ш(A/\mathbf{Q})[p^\infty]$ *has order that is* ***not a perfect square****.*

What is the Abelian Variety *A*?

Let R be the Weil restriction of scalars of E from K down to $\mathbf{Q}$, so R is an abelian variety over $\mathbf{Q}$ of dimension p (i.e., the degree of K). Then A is the kernel of the map induced by trace:

$$0 \longrightarrow A \longrightarrow R \longrightarrow E \longrightarrow 0$$

Note that

- A has dimension $p - 1$
- A is isomorphic over K to a product of copies of E
- Our hypothesis on ℓ and Kato's finiteness theorems imply that $A(\mathbf{Q})$ and $\#\text{Ш}(A/\mathbf{Q})$ are both finite.

Proof Sketch (1): Exact Sequence of Neron Models

Neron

The exact sequence

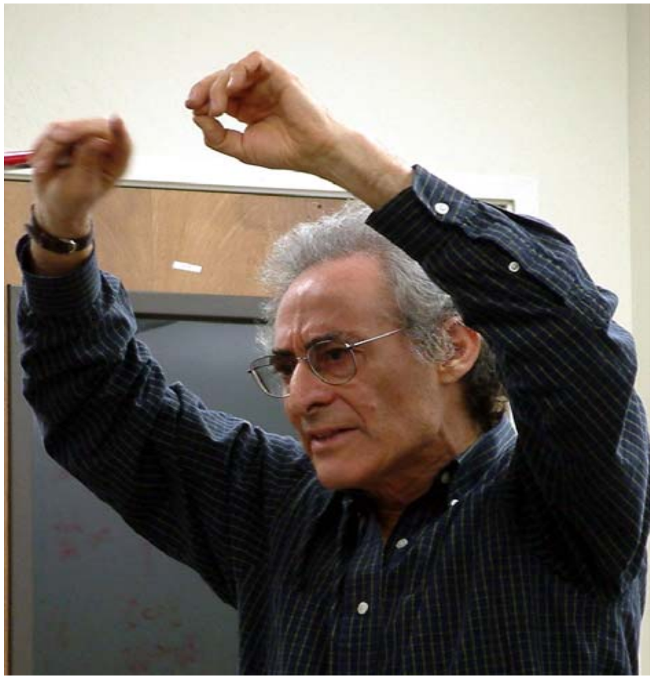
$$0 \to A \to R \to E \to 0$$

extends to an exact sequence of *Néron models* (and hence sheaves for the étale topology) over $\mathbf{Z}$:

$$0 \to \mathcal{A} \to \mathcal{R} \to \mathcal{E} \to 0.$$

To check this, we use that formation of Néron models commutes with unramified base change and Prop. 7.5.3(a) of [*Néron Models*, 1990].

Proof (2): Mazur's Etale Cohomology Sha Theorem



Mazur's *Rational Points of Abelian Varieties with Values in Towers of Number Fields*:

For $F = A, R, E$ let $\mathcal{F} = \text{Néron}(F)$. Then

$$H^1_{\text{ét}}(\mathbf{Z}, \mathcal{F})[p^\infty] \cong Ш(F/\mathbf{Q})[p^\infty]$$

In general this is not true, but our hypothesis on p and ℓ are exactly strong enough to kill the relevant error terms.

Proof (3): Long Exact Sequence

The long exact sequence of étale cohomology begins

$$0 \to A(\mathbf{Q}) \to R(\mathbf{Q}) \to E(\mathbf{Q}) \xrightarrow{\delta} H^1_{\text{ét}}(\mathbf{Z}, \mathcal{A}) \to H^1_{\text{ét}}(\mathbf{Z}, \mathcal{R}) \to H^1_{\text{ét}}(\mathbf{Z}, \mathcal{E}) \to H^2_{\text{ét}}(\mathbf{Z}, \mathcal{A})$$

Take the p-power torsion in this exact sequence then use Mazur's theorem. Next analyze the cokernel of δ...

Proof (4): Apply Kato's Finiteness Theorems

We have $\mathrm{Coker}(\delta) = E(\mathbf{Q})/pE(\mathbf{Q})$ since

$$L(E, \chi_{p,\ell}, 1) \neq 0 \quad \text{and} \quad a_\ell \not\equiv \ell + 1 \pmod{p}.$$

(To see this requires chasing some diagrams.)

Also $H^2_{\text{ét}}(\mathbf{Z}, \mathcal{A})[p^\infty] = 0$ (proof uses Artin-Mazur duality).

Both of these steps use Kato's finiteness theorem in an essential way. Putting everything together yields the claimed exact sequence

$$0 \to E(\mathbf{Q})/pE(\mathbf{Q}) \to Ш(A/\mathbf{Q})[p^\infty] \to Ш(E/K)[p^\infty] \to Ш(E/\mathbf{Q})[p^\infty] \to 0.$$

What Next?

- Remove the hypothesis that p<25000, i.e., prove a nonvanishing result about twists of newforms of large degree. (One currently only has results for degrees 2 and in some cases 3.)
- Prove that the Birch and Swinnerton-Dyer conjecture predicts that $\#Ш(A/F)$ has order divisible by p, in agreement with our construction.
- Replace E by an abelian variety of dimension bigger than 1.

Thank you for coming!

Acknowledgements:

Kevin Buzzard, Cristian Gonzalez, Barry Mazur,
Bjorn Poonen, Ken Ribet, and Michael Stoll

For more details including an accepted paper:

`http://modular.fas.harvard.edu/papers/nonsquaresha/.`