Questions About Finiteness of Shafarevich-Tate Groups of Higher Rank Elliptic Curves

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Abstract

1 Introduction

Conjecture 1.1. *There are infinitely many pairs* (E, p) *with* E *an elliptic curve over* $\mathbb{Q}$ *such that* $E(\mathbb{Q})$ *has rank* ≥ 2*, and the* p*-primary part* $Ш(E/\mathbb{Q})(p)$ *of* $Ш(E/\mathbb{Q})$ *finite.*

We will make use of the following two unproved conjectures, which are consequences of the Birch and Swinnerton-Dyer conjecture:

Conjecture 1.2 (Parity Conjecture)**.** *The parity of the rank of* $E(\mathbb{Q})$ *is even if and only if the sign in the functional equation for* $L(E, s)$ *is* $+1$*.*

Conjecture 1.3 (Squareness)**.** *For any prime* p*, the order of* $Ш(E/\mathbb{Q})[p]$ *is a perfect square.*

Remark 1.4. Conjecture 1.3 is implied by finiteness of $Ш(E/\mathbb{Q})(p)$, because of the Cassels-Tate pairing. However, it is an a priori weaker statement, since we could have $Ш(E/\mathbb{Q}) \approx (\mathbb{Q}_p/\mathbb{Z}_p)^2$, in which case $\#Ш(E/\mathbb{Q})[p] = p^2$, but $Ш(E/\mathbb{Q})$ is not finite. Conjecture 1.3 is still very difficult; we only know it holds for curves with $\mathrm{ord}_{s=1} L(E, s) \leq 1$ and for specific examples of curves of rank ≥ 2.

We prove Conjecture 1.1, assuming either Conjecture 1.2 or Conjecture 1.3. More specifically, we can prove the following three statements:

Theorem 1.5.

1. *Assuming Conjecture 1.2 or Conjecture 1.3 for* $p = 2$*. Then there are infinitely many elliptic curves of rank exactly* 2 *with* $Ш(E/\mathbb{Q})(2) = 0$*.*

2. *Same as 2, with* $p = 2$ *replaced by* $p = 3$ *(in both the hypothesis and conclusion).*

3. Same as 2, with rank exactly 2 replaced by rank exactly 3.

We may also relax the assumptions somewhat; we only need the assumptions for curves in the families we use for these average Selmer results ($\mathcal{F}_1$, elliptic curves with one marked point, and $\mathcal{F}_2$, elliptic curves with two marked points).

2 Proofs

Proof of part of Theorem 1.5. We prove 2. The arguments for the other two are identical, using the appropriate Selmer averages.

In the family $\mathcal{F}_1$ of elliptic curves with one marked point, we use congruence conditions to construct a positive density family $\mathcal{F}$ such that half have root number $+1$ and half have root number -1. By [], the average cardinality of the 2-Selmer group in this family is 6. (This average cardinality result is a key new input that makes it possible to prove this theorem.)

Dokchitsers' result [DD09] states that the root number gives the parity of the p-Selmer rank, i.e., root number $+1$ means even p-Selmer rank. We can ensure that our curves satisfy the hypothesis of Dokchiters (explain how). So in this case, half of the family has even 2-Selmer rank and half has odd. (This is another key nontrivial new input that makes this result possible.)

Let p_i denote the proportion of curves in $\mathcal{F}$ with 2-Selmer rank i. We know (shown elsewhere) that $p_0 = 0$ (this is a statement about specialization of rank for curves in a family). We also have

$$1/2 = p_1 + p_3 + p_5 + \cdots$$

and

$$1/2 = p_2 + p_4 + p_6 + \cdots .$$

Thus

$$6 = 2p_1 + 4p_2 + 8p_3 + 16p_4 + \cdots .$$

If $p_2 = 0$, then we would have

$$6 = 2p_1 + 8p_3 + 16p_4 + \cdots \geq 2 \cdot 1/2 + 16 \cdot 1/2 = 9,$$

which is a contradiction. So $p_2 > 0$, i.e., a positive proportion of curves in $\mathbb{F}$ have 2-Selmer rank 2 and root number $+1$. (In fact, we have that $p_2 \geq 1/4$.)

Since 100% of curves in $\mathbb{F}$ have rank ≥ 1 (as mentioned above), Conjecture 1.2 implies that 100% of curves with 2-Selmer rank 2 and root number $+1$ (so p_2 proportion of $\mathbb{F}$) must have algebraic rank 2, hence for these curves $Ш(E/\mathbb{Q})[2] = 0$.

Alternatively, assuming instead that $Ш(E/\mathbb{Q})[2]$ is a square also implies that 100% of curves with 2-Selmer rank 2 in $\mathcal{F}$ must have algebraic rank 2 and trivial $Ш(E/\mathbb{Q})[2]$.

Clearly (and this time it really is "clearly"!) trivial $Ш(E/\mathbb{Q})[2]$ implies trivial $Ш(E/\mathbb{Q})(2)$, and a group of order 1 is finite! So in fact, we have produced a fairly large proportion of curves (out of all curves of rank at least 1) with rank 2 and finite $Ш(E/\mathbb{Q})(2)$. □

3 Other related thoughts and questions

1. It seems unlikely that we could remove the conditional assumptions from Theorem 1.5, since then we would have produced a lot of rank 2 curves without knowing where the second point is coming from.

2. Can we vary p instead of the elliptic curve? Probably not, with these sorts of techniques.

3. Can we parametrize smaller families, e.g., the one $y(y+1) = x(x-1)(x+a)$, in order to get Selmer average results for them?

4. Can we parametrize families of elliptic curves where the root number is always $+1$ or -1?

5. Question: are we also (conditionally) showing that for a positive proportion of all elliptic curves with rank ≥ 2 that $Ш(E/\mathbb{Q})(6) = 0$? That would be **extremely surprising** (at least, to me), even conditionally.

References

[DD09] T. Dokchitser and V. Dokchitser, *Root numbers and parity of ranks of elliptic curves*, `http://arxiv.org/abs/0906.1815`.