THE EFFECTIVE COMPUTATION OF ITERATED INTEGRALS AND CHOW-HEEGNER POINTS ON TRIPLE PRODUCTS

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ABSTRACT. FIXME

1. INTRODUCTION

Let $E_{/\mathbf{Q}}$ be an elliptic curve of conductor N and $f = \sum_{n \geq 1} a_n e^{2\pi i n z} \in S_2(\Gamma_0(N))$ the weight two eigenform associated to E by modularity. The conjecture of Birch and Swinnerton-Dyer (BSD) predicts that for any number field H, the rank of the Mordell-Weil group $E(H)$ of H-rational points on E is equal to the order of vanishing $r_{\mathrm{an}}(E_{/H})$ of the L-series $L(E_{/H}, s)$ at its central critical point $s = 1$. In particular, this raises the question:

If $L(E_{/H}, 1) = 0$, how can one construct a non-torsion rational point $P \in E(H)$?

A fruitful approach to this question is afforded by the theory of *Heegner points* over imaginary quadratic fields K. Work of Gross, Zagier and Kolyvagin using this theory provides what is (at present) perhaps the best theoretical evidence for BSD; see [GrZa], [Ko], as well as the more recent [Zh].

Let $\mathfrak{H}^* = \{z \in \mathbf{C} : \mathrm{Im}(z) > 0\} \cup \mathbf{P}^1(\mathbf{Q})$ denote the extended upper-half plane, topologized in the usual manner (*cf.*, for example, [DS, Ch. 2]). For $z \in \mathfrak{H}^*$, let γ_z be a choice of continuous path between the cusp at $\infty \in \mathbf{P}^1(\mathbf{Q})$ and z. Heegner points may be defined complex-analytically as the images of quadratic irrationalities $z \in \mathfrak{H} \cap K$ under the modular parametrization

$$(1.0.1) \qquad \varphi : X_0(N)(\mathbf{C}) \simeq \Gamma_0(N)\backslash\mathfrak{H}^* \longrightarrow E(\mathbf{C}) \simeq \mathbf{C}/\Lambda_E, \ z \mapsto \int_{\gamma_z} \omega_f = c \sum_{n \geq 1} \frac{a_n}{n} e^{2\pi i n z}.$$

Here we assume for simplicity that E is the strong Weil curve in its isogeny class. Thus the regular differential form associated with f is $\omega_f = 2\pi i c \cdot f(z)dz$, where $c \in \mathbf{Z}$ is the *Manin constant*, conjecturally equal to 1 and known to be so when $N < 60000$, *cf.* [ARS].

Heegner points, although *a priori* defined over $\mathbf{C}$, are known to be algebraic. More precisely, by the theory of complex multiplication they are defined over suitable abelian extensions H of the imaginary quadratic field K. Conversely, denoting by $\bar{\mathbf{Q}}$ the algebraic closure of $\mathbf{Q}$ in $\mathbf{C}$, if $z \in \mathfrak{H} \cap \bar{\mathbf{Q}}$ is such that $[\mathbf{Q}(z) : \mathbf{Q}] > 2$, then the image $\varphi(z)$ lies in $E(\mathbf{C}) \setminus E(\bar{\mathbf{Q}})$. (See [Wa, Ch. 3], where it is shown that $j(z)$ is trancendental for $z \in \mathfrak{H} \cap \bar{\mathbf{Q}}$ with $[\mathbf{Q}(z) : \mathbf{Q}] > 2$. Since the pair of rational functions $(j(z), j(Nz))$ give coordinates for a (singular) affine model of $X_0(N)$ over $\mathbf{Q}$, it follows that the class of z in $X_0(N)(\mathbf{C})$ is a point with transcendental coordinates. The claim follows because φ is a finite map.)

Various attempts have been made to find other explicit analytic formulas for rational points on E over number fields H:

- In [BD] and [Gr], the complex upper half-plane $\mathfrak{H}^*$ is replaced by the p-adic upper half-plane, regarded as a rigid analytic variety. Using Coleman's p-adic path integrals, this work gives formulas for (and presents explicit calculations of) p-adic incarnations of the *same* Heegner points considered in (1.0.1).
- Again exploiting p-adic path integrals, [D1] suggests a construction of points on E defined over $\mathbf{C}_p$, the completion of an algebraic closure of $\mathbf{Q}_p$. These *Stark-Heegner*

points are conjectured to be H-rational for suitable abelian extensions of *real* quadratic fields K for which p is inert. See [DP] for the numerical verification of these conjectures.

- Let $E_{/F}$ be a modular elliptic curve over a totally real number field F. Using cohomological constructions similar to those of [D1], but replacing (1.0.1) by a formula involving integrals of complex-analytic differential forms of *higher order* associated to Hilbert modular forms, [D2, Ch. VIII] proposes another construction of points in $E(\mathbf{C})$. These points are conjectured to be defined over suitable abelian extensions H of quadratic extensions K/F which have exactly one non-real archimedean place. See also [DL] for numerical calculations providing evidence for these conjectures.

- Let $E_{/F}$ be an elliptic curve defined over a number field F, and assume E has complex multiplication by an imaginary quadratic field K.[1] As replacements for the set of Heegner points $z \in \mathfrak{H} \cap K$, [BDP2] defines, for any integer $r \geq 0$, a collection of generalized Heegner cycles on a $(2r + 1)$-dimensional manifold $W_r \times E^r$, where W_r is a Kuga-Sato variety. Here the role of the modular parametrization (1.0.1) is played by Abel-Jacobi maps from the Chow groups of null-homologous cycles on $W_r \times E^r$ to certain generalized Jacobians.

Here we follow a different approach, using K.-T. Chen's iterated path integrals [Ch]. A defect of the functional

$$\pi_1(X_0(N)(\mathbf{C}); \infty) \to \mathbf{C}, \text{ defined by}$$

$$\gamma \mapsto \int_\gamma \omega_f$$

is that it factors through the maximal abelian quotient $H_1(X_0(N)(\mathbf{C}), \mathbf{Z})$ of the fundamental group of the Riemann surface underlying the modular curve $X_0(N)$. Chen observed that it is nevertheless possible to obtain non-abelian information about this fundamental group using integration. His theory of iterated integrals give rise to functionals

$$J : \pi_1(X_0(N)(\mathbf{C}); \infty) \longrightarrow \mathbf{C}$$

which do not necessarily factor through $H_1(X_0(N)(\mathbf{C}), \mathbf{Z})$.

The goal of this paper is to describe an algorithm for the computation of *Chow-Heegner points*, following [DRS]. In this construction one replaces the integral appearing in (1.0.1) by an *iterated* integral of *two* modular forms. In order to describe it more precisely, we fix cuspidal eigenforms $f, g \in S_2(\Gamma_0(N))$ and assume f is a newform with rational Fourier coefficients. To g is associated a collection of one-dimensional algebraic cycles $T_{g,n}$ for all integers $n \geq 1$ on $X_0(N) \times X_0(N)$. Each of these cycles gives rise to a class in $\mathrm{Pic}(X_0(N))$, essentially by intersecting $T_{g,n}$ with the diagonal in $X_0(N) \times X_0(N)$, and to an associated rational point $P_{g,n}$ of the Jacobian $J_0(N)$; the construction of $T_{g,n}$ and $P_{g,n}$ is described more precisely in §2.3 and §3.2 below. Write A_f for the modular abelian variety attached to f by the Eichler-Shimura construction. The image $P_{g,f,n}$ of $P_{g,n}$ in $A_f(\mathbf{Q})$ is called the *Chow-Heegner point* attached to the eigenforms g and f and the integer n.

Work of Yuan–Zhang–Zhang [YZZ][2] gives a criterion in terms of L-functions (*cf.* [DRS, Corollary 1.4]) for when the points $P_{g,f,n}$ are non-torsion, which explains why Chow-Heegner points are of interest in the context of BSD.

Although the Hodge class $\xi_{g,n}$ in the cohomology of $X_0(N) \times X_0(N)$ associated to the cycle $T_{g,n}$ is easy to compute using linear algebra, the cycle $T_{g,n}$ itself is much more difficult to compute explicitly. This poses a challenge if one wishes to calculate the points $P_{g,f,n}$ numerically. However, it is shown in [DRS] that $P_{g,f,n}$ can be computed as an element of the analytic curve $A_f(\mathbf{C})$ using *only* the data of the Hodge class $\xi_{g,n}$, in terms of a certain iterated

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[1]FIXME: Do you mean the complex multiplication (the extra endomorphisms) to be defined over F or $\bar{F}$?

[2]FIXME: Is this the right citation?

path integral in the sense of [Ch]. This formula is amenable to numerical computation, which we have implemented using the free software package `Sage` [S$^+$09]. Our algorithm makes these points readily computable in practice for elliptic curves A_f of small conductor.

While the analytic formula is not the only way of computing the points $P_{g,f,n}$ (see the Appendix) our approach has certain advantages: it applies to a broad class of eigenforms g, and in principle[3] works for an arbitrary Hodge cycle[4]. In future work, we hope to extend the algorithm described below to compute Chow-Heegner points [DRS2] associated to exceptional Hodge classes on products of Kuga-Sato varieties arising from CM forms. The rationality of Chow-Heegner points computed in this manner could provide numerical evidence for certain open cases of the Hodge conjecture.

Plan of the paper. In §2 we recall necessary facts about itegrated integrals and related ingredients for our main algorithm. In §3 we specialize to the case of modular curves, define the points $P_{g,f,n}$ precisely, and write down an explicit analytic formula for them. In §4 we describe in detail an algorithm for evaluating this formula numerically. The algorithm is illustrated with numerical examples in §5, which also includes a table of triple Chow-Heegner points on rank one elliptic curves of small conductor.

Source code. Our `Sage` implementation of the algorithm described in this paper can be found at[5].

Acknowledgments. To be added.[6]

2. Preliminaries

2.1. Let X be a smooth, complete algebraic curve of genus $g \geq 2$ over $\mathbf{Q}$, and let $Y = X \setminus \{\infty\}$ be the complement of a single point in $X(\mathbf{Q})$. For a smooth variety V defined over $\mathbf{Q}$ (such as X or Y) we denote by V^{an} the complex manifold $V(\mathbf{C})$ with its analytic topology.

The *de Rham cohomology* $H^1_{\mathrm{dR}}(X^{\mathrm{an}}, \mathbf{C})$ is the cohomology of the de Rham complex of smooth $\mathbf{C}$-valued differential forms on X^{an}. Since the Riemann surface X^{an} is algebraic and defined over $\mathbf{Q}$, there is a canonical isomorphism

$$H^1_{\mathrm{dR}}(X^{\mathrm{an}}, \mathbf{C}) = H^1_{\mathrm{dR}}(X/F) \otimes \mathbf{C}.$$

Here

$$H^1_{\mathrm{dR}}(X/\mathbf{Q}) := \mathbf{H}^1(0 \to \mathcal{O}_X \to \Omega^1_X \to 0)$$

is the *algebraic de Rham cohomology* of $X/\mathbf{Q}$, defined as the hypercohomology of the de Rham complex of sheaves of regular differential forms on X.

Because X is a curve, the group $H^1_{\mathrm{dR}}(X/\mathbf{Q})$ has a particularly simple description using

$$\Omega^1_{\mathrm{II}}(X) := \text{rational 1-forms on } X \text{ with vanishing residues at all points of } X.$$

Such 1-forms are called *differentials of the second kind* (on X). Observe that $\Omega^1_{\mathrm{II}}(X) = \Omega^1_{\mathrm{II}}(Y)$, by the residue formula. Thus there is a canonical isomorphism

$$H^1_{\mathrm{dR}}(X/\mathbf{Q}) = \Omega^1_{\mathrm{II}}(Y)/d\mathbf{Q}(X),$$

where $\mathbf{Q}(X)$ is the field of rational functions on X. Using Riemann-Roch one sees that $\Omega^1_{\mathrm{II}}(Y)/d\mathbf{Q}(X) \cong \Omega^1(Y)/d\Gamma(Y, \mathcal{O}_Y)$. So $H^1_{\mathrm{dR}}(X/\mathbf{Q})$ can also be computed as the space of *regular* 1-forms on Y, modulo exact forms. For computational purposes, the latter description is the most useful: we will compute with classes in $H^1_{\mathrm{dR}}(Y)$ using rational 1-forms on X, regular

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[3]FIXME: and in practice if we succeed in implementing it!

[4]FIXME: what's the best way of saying this?

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[6]FIXME

away from the point ∞. These are amenable to computation via their Laurent expansions about ∞.

Later on we shall make use of the symplectic *Poincaré pairing*

$$\langle , \rangle : H^1_{\mathrm{dR}}(X^{\mathrm{an}}, \mathbf{C}) \times H^1_{\mathrm{dR}}(X^{\mathrm{an}}, \mathbf{C}) \to \mathbf{C},$$

defined by

$$\langle \omega, \eta \rangle := \frac{1}{2\pi i} \int_X \omega \wedge \eta$$

for (the cohomology classes of) smooth $\mathbf{C}$-valued 1-forms ω, η on X^{an}. If ω and η are differentials of the second kind on X, regular away from the cusp ∞, then the induced pairing on $H^1_{\mathrm{dR}}(X/\mathbf{Q})$ can be computed as

$$\langle \omega, \eta \rangle = \mathrm{res}_\infty(F_\omega \cdot \eta) = -\mathrm{res}_\infty(\omega \cdot F_\eta),$$

where F_ν denotes a local primitive of the differential ν at ∞.

2.2. We now turn to recalling the definition and basic properties of iterated integrals; see [Ch], [H1], [H2] for more details.

Fix a base point $o \in Y^{\mathrm{an}}$ and dente by $\Gamma := \pi_1(Y^{\mathrm{an}}; o)$ the fundamental group of the Riemann surface Y^{an}. We write $I \subset \mathbf{Z}[\Gamma]$ for augmentation ideal of the integral group ring of Γ. Recall that $H_1(X^{\mathrm{an}}, \mathbf{Z}) = H_1(Y^{\mathrm{an}}, \mathbf{Z}) \cong \Gamma^{\mathrm{ab}}$, as can be seen from the well-known presentation for the fundamental group of a Riemann surface, and that this abelian group is naturally identified with I/I^2.

Definition 2.2.1. The *path space* on Y based at o, denoted $\mathbf{P}(Y; o)$, is the set of piecewise-smooth paths

$$\gamma : [0,1] \longrightarrow Y^{\mathrm{an}}, \quad \text{with } \gamma(0) = o.$$

Let

$$\pi : \tilde{Y} \to Y^{\mathrm{an}}, \qquad \text{resp.} \qquad \pi : \tilde{X} \to X^{\mathrm{an}},$$

denote the universal covering space of Y^{an} (resp. X^{an}) corresponding to the basepoint o. The group Γ acts on $\tilde{Y}$ transitively and without fixed points, and the map $\gamma \mapsto \gamma(1)$ identifies the quotient $\tilde{Y}/\Gamma$ with Y^{an}.

Recall a closed, $\mathbf{C}$-valued smooth 1-form (resp. a meromorphic 1-form of the second kind) η on X^{an} admits a smooth (resp. meromorphic) primitive $F_\eta : \tilde{X} \to \mathbf{C}$, defined by the rule

$$F_\eta(\gamma) := \int_0^1 \gamma^* \eta.$$

Definition 2.2.2. The *basic iterated integral* attached to an ordered n-tuple $(\omega_1, \ldots, \omega_n)$ of smooth 1-forms on Y^{an} is the function $\mathbf{P}(Y; o) \to \mathbf{C}$, denoted $\int \omega_1 \cdot \omega_2 \cdot \ldots \cdot \omega_n$, defined by

$$\gamma \mapsto \int_\gamma \omega_1 \cdot \omega_2 \cdot \ldots \cdot \omega_n := \int_\Delta (\gamma^* \omega_1)(t_1)(\gamma^* \omega_2)(t_2) \cdots (\gamma^* \omega_n)(t_n),$$

where Δ is the simplex in $[0,1]^n$ defined by $0 \leq t_n \leq t_{n-1} \leq \cdots \leq t_1 \leq 1$. The integer n is called the *length* of this basic iterated integral.

Example 2.2.3. When $n = 2$, the basic iterated integral attached to ω and η can be computed by the formula

$$\int_\gamma \omega \cdot \eta = \int_\gamma \omega F_\eta = \int_0^1 \gamma^*(\omega F_\eta).$$

In the expression in the middle, we abusively use the same notation ω for the differential $\pi^* \omega$ on $\tilde{Y}$. The 1-form ωF_η is to be integrated along a lift of γ to $\tilde{Y}$, which is unique once a lift of o to $\tilde{Y}$ is specified.

Definition 2.2.4. An *iterated integral* is a linear combination of basic iterated integrals, viewed as a function on $\mathbf{P}(Y; o)$. Its *length* is defined to be the maximum of the lengths of its constituent basic iterated integrals. It is said to be *homotopy invariant* if its value on any path γ depends only on the homotopy class of γ.

A homotopy invariant iterated integrals defines a $\mathbf{C}$-valued function on Γ, and by extending linearly induces a homomorphism of abelian groups $\mathbf{Z}[\Gamma] \to \mathbf{C}$. Observe that a homotopy invariant iterated integral of length $\leq n$ vanishes on the $(n+1)^{\text{st}}$ power I^{n+1} of the augmentation ideal in $\mathbf{Z}[\Gamma]$, and hence gives rise to a well-defined element of $\operatorname{Hom}(I/I^{n+1}, \mathbf{C})$. The natural map

(2.2.1) $\qquad$ (homotopy invariant iterated integrals of length $\leq n$) $\longrightarrow \operatorname{Hom}(I/I^{n+1}, \mathbf{C})$

is an isomorphism; see, for example, [H2].

We will be interested in numerically evaluating certain homotopy invariant iterated integrals on Y of length ≤ 2. Suppose ω and η are two differentials of the second kind on X, regular on Y, representing cohomology classes $\omega, \eta \in H^1_{\mathrm{dR}}(X/\mathbf{Q})$ in the manner of §2.1. The basic iterated integral $\int \omega \cdot \eta$ of length 2 is not generally homotopy invariant. But when either ω or η is holomorphic on X — i.e., has no pole at ∞ — a suitable modification of $\int \omega \cdot \eta$ will be homotopy invariant, as we now explain.

Recall that a differential on a Riemann surface is said to have a *logarithmic pole* at a point if its expansion in terms of a local parameter q at this point is of the form $\sum_{n=0}^{\infty} a_n q^n \frac{dq}{q}$. When ω is holomorphic at ∞, we let $\alpha_{\omega,\eta}$ be a meromorphic 1-form on X that is regular on Y and is such that the induced differential $\omega F_\eta - \alpha_{\omega,\eta}$ on $\tilde{X}$ has at worst a logarithmic pole at (any point lying over) ∞. This condition is well-posed because the principal part of ωF_η at $\tilde{x} \in \tilde{X}$ depends only on the image x of $\tilde{x}$; see [DRS, §2]. The form $\alpha_{\omega,\eta}$ exists — and in fact can even be taken to be algebraic and defined over $\mathbf{Q}$ — by Riemann-Roch. If ω is not holomorphic at ∞ but η is, then we define $\alpha_{\omega,\eta} := -\alpha_{\eta,\omega}$.

Lemma 2.2.5. *Let ω and η be as above, and assume that either ω or η is holomorphic at ∞. Then:*

    i. *The iterated integral $J_{\omega,\eta} := \int \omega \cdot \eta - \alpha_{\omega,\eta}$ is homotopy-invariant.*

    ii. *Suppose that ω and η represent integral cohomology classes. Using (2.2.1), identify $J_{\omega,\eta}$ with a homomorphism of abelian groups $I/I^3 \to \mathbf{C}$. The restriction of $J_{\omega,\eta}$ to I^2/I^3 is $\mathbf{Z}$-valued and can be identified with $\omega \otimes \eta$, viewed as an element of*

$$H^1(X^{\mathrm{an}}, \mathbf{Z}) \otimes H^1(X^{\mathrm{an}}, \mathbf{Z}) \cong (H_1(X^{\mathrm{an}}, \mathbf{Z}) \otimes H_1(X^{\mathrm{an}}, \mathbf{Z}))^\vee = (I/I^2 \otimes I/I^2)^\vee = (I^2/I^3)^\vee.$$

    (Here $A^\vee$ denotes $\operatorname{Hom}(A, \mathbf{Z})$, for any abelian group A.)

Proof. The homotopy invariance of $J_{\omega,\eta}$ follows from the fact that $J_{\omega,\eta}(\gamma) = \int_\gamma \omega F_\eta - \alpha_{\omega,\eta}$, and the 1-form on $\tilde{X}$ in the integrand is holomorphic when restricted to $\tilde{Y}$. For the second claim, see the discussion at the beginning of §2 of [DRS], and *loc. cit.*, Lemma 1.1(2). [7] $\qquad\square$

Remark 2.2.6. Note that if ω and η are both holomorphic at ∞, then we can take $\alpha_{\omega,\eta} = 0$.

Now consider an integral Hodge class $\xi \in H^1(X^{\mathrm{an}}, \mathbf{Z}) \otimes H^1(X^{\mathrm{an}}, \mathbf{Z})$. The Hodge condition implies that we can choose a basis $\{\omega_i\}$ such that when we write $\xi = \sum c_{i,j} \omega_i \otimes \omega_j$, then either ω_i or ω_j is holomorphic at ∞ whenever $c_{i,j} \neq 0$. By the previous lemma, the iterated integral $J_\xi = \sum J_{\omega_i,\omega_j}$ is homotopy invariant and induces a homomorphism

$$J_\xi : H_1(X^{\mathrm{an}}, \mathbf{Z}) = I/I^2 \to \mathbf{C}/\mathbf{Z}.$$

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[7]FIXME: Should we say more about why this is true when η but not ω is holomorphic?

Fix an auxiliary holomorphic 1-form $\rho \in H^{1,0}(X_{\mathbf{C}}) \subset H^1(X^{\mathrm{an}}, \mathbf{C})$. Denote by Λ the period lattice $\langle \int_\gamma \rho : \gamma \in H_1(X^{\mathrm{an}}, \mathbf{Z}) \rangle$. The class $\gamma_\rho \in H_1(X^{\mathrm{an}}, \mathbf{C})$ that is Poincaré dual to ρ actually belongs to $H_1(X^{\mathrm{an}}, \mathbf{Z}) \otimes \Lambda$. Consequently $J_\xi(\gamma_\rho)$ is a well-defined element of $\mathbf{C}/\Lambda$.

2.3. Let X_1, X_2 denote copies of X, and X_{12} the diagonal copy of X in $X_1 \times X_2$. To a codimension-1 algebraic cycle $Z \subset X \times X = X_1 \times X_2$ (defined over $\mathbf{Q}$) we associate the divisor class

$$D_Z = (Z \cap X_{12}) - (Z \cap X_1) - (Z \cap X_2).$$

Define a corresponding degree-zero divisor class

$$P_Z = D_Z - \deg(D_Z) o \in \mathrm{Pic}^0(X).$$

(Recall that $o \in X(\mathbf{Q})$ is a fixed base point.)

We now state the iterated integral formula from [DRS] for the image of P_Z under the Abel-Jacobi map

$$\mathrm{AJ}_X : \mathrm{Pic}^0(X) \to \Omega^1(X^{\mathrm{an}})^\vee / H_1(X^{\mathrm{an}}, \mathbf{Z}).$$

Let ϵ_o be the projector on $\mathrm{Pic}(X \times X)$ defined by

$$\epsilon_o(Z) = Z - i_{1*}\pi_{1*} - i_{2*}\pi_{2*}$$

where $\pi_1, \pi_2 : X \times X \rightrightarrows X$ are the projections and $i_1, i_2 : X \rightrightarrows X \times X$ are the inclusions of "vertical and horizontal" copies of X over the basepoint o.

Let

$$\mathrm{cl}(\epsilon_o -) : \mathrm{Pic}(X \times X) \to H^1(X^{\mathrm{an}}, \mathbf{Z}) \otimes H^1(X^{\mathrm{an}}, \mathbf{Z})$$

denote the composition of the cycle class map and the projector ϵ_o. (The effect of ϵ_0 is to annihilate the $H^2 \otimes H^0$ and $H^0 \otimes H^2$ factors in the Künneth decomposition of the second Betti cohomology group of $X \times X$.) Suppose $\mathrm{cl}(\epsilon_o Z)$ is represented by $\sum_{i,j} c_{i,j} \omega_i \otimes \omega_j$, where ω_i or ω_j is holomorphic at ∞ whenever $c_{i,j} \neq 0$.

Theorem 2.3.1 ([DRS], Corollary 3.6). *The image* $\mathrm{AJ}_X(P_Z) \in \Omega^1(X^{\mathrm{an}})^\vee / H_1(X^{\mathrm{an}}, \mathbf{Z})$ *is represented by the linear functional that maps* $\rho \in \Omega^1(X^{\mathrm{an}})$ *to*

$$\sum J_{\omega_i, \omega_j}(\gamma_\rho) = \sum_{i,j} c_{i,j} \int_{\gamma_\rho} (\omega_i \cdot \omega_j - \alpha_{\omega_i, \omega_j}) + \deg(D_Z) \int_o^\infty \rho \in \mathbf{C},$$

where $\gamma_\rho \in H_1(X^{\mathrm{an}}, \mathbf{C})$ *is Poincaré dual to* $\rho \in H^{1,0}(X^{\mathrm{an}}) \subset H^1_{\mathrm{dR}}(X^{\mathrm{an}}, \mathbf{C})$. $\square$

3. Chow-Heegner points on modular curves

We now specialize the discussion of the preceding section to the case of classical modular curves X. We shall define certain rational points on an arbitrary elliptic curve $E_{/\mathbf{Q}}$ called *Chow-Heegner points*, such that the corresponding points of $E(\mathbf{C}) \cong \mathbf{C}/\Lambda_E$ can be computed using iterated path integrals via Theorem 2.3.1.

3.1. Let $N \geq 1$ be an integer and let $X = X_0(N)$ denote the canonical model over $\mathbf{Q}$ of the classical modular curve of level N; write $J_0(N)$ for the Jacobian of $X_0(N)$. With this choice of X we place ourselves in the setup of §1, taking the ground field F to be $\mathbf{Q}$ and the point $\infty \in X(\mathbf{Q})$ to be the usual cusp at infinity. Thus $Y := X_0(N) - \{\infty\}$. (Note that Y differs from the usual open modular curve $Y_0(N)$ when $N > 1$.) For the moment we do not specify the basepoint $o \in Y^{\mathrm{an}}$ used for topological constructions.

We write $S_2(\Gamma_0(N))$ for the space of cuspidal weight 2 modular forms for $\Gamma_0(N)$. This space is canonically isomorphic to the vector space $\Omega^1(X^{\mathrm{an}})$ of holomorphic 1-forms on X, via the map which associates to a modular form $f : \mathfrak{H}^* \to \mathbf{C}$ the 1-form $\omega_f = 2\pi i f(z)\, dz$ on X. We recall that if f has Fourier expansion $\sum_{n \geq 1} a_n q^n$ then a Laurent expansion for ω_f near ∞ is given by $\sum_{n \geq 1} a_n q^n \frac{dq}{q}$.

3.2. Let E be an elliptic curve over $\mathbf{Q}$ of conductor N whose isogeny class corresponds to a newform $f \in S_2(\Gamma_0(N))$ with rational Fourier coefficients. In particular there is a *modular parametrization* $\pi_E : J_0(N) \to E$, a morphism of abelian varieties defined over $\mathbf{Q}$. We will assume for the remainder of this paper that E is *optimal*, i.e. $\ker \pi_E$ is connected. In this case, assuming the Manin constant $c = 1$, the Néron lattice of E coincides with the period lattice Λ_f of the differential $\omega_f = 2\pi i f(z)\mathrm{d}z \in \Omega^1(X^{\mathrm{an}})$ corresponding to f. The map π_E can be computed on complex points explicitly, using the Abel-Jacobi isomorphism

$$\mathrm{AJ}_X : J_0(N)(\mathbf{C}) \cong \Omega^1(X^{\mathrm{an}})^\vee / H_1(X^{\mathrm{an}}, \mathbf{Z}),$$

the Weierstrass uniformization $W : \mathbf{C}/\Lambda_f \cong E(\mathbf{C})$, and the analytic parametrization

$$\pi_E^{\mathrm{an}} : \Omega^1(X^{\mathrm{an}})^\vee / H_1(X^{\mathrm{an}}, \mathbf{Z}) \to \mathbf{C}/\Lambda_f.$$

The map π_E^{an} sends the coset of a functional on $\Omega^1(X^{\mathrm{an}})$ to the evaluation of that functional at ω_f. Thus for $P_{\mathbf{C}} \in J_0(N)(\mathbf{C})$ we have

$$(3.2.1) \qquad \pi_{E,\mathbf{C}}(P_{\mathbf{C}}) = W(\pi_E^{\mathrm{an}}(\mathrm{AJ}_X(P_{\mathbf{C}}))) = W(\mathrm{AJ}_X(P_{\mathbf{C}})(\omega_f)).$$

Let $\mathbf{T} = \mathbf{Z}[\ldots, T_n, \ldots]$ be the Hecke algebra of level N, and let $\mathbf{T}_0$ denote the subalgebra generated by Hecke operators prime to the level N. Then $\mathbf{T}_0 \otimes \mathbf{Q}$ factors as a product $\prod_h K_h$ where h runs over newforms of all levels M dividing N and $K_h = \mathbf{Q}(\{a_n(h)\}_{n\geq1})$ is the number field generated by the Hecke eigenvalues of h. For a divisor M of N and a newform $g \in S_2(\Gamma_0(M))$, denote by

$$T_g' \in \mathbf{T}_0 \otimes \mathbf{Q} \cong \prod_h K_h$$

the idempotent with 1 in the K_g component and 0 elsewhere. Define T_g to be the image of T_g' via the map $\mathbf{T}_0 \otimes \mathbf{Q} \hookrightarrow \mathbf{T} \otimes \mathbf{Q}$. For any positive integer n, define $T_{g,n} := T_g \cdot T_n \in \mathbf{T} \otimes \mathbf{Q}$ as the product of T_g and T_n. Identifying $\mathbf{T}$ as an algebra of correspondences on X, $T_{g,n}$ is associated to a divisor class in $\mathrm{Pic}(X \times X) \otimes \mathbf{Q}$, which we also denote by $T_{g,n}$. In §2.3 a point $P_Z \in \mathrm{Pic}^0(X) = J_0(N)(\mathbf{Q})$ was attached to a divisor Z on $X \times X$. This construction gives rise to a map

$$\mathrm{Pic}(X \times X) \otimes \mathbf{Q} \quad \to \quad J_0(N)(\mathbf{Q}) \otimes \mathbf{Q}$$
$$Z \quad \mapsto \quad P_Z.$$

In particular we have a point $P_{T_{g,n}} \in J_0(N)(\mathbf{Q}) \otimes \mathbf{Q}$.

Note that both the divisor class $P_{T_{g,n}}$ and the projector ϵ_o depend on the basepoint o, so we must now specify a particular choice of o. In the following definition, *we fix the choice of basepoint o to be the cusp $0 \in X$*, which is distinct from ∞ because $N > 1$.

Definition 3.2.1. The *Chow-Heegner point* $P_{g,f,n}$ corresponding to the pair (g, f) of modular forms and the integer $n \geq 1$ is the element $\pi_E(P_{T_{g,n}}) \in E(\mathbf{Q}) \otimes \mathbf{Q}$.

For a general discussion of the notion of Chow-Heegner point, of which the construction above is a special case, see the introduction of [BDP2].

Remark 3.2.2. To avoid the troublesome tensor products with $\mathbf{Q}$ in the definition of Chow-Heegner points $P_{g,f,n}$, one must work with an actual algebraic cycle on $X \times X$, as opposed to $T_{g,n}$ which is merely a $\mathbf{Q}$-linear combination of such cycles. Define the "denominator" $d_{g,n}$ of $T_{g,n} \in \mathbf{T} \otimes \mathbf{Q}$ to be the smallest positive integer such that $d_{g,n}T_{g,n}$ lies in the image of $\mathbf{T}$ under the inclusion $\mathbf{T} \hookrightarrow \mathbf{T} \otimes \mathbf{Q}$. Then $d_{g,n}P_{g,f,n}$ belongs to $E(\mathbf{Q})$; see also §4.7.

However, we regard $d_{g,n}P_{g,f,n}$ as well-defined only modulo the torsion subgroup $E(\mathbf{Q})^{\mathrm{tor}}$. The reason for this is that to obtain a computable formula for $P_{g,f,n}$ one can take the basepoint o to be *any* cusp other than ∞. There being no distinguished choice, one is left with an ambiguity valued in the image in $E(\mathbf{Q})$ of the (torsion) cuspidal subgroup of $J_0(N)$; see (3.3.2) below and the subsequent remarks.

Finally, note that if eigenvalue $a_n(g)$ of T_n acting on the g-isotypic component is rational, then $T_{g,n}$ is rationally equivalent to $a_n(g)T_{g,1}$, and so $P_{g,f,n} = a_n(g)P_{g,f,1}$[8]. Thus, the points $P_{g,f,n}$ for $n > 1$ are only of interest when g has irrational Fourier coefficients or is an oldform.

3.3. Theorem 2.3.1 gives rise to an explicit formula for a triple Chow-Heegner point $P_{g,f,n}$ in terms of iterated integrals. This formula is stated in terms of the components of the cohomology class $\mathrm{cl}(\epsilon_o T_{g,n}) \in H^1_{\mathrm{dR}}(X/\mathbf{Q})^{\otimes 2}$ expressed as a sum of pure tensors. In this section we find an explicit formula for $\mathrm{cl}(\epsilon_o T_{g,n})$.

The action of the Hecke algebra $\mathbf{T}_0$ on modular forms extends to an action on the de Rham cohomology of X. Under this action, we have

$$H^1_{\mathrm{dR}}(X/\mathbf{Q}) \cong H^1_{\mathrm{dR}}(X/\mathbf{Q})[g_1] \oplus \cdots \oplus H^1_{\mathrm{dR}}(X/\mathbf{Q})[g_n],$$

indexed by Galois conjugacy classes of newforms of all levels M dividing N. Let $g \in S_2(\Gamma_0(M))$ be a newform and suppose $\{\omega_{g,1}, \ldots, \omega_{g,2k}\}$ is a collection of differentials of the second kind on X representing a basis for $H^1_{\mathrm{dR}}(X/\mathbf{Q})[g]$. Write $T_n\omega_{g,i} = \sum_j a^n_{ij}\omega_{g,j}$, and denote by A_n and B the matrices $(a^n_{ij})_{1 \le i,j \le 2k}$ and $(\langle \omega_{g,i}, \omega_{g,j} \rangle)_{1 \le i,j \le 2k}$, respectively, where $\langle \, , \, \rangle$ denotes the Poincare pairing. Then we have the following:

Lemma 3.3.1. $\mathrm{cl}(\epsilon_o T_{g,n}) = \sum_{i,j} c^n_{ij}\omega_{g,i} \otimes \omega_{g,j}$, where $(c^n_{ij})_{1 \le i,j \le 2k} = -B^{-1}A_n$.

Proof. The projector ϵ_o acts on $H^2_{\mathrm{dR}}(X \times X)$ by annihilating the $H^0_{\mathrm{dR}}(X) \otimes H^2_{\mathrm{dR}}(X)$ and $H^2_{\mathrm{dR}}(X) \otimes H^0_{\mathrm{dR}}(X)$ components of the Künneth decomposition, so we have $\mathrm{cl}(\epsilon_o T_{g,n}) \in H^1_{\mathrm{dR}}(X) \otimes H^1_{\mathrm{dR}}(X)$. Note from the definition that $T_{g,n}$ acts on $H^1_{\mathrm{dR}}(X)[h]$ as T_n if $h = g$ and 0 otherwise, so $\mathrm{cl}(\epsilon_o T_{g,n})$ is equal to the image of T_n under the identification:

$$\mathrm{End}(H^1_{\mathrm{dR}}(X)[g]) \simeq H^1_{\mathrm{dR}}(X)[g]^\vee \otimes H^1_{\mathrm{dR}}(X)[g] \simeq H^1_{\mathrm{dR}}(X)[g] \otimes H^1_{\mathrm{dR}}(X)[g]$$

The first map is the canonical isomorphism of finite dimensional vector spaces, and the second is induced from the inverse of the identification $H^1_{\mathrm{dR}}(X)[g] \simeq H^1_{\mathrm{dR}}(X)[g]^\vee$ via the map $v \mapsto (w \mapsto \langle v, w \rangle)$. The remainder of the proof is a straightforward exercise in linear algebra, and is left to the reader. $\square$

Combining the previous results, we obtain the following formula for $P_{g,f,n}$. Let γ_f be the Poincaré dual of ω_f and let $\omega_{g,1}, \ldots, \omega_{g,2k}$ be differentials of the second kind that give rise to a symplectic basis for the g-isotypic $\mathbf{Q}$-subspace $H^1_{\mathrm{dR}}(X/\mathbf{Q})[g]$. That is, for $1 \le i \le k$ we have $\langle \omega_{g,i}, \omega_{g,i+k} \rangle = 1$ and $\langle \omega_{g,i}, \omega_{g,j} \rangle = 0$ for $j \ne i+k$. Assume moreover that this basis is adapted to the Hodge filtration, in the sense that the 1-forms $\omega_{g,1}, \ldots, \omega_{g,k}$ are assumed to be regular at ∞. Since $H^0(X, \Omega^1(X/\mathbf{Q}))[g] \subset H^1_{\mathrm{dR}}(X/\mathbf{Q})[g]$ is a maximal isotropic subspace with respect to the Poincaré pairing, we can simply take any basis $\omega_{g,1}, \ldots, \omega_{g,k}$ for $H^0(X, \Omega^1(X/\mathbf{Q}))[g] = S_2(\Gamma_0(N))[g]$ and extend it to a symplectic basis of the sort desired. Write

$$(3.3.1) \qquad \mathrm{cl}(\epsilon_o T_{g,n}) = \sum_{i,j}^{2k} c^n_{ij}\omega_{g,i} \otimes \omega_{g,j}$$

where the coefficients can be computed as in Lemma 3.3.1. Since we chose the basis to be symplectic, the matrix B is the standard symplectic matrix. Additionally, because T_n preserves the holomorphic subspace of $H^1_{\mathrm{dR}}(X/\mathbf{Q})$, then the computation of $-B^{-1}A_n$ shows that $c^n_{ij} = 0$ when $k+1 \le i,j \le 2k$. Thus, the Hodge class in (3.3.1) is of the form discussed after Remark 2.2.6.

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[8]FIXME: Is this true? Certainly, $T_{g,n}$ and $a_n(g)T_{g,1}$ induce the same endomorphism of $H^1_{\mathrm{dR}}(X)[g]$, and $P_{g,f,n} = a_n(g)P_{g,f,1}$, but is it true that $T_{g,n}$ and $a_n(g)T_{g,1}$ are rationally equivalent cycles?

Combining (3.3.1) with (3.2.1) and Theorem 2.3.1, we obtain the following formula for the point $P_{g,f,n}$:

$$(3.3.2) \qquad P_{g,f,n} = W\left(\sum_{i,j}^{2k} c_{ij}^n \left(\int_{\gamma_f} \omega_{g,i} \cdot \omega_{g,j} - \alpha_{\omega_{g,i},\omega_{g,j}}\right)\right).$$

Note that (3.3.2) omits the term $\deg(D_{T_{g,n}}) \int_o^\infty \omega_f$ from the formula in Theorem 2.3.1. This is justified by our choice of basepoint o, for by a theorem of FIXME [?], the difference $[\infty] - [0]$ is a torsion point of $J_0(N)(\mathbf{Q})$. Since $P_{g,f,n}$ is defined as an element of $E(\mathbf{Q}) \otimes \mathbf{Q}$, torsion can be disregarded.

We emphasize that by Lemma 2.2.5 (and the discussion immediately following it), the right-hand side of (3.3.2) depends only on the *homology* class $\gamma_f \in H_1(Y^{\mathrm{an}}, \mathbf{Z}) = H_1(X^{\mathrm{an}}, \mathbf{Z})$ Poincaré dual to ω_f. It can therefore be evaluated by lifting γ_f arbitrarily to an element $\tilde{\gamma}_f \in \pi_1(Y^{\mathrm{an}}; o)$ and evaluating the length ≤ 2 homotopy invariant iterated integral (3.3.2) on any loop in the homotopy class $\tilde{\gamma}_f$.

Remark 3.3.2. When $n = 1$, formula (3.3.2) greatly simplifies. In the notation of Lemma 3.3.1, A_1 is the identity matrix and B is the standard symplectic matrix by our choice of basis, and so if we write $\eta_{g,i}$ for $\omega_{g,i+k}$, then

$$P_{g,f,1} = W\left(\sum_{i=1}^k \left(\int_{\gamma_f} \omega_{g,i} \cdot \eta_{g,i} - \eta_{g,i} \cdot \omega_{g,i} - 2\alpha_{\omega_{g,i},\eta_{g,i}}\right)\right)$$

3.4. Although $P_{g,f,n}$ is defined relative to the choice of base point o as the cusp 0 on Y, it will be convenient to *compute* it relative to a different choice of basepoint.

Lemma 3.4.1. *The right hand side of* (3.3.2) *is independent of the choice of $o \in Y(\mathbf{Q})$.*

Proof. Changing the basepoint from o to o' amounts to conjugating the representative path γ_f for the homology class Poincaré dual to ω_f by a path β from o to o'. This manifestly does not affect the value of the integral of the meromorphic 1-form $\alpha_{\omega_{g,i},\omega_{g,j}}$. Thus the issue is whether we have an identity

$$(3.4.1) \qquad \int_{\gamma_f} \omega_{g,i} \cdot \omega_{g,j} \overset{?}{=} \int_{\beta\gamma_f\beta^{-1}} \omega_{g,i} \cdot \omega_{g,j}.$$

But by [H1, Exer. 8], for any 1-forms ω, η, loop γ, and path β, we have

$$(3.4.2) \qquad \int_{\beta\gamma\beta^{-1}} \omega \cdot \eta = \int_\gamma \omega \cdot \eta + \begin{vmatrix} \int_\gamma \omega & \int_\gamma \eta \\ \int_\beta \omega & \int_\beta \eta \end{vmatrix}.$$

In our situation, the determinants expressing the difference between the two sides of (3.4.1) vanish. Indeed, $\int_{\gamma_f} \omega_{g,i} = \langle \omega_f, \omega_{g,i} \rangle = 0$, since the decomposition into isotypic components for the action of the Hecke algebra is orthogonal with respect to the Poincaré pairing. $\square$

3.5. We record a fundamental property of the point $P_{g,f,1}$.

Theorem 3.5.1 ([DRS], Corollary 1.4)**.** *Assume that f and g are rational newforms of level N, and the local signs $\varepsilon_p(g,g,f)$ of Garrett's triple product L-function $L(g,g,f,s)$ are $+1$ at the primes $p \mid N$. Then the point $P_{g,f,1} \in E(\mathbf{Q}) \otimes \mathbf{Q}$ is nonzero (or equivalently, the point $\pi_E(d_g P_{T_g}) \in E(\mathbf{Q})$ is non-torsion) if and only if the following conditions hold:*

    i. $L(f,1) = 0$,

    ii. $L'(f,1) \neq 0$, *and*

    iii. $L(f \otimes \mathrm{Sym}^2(g), 2) \neq 0$.

Remark 3.5.2. If g is not a newform with rational Fourier coefficients, the point $P_{g,f,1}$ can be zero even when the conditions of the above theorem are satisfied. In general, one expects $P_{g,f,n}$ to be nonzero for some value of n if and only if conditions i-iii hold.[9]

4. Algorithm for effective computation of triple Chow-Heegner points

We now turn to the question of numerically evaluating formula (3.3.2) for a triple Chow-Heegner point $P_{g,f,n} \in E(\mathbf{Q}) \otimes \mathbf{Q}$ for an elliptic curve $E = E_f$. We retain all the notation from §§2–3. The following ingredients occur in the (3.3.2):

1. The Poincaré dual $\gamma_f \in H_1(X, \mathbf{C})$ of $\omega_f \in H^1_{\mathrm{dR}}(X^{\mathrm{an}}, \mathbf{C})$.
2. A collection of rational differentials of the second kind $\omega_{g,1}, \ldots, \omega_{g,2k}$ on X, regular away from ∞, whose images in $H^1_{\mathrm{dR}}(X/\mathbf{Q})$ are a symplectic basis for the g-isotypic component $H^1_{\mathrm{dR}}(X/\mathbf{Q})[g]$. This basis should be adapated to the Hodge filtration on $H^1_{\mathrm{dR}}(X/\mathbf{Q})[g]$ in the sense that $\omega_{g,1}, \ldots, \omega_{g,k}$ are holomorphic everywhere on X, including the point ∞.
3. The coefficients c^n_{ij} arising from the action of T_n on $H^1_{\mathrm{dR}}(X)$ with respect to the basis $\omega_{g,1}, \ldots, \omega_{g,2k}$.
4. Meromorphic differentials $\alpha_{\omega_{g,i}, \omega_{g,j}}$ on X, regular on Y, such that $\omega_{g,i} F_{\eta_{g,j}} - \alpha_{\omega_{g,i}, \eta_{g,j}}$ has at worst a logarithmic pole at (any point lying over) ∞ for $1 \leq i \leq k$.

These data must be "known" in a sufficiently concrete form to evaluate the iterated integrals occuring in (3.3.2). It is also desirable to know

5. the denominator $d_{g,n}$ of the projector onto the g-isotypic component of the cohomology of X.

This last item will allow for the computation of a point in $E(\mathbf{Q})$, as opposed to one in $E(\mathbf{Q}) \otimes \mathbf{Q}$. This section is devoted to methods of computing these five ingredients.

4.1. Evaluating iterated integrals.
Let $J = \int \omega \cdot \eta - \alpha_{\omega, \eta}$ be a homotopy-invariant iterated integral of length ≤ 2 on Y, expressed in terms of differentials of the second kind on X, regular on Y. We seek to compute the right-hand side of formula (3.3.2), which is a $\mathbf{Q}$-linear combination of $J(\gamma)$ for various choices of J and (the homotopy class of) a path $\gamma \in \pi_1(Y; o)$. As remarked earlier, (3.3.2) actually depends only on the homology class γ_0 of γ. This homology class belongs to $H_1(Y^{\mathrm{an}}, \mathbf{Z}) = H_1(X^{\mathrm{an}}, \mathbf{Z})$, which is the abelianization of the quotient $\pi_1(X^{\mathrm{an}}, o) = \bar{\Gamma}_0(N)$ of $\Gamma_0(N)$ by the smallest normal subgroup containing the elliptic and parabolic elements.

To evaluate $J(\gamma_0)$ for $\gamma_0 \in H_1(Y^{\mathrm{an}}, \mathbf{Z})$, choose the basepoint o away from the set S of elliptic points on $Y_0(N)^{\mathrm{an}} \subset Y^{\mathrm{an}}$ and lift γ_0 arbitrarily to a path $\tilde{\gamma} \in \pi_1(Y_0(N)^{\mathrm{an}} \setminus S, o)$. For each elliptic point $x \in S$, let $e_x = |\mathrm{Stab}_{\Gamma_0(N)}(x)/\{\pm 1\}|$ denote the index of x (which is either 2 or 3) and γ_x be a sufficiently small counterclockwise loop around x. Writing H for the normal subgroup of $\pi_1(Y_0(N)^{\mathrm{an}} \setminus S, o)$ generated by $\{\gamma_x^{e_x}, x \in S\}$, there is a natural isomorphism $\Gamma_0(N) \simeq \pi_1(Y_0(N)^{\mathrm{an}} - S; o)/H$.

We may regard then $\tilde{\gamma}$ as an element of $\Gamma_0(N)$; this causes no ambiguity because H lies in the kernel of the natural projection $H_1(Y_0(N)^{\mathrm{an}} - S, \mathbb{Z}) \to H_1(Y_0(N)^{\mathrm{an}}, \mathbb{Z}) \to H_1(Y^{\mathrm{an}}, \mathbb{Z})$. The path $\tilde{\gamma}$ can then also be viewed as a path in $\mathfrak{H}$ from τ_0 to $\gamma\tau_0$, where $\gamma \in \Gamma_0(N)$ is a lift of γ_0.

Lemma 4.1.1. *Suppose γ_0 is Poincaré-dual to ρ. As an element of $\mathbf{C}/\Lambda_\rho$, we have*

$$J(\gamma_0) = \int_{\tau_0}^{\gamma\tau_0} \omega F_\eta - \alpha_{\omega, \eta}$$

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[9]FIXME: Is this right?

where we conflate 1-forms on X with their pullbacks to $\mathfrak{H}^* = \mathfrak{H} \cup \{\infty\}$. Moreover, F_η has Laurent expansion about $\infty \in \mathfrak{h}^*$ given by formally integrating the Laurent expansion of η about the cusp $\infty \in X$.

Proof. This follows from the preceding discussion, using the definition of iterated integrals and the homotopy invariance of J. □

Given any differential form λ of the second kind on X, and any $\gamma \in \Gamma_0(N)$, let

$$I(\lambda; \gamma) := \int_{\tau_0}^{\gamma \tau_0} \lambda.$$

(As above, in the righthand side of this expression λ is conflated with its pullback to $\mathfrak{H}^*$.) By the residue formula, this expression is independent of the choice of path on the upper half-plane $\mathfrak{H}$ from τ_0 to $\gamma \tau_0$. The $\Gamma_0(N)$-invariance of λ also shows that this expression is independent of the choice of base point $\tau_0 \in \mathfrak{H}$, which justifies suppressing τ_0 from the notation.

If λ instead denotes a differential of the second kind on $\tilde{X}$ then the integral above still makes sense but depends on both the basepoint o and the chosen lift of o to $\tau_0 \in \mathfrak{H}$. We will primarily be interested in evaluating such integrals in the context of (3.3.2), for which the choice of basepoint is ultimately irrelevant. (This is because the Poincaré dual of the homology class of γ, is orthogonal to the 1-forms in the iterated integral giving rise to the path integral we seek to evaluate; *cf.* Lemma 3.4.1.) However, as we are about to see, for the purposes of algorithmic efficiency it is necessary to break up the path of integration into pieces that can be computed relatively quickly. The integrals over these pieces may no longer be basepoint-independent: when we express γ as a product of computationally-amenable elements $\gamma^{(j)} \in \Gamma_0(N)$, the corresponding homology classes may no longer lie in (the Poincaré dual of) the orthogonal complement of $H^1_{\mathrm{dR}}(X/\mathbf{Q})[g]$. Thus for a general meromorphic 1-form λ on $\tilde{X}$ and a general $\gamma \in \Gamma_0(N)$, we adopt the notation

$$I_{\tau_0}(\lambda; \gamma) = \int_{\tau_0}^{\gamma \tau_0} \lambda$$

to emphase the dependence on the choice of basepoint.

By meromorphicity, for λ as above (defined on either X or $\tilde{X}$) the integral $I(\lambda; \gamma)$ or $I_{\tau_0}(\lambda; \gamma)$ can be computed by integrating term-wise a Laurent expansion for λ using the fundamental theorem of calculus. Thus, in practice, one computes the Laurent expansion for the primitive F_λ about $\infty \in X$ (or a choice of $\tilde{\infty} \in \tilde{X}$ lying over ∞), regarded as function given by a convergent power series in $q = e^{2\pi i \tau}$ on $\mathfrak{h}$, and evaluates it at τ_0 and $\tau_0' = \gamma \tau_0$. The larger the imaginary parts of τ_0 and τ_0' are, the faster this series converges and the fewer coefficients of the Laurent series of λ are necessary to approximate $I(\lambda; \gamma)$ or $I_{\tau_0}(\lambda; \gamma)$ to a give degree of accuracy. Writing $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right)$, it is well-known that the best compromise between $\mathrm{Im}(\tau_0)$ and $\mathrm{Im}(\tau_0')$ is achieved when we choose $\tau_0 = -\frac{d}{c} + \frac{1}{|c|} i$ (*cf.*, for example, [Cr, p. 35]). This *optimal basepoint for* γ will be denoted τ_γ^*.

With this remark in mind, we take the following approach to computing $J(\gamma_0)$ as in the lemma above. First compute Laurent expansions for the differentials $\omega, \eta, \alpha_{\omega, \eta}$. Then find a "good" expression for the homology class $\gamma_0 \in H_1(X^{\mathrm{an}}, \mathbf{C})$, writing it as a $\mathbf{C}$-linear combination of classes $\gamma_0^{(j)} \in H_1(X^{\mathrm{an}}, \mathbf{Z})$ that lift to elements $\gamma^{(j)} \in \Gamma_0(N)$ with small lower-left entries cN. Finally, calculate approximations to the integrals $I_{\tau_0}(\omega F_\eta; \gamma^{(j)})$ and $I(\alpha_{\omega, \eta}; \gamma^{(j)})$. The appropriate linear combination of these integrals is an (approximate) representative for the coset $J(\gamma_0) \in \mathbf{C}/\Lambda_\rho$.)

To calculate $I(\alpha_{\omega, \eta}; \gamma^{(j)})$, one is free to change the basepoint from τ_0 to the optimal basepoint $\tau_j^* := \tau_{\gamma^{(j)}}^*$ for $\gamma^{(j)}$, since $\alpha_{\omega, \eta}$ is defined on X and not only on $\tilde{X}$. The same is *not* true for ωF_η. To evaluate $I_{\tau_0}(\omega F_\eta; \gamma^{(j)})$ we appeal to the following lemma.

Lemma 4.1.2. $I_{\tau_0}(\omega F_\eta; \gamma^{(j)}) = I_{\tau_j^*}(\omega F_\eta; \gamma^{(j)}) - I(\eta; \gamma^{(j)}) \int_{\tau_0}^{\tau_j^*} \omega.$

Observe that every term on the righthand side can be computed using the fundamental theorem of calculus, evaluating power series only at the points τ_0 and τ_j^*. In particular, taking $\tau_0 = i/N$, each such evaluation converges at least as fast as an evaluation at τ_j^*, so this formula for the integral is "optimally efficient".

Proof of Lemma 4.1.2. Since $\lambda = \omega F_\eta$ is a holomorphic 1-form on $\mathfrak{H}$, its integral along a closed contour vanishes. Thus

$$I_{\tau_0}(\lambda; \gamma^{(j)}) = I_{\tau_j^*}(\lambda; \gamma^{(j)}) + \int_{\tau_0}^{\tau_j^*} \lambda - \int_{\gamma^{(j)}\tau_0}^{\gamma^{(j)}\tau_j^*} \lambda.$$

To evaluate the second term on the righthand side, we observe that ω comes from a 1-form on X, so it is $\Gamma_0(N)$-invariant; it thus pulls back to itself along the fractional linear transformation defined by $\gamma^{(j)}$. On the other hand,

$$I(\eta; \gamma^{(j)}) = \int_{\gamma^{(j)}} \eta = F_\eta(\gamma^{(j)}\tau) - F_\eta(\tau), \qquad \text{for all } \tau \in \mathfrak{H}.$$

Hence $(\gamma^{(j)})^* F_\eta = F_\eta + I(\eta; \gamma^{(j)})$. So $(\gamma^{(j)})^* \lambda = \lambda + I(\eta; \gamma^{(j)})\omega$, and we find

$$\int_{\gamma^{(j)}\tau_0}^{\gamma^{(j)}\tau_j^*} \lambda = \int_{\tau_0}^{\tau_j^*} (\gamma^{(j)})^* \lambda = \int_{\tau_0}^{\tau_j^*} \lambda + I(\eta; \gamma^{(j)}) \int_{\tau_0}^{\tau_j^*} \omega,$$

which yields the lemma. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Remark 4.1.3. We warn the reader that possibly $I_{\tau_0}(\omega F_\eta; \gamma^{(j)}) \neq \int_{\gamma^{(j)}} \omega \cdot \eta$ (regarding $\gamma^{(j)}$ as an element of $\pi_1(Y^{\mathrm{an}}; o)$). Indeed, the iterated integral $\omega \cdot \eta$ need not even be homotopy invariant (!) so $\int_{\gamma^{(j)}} \omega \cdot \eta$ is ill-defined. In particular, one *cannot* relate $I_{\tau_0}(\omega F_\eta; \gamma^{(j)})$ to $I_{\tau_j^*}(\omega F_\eta; \gamma^{(j)})$ using the change-of-basepoint formula (3.4.2) for iterated integrals.

To efficiently evaluate the integrals in (3.3.2) using the method just explained, it is therefore necessary to know:

    a. Laurent expansions about ∞ for a symplectic basis $\omega_{g,1}, \ldots, \omega_{g,2k}$ of $H^1_{\mathrm{dR}}(X/\mathbf{Q})[g]$ and the forms $\alpha_{\omega_{g,i},\omega_{g,j}}$; and,

    b. the homology class γ_f as a $\mathbf{C}$-linear combinationation of class $\gamma_0^{(j)}$ whose lifts to $\Gamma_0(N)$ have small lower-left entries cN.

In the rest of this section we turn to the task of computing these data.

4.2. Calculating a symplectic basis for $H^1_{\mathrm{dR}}(X/\mathbf{Q})[g]$. The calculation of a basis for the de Rham cohomology can be carried out by first writing down a modular function u — that is, a rational function on $X = X_0(N)$ — which is regular away from ∞. Such a function exists by Riemann-Roch and a q-expansion for one such function can sometimes be computed explicitly using the Dedekind eta-function, as explained in the next subsection.

Using a modular symbol algorithm, one can compute q-expansions for a $\mathbf{C}$-basis of $S_2(\Gamma_0(N))$ consisting of cusp forms with rational Fourier coefficients; *cf.* [Stn], for example. Write $\omega_1, \ldots, \omega_t$ for the corresponding holomorphic 1-forms on X, where for convenience we denote by $t = \dim_{\mathbf{C}} S_2(\Gamma_0(N))$ the genus of X.

Define $\eta_i = u\omega_i$, which is a differential of the second kind by the residue theorem, and let $\mathcal{B} = \{\omega_1, ..., \omega_t, \eta_1, ..., \eta_t\} \subset H^1_{\mathrm{dR}}(X/\mathbf{Q})$ be the corresponding set of cohomology classes. A simple application of Riemann-Roch shows the following.

Lemma 4.2.1. *The set $\mathcal{B}$ is basis for $H^1_{\mathrm{dR}}(X/\mathbf{Q})$ whenever ∞ is not a Weierstrass point on X and u has a pole of order $t + 1$ (i.e., the smallest possible) at ∞.*

Proof. Since ∞ is not a Weierstrass point on X, we may assume that $\mathrm{ord}_\infty(\omega_i) = i-1$, and thus $\mathrm{ord}_\infty(\eta_i) = i-t-2$. For any differential of the second kind ω', we can find a linear combination of $\eta_1, \ldots, \eta_t$ and dh for an appropriate rational function h having the same principal part as ω'. Thus the difference is holomorphic, and lies in the span of $\{\omega_1, \ldots, \omega_t\}$.   □

Remark 4.2.2. By a result of Ogg [O], the cusp ∞ is not a Weierstrass point when the level N is prime, or more generally when $N = pM$ for prime p and an integer $M \geq 1$ such that $X_0(M)$ has genus zero and $p \nmid M$. Even if u has a pole of order $> g(X) + 1$, the set $\mathcal{B}$ may still be a basis of $\mathrm{H}^1_{\mathrm{dR}}(X/\mathbf{Q})$. This can be checked by computing the matrix for the Poincare pairing, and in every example we have computed this is the case.

When ∞ is a Weierstrass point, there is a rational function with a single pole at ∞ of order $\leq g(X)$. When u is taken to be such a function, then the set $\mathcal{B}$ will never be a basis. Indeed, since ∞ is a Weierstrass point, there exists a holomorphic differential form ω with order of vanishing $\geq g(X)$ at ∞. Then $u\omega$ is still holomorphic, and thus lies in the span of $\{\omega_1, \ldots, \omega_t\}$. But $u\omega$ also is in the span of $\{\eta_1, \ldots, \eta_t\}$ by definition of the η_i, giving rise to a linear dependence relation. Hence, in order for $\mathcal{B}$ to be a basis, it is necessary for u to have a pole at ∞ of order greater than the order of vanishing at ∞ of any holomorphic differential.

Given one basis $\mathcal{B}$ for $\mathrm{H}^1_{\mathrm{dR}}(X/\mathbf{Q})$ — for example, one computed as above — it is then a matter of linear algebra to produce a better basis that is adapted to the action of the Hecke algebra. Note that the usual formula for the action of the Hecke algebra $\mathbf{T}_0$ on holomorphic modular forms in terms of q-expansions extends to *weakly holomorphic modular forms* (i.e., meromorphic 1-forms on X with possible poles only at the cusps), and preserves the subspace of differentials regular on Y. In particular one can compute the action of $\mathbf{T}_0$ on any 1-form representing an element of $\mathrm{H}^1_{\mathrm{dR}}(X/\mathbf{Q})$. Using q-series for the elements of the basis $\mathcal{B}$, we can thus write down the matrix $[T_p] \in \mathrm{Mat}_{2t \times 2t}(\mathbf{Q})$ that describes the action of any $T_p \in \mathbf{T}_0$. After identifying $\mathrm{H}^1_{\mathrm{dR}}(X/\mathbf{Q}) \approx \mathbf{Q}^{2t}$ via the basis $\mathcal{B}$, by finding the eigenspaces of finitely many such matrices one can write down $\mathbf{Q}$-bases for each isotypic component of $\mathrm{H}^1_{\mathrm{dR}}(X/\mathbf{Q})$. As is shown in [Stn], the Hecke algebra $\mathbf{T}_0$ is generated as a $\mathbf{Z}$-module by T_i for $1 \leq i \leq \frac{m}{6} - \frac{m-1}{N}$, where $m = [\Gamma(1) : \Gamma_0(N)]$. This gives an upper bound on the number of matrices needed, although in practice considerably fewer are necessary. Using these it is simple linear algebra to produce the desired symplectic basis $\omega_{g,1}, \ldots, \omega_{g,2k}$ for each isotypic component $\mathrm{H}^1_{\mathrm{dR}}(X/\mathbf{Q})[g]$.

Remark 4.2.3. FIXME: Add a remark about the alternative method which avoids computing Hecke actions on differentials explicitly. (M-symbols, periods, blah blah) Note from Mike: I have been unable to implement this method without ending up with complex matrices, rather than ones with rational entries. Sage cannot compute eigenspaces of complex matrices, so I'm not sure if this method can produce explicit bases of isotypic components. The computation can be done without explicitly finding isotypic bases, but it is far less efficient.

4.3. Modular units and η-products. The preceding discussion raises the question of how to compute the q-expansion about ∞ of a rational function u used to write down an initial choice of basis $\mathcal{B}$ for $\mathrm{H}^1_{\mathrm{dR}}(X/\mathbf{Q})$.

Recall that a *modular unit* (for $\Gamma_0(N)$) is a modular function $u \in \mathbf{C}(X)^\times$ whose associated divisor is supported on the cusps of $X = X_0(N)$. Denote by U the multiplicative group of modular units.

Definition 4.3.1. The *eta group* U_η is the group of rational functions $u \in \mathbf{Q}(X)$ of the form

$$u(q) = \prod_{0 < d \mid N} \eta(q^d)^{r_d},$$

where $\eta(q) = q^{1/24} \prod_{n>0}(1 - q^n)$ is the classical eta function, and $\{r_d\}_{d \mid N}$ is a collection of integers satisfying the following conditions.

i. $\sum_{d|N} r_d = 0$,

ii. $\prod_{d|N} d^{r_d} \in \mathbf{Q}^\times$ is a square,

iii. $(n_d) := A_N \cdot (r_d)$ is a vector (indexed by divisors d of N) of integers divisible by 24, where A_N is the $\sigma(N) \times \sigma(N)$-matrix whose entry indexed by (d, d') is $\frac{N \cdot (d,d')^2}{dd'(d',N/d')}$.

Work of Newman and Ligozat shows that such functions are indeed modular units on X; that is, $U_\eta \subset U$. In fact more is true:

Proposition 4.3.2. $\mathbf{Q} \otimes U_\eta = \mathbf{Q} \otimes U$.

Proof. It easy to see that the set $\{\frac{a}{d} : d \mid N, a \in (\mathbf{Z}/(d, N/d)\mathbf{Z})^\times\} \subset \mathbf{P}^1(\mathbf{Q})$ is a complete set of representatives of the cusps of X. The subspace $\mathbf{Q} \otimes U_\eta \subset \mathbf{Q} \otimes U$ coincides with $\mathbf{Q} \otimes U'$, where $U' \subset U$ consists of modular units that have the same valuation at any two cusps a/d, a'/d with the same denominator; *cf.* [G, Prop. 2]. This implies the proposition in light of the next lemma, since an element $u \in U \subset \mathbf{Q}(X)$ has the same valuation at any two Galois-conjugate cusps. $\square$

Lemma 4.3.3. *Let $d|N$. Then the cusp $1/d$ is rational if and only if $(d, N/d) = 1$. More generally, the Galois orbit of the cusp $1/d$ is the set of cusps a/d with a relatively prime to $(d, N/d)$.*

Proof of the lemma. We prove the first statement using the results of [Stv, §1.3]. Namely, it is known that the cusps of X are rational over $\mathbf{Q}(\zeta_N)$, and the Galois action can be described explicitly as follows [Stv, Thm. 1.3.1]: if τ is the automorphism of $\mathbf{Q}(\zeta_N)$ that sends $\zeta_N \mapsto \zeta_N^n$ and $n' \in \mathbf{Z}$ is chosen so that $nn' \equiv 1 \pmod{N}$ then τ sends the cusp x/y to $x/n'y$. In particular, it follows straightforwardly that $[1/d]^\tau = [n'/d]$. Thus it suffices to prove that the cusps n'/d and $1/d$ coincide for all n' relatively prime to N, if and only if $(d, N/d) = 1$. This fact can be shown by an elementary argument using the conditions for the integer matrix sending $1/d$ to n'/d to be in $\Gamma_0(N)$. The second statement is proved similarly[10]. $\square$

By the Riemann-Roch theorem, there exist nonconstant rational functions on X that are regular away from ∞. The proposition implies that an integer power of such a function belongs to the subgroup $U_\eta \subset U$, which yields the following.

Corollary 4.3.4. *There exists an eta product $u \in U_\eta$ that is regular away from ∞.* $\square$

It is thus possible to compute the rational function u required in the compation of a basis for $H^1_{\mathrm{dR}}(X)$ as an eta product. A practical approach to finding the vector $(r_d)_{d|N}$ giving rise to the u we seek is to apply a mixed-integer linear programming algorithm: one minimizes the pole order $-n_N$ of u at ∞ subject to the criteria of Newman-Ligozat in Definition 4.3.1 and the condition that the orders n_d of u at other cusps are non-negative.

Remark 4.3.5. To determine the complexity of the algorithm described in this paper (see §4.8), it is necessary to bound effectively (as a function of N) the order of the pole at ∞ of the eta quotient u in Corollary 4.3.4. This can be done by examining the proof of Corollary 4.3.4. By the Riemann-Roch theorem, there is a nonconstant rational function w on X that is regular on Y and has a pole of order $\leq t = \mathrm{genus}(X)$ at ∞. From the formula for $\mathrm{genus}(X)$ as a function on N, one can thus extract the bound $-\mathrm{ord}_\infty(w) = O(N \log\log N)$; *cf.* for example [CWZ]. (We adopt the convention that unless decorated with a subscript, an expression $O(-)$ denotes a bound with an absolute implied constant.) An examination of the proof of [G, Prop. 2], which was invoked to show Proposition 4.3.2, actually gives the more precise result that w^μ belongs to U_η for an integer $\mu = O(\det A_N)$. Combining this with the explicit formula [G, Prop. 1] for $\det A_N$, one deduces the estimate $-\mathrm{ord}_\infty(u) = O(N^{2\sigma_0(N)+2})$, where $\sigma_0(N)$ denotes the number of positive divisors of N.

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[10]FIXME: double check this!

4.4. Computing the Poincaré dual γ_f of ω_f. Assume that $\{\gamma^{(j)}\}$ is a collection of elements of $\Gamma_0(N)$ with small lower-left entries cN, whose homology classes $\gamma_0^{(j)}$ generate $H_1(X^{\mathrm{an}}, \mathbf{Z})$. By a brute-force search it is straightforward to find such elements $\gamma^{(j)}$ in practice. (For small N, often one need take c no greater than 2 or 3.)

For any $m \in H_1(X^{\mathrm{an}}, \mathbf{C})$, write η_m for the Poincaré dual of c; conversely, for any differential η of the second kind on X, let $m_\eta \in H_1(X^{\mathrm{an}}, \mathbf{C})$ denote the Poincaré dual of its cohomology class. We normalize the Poincaré duality isomorphism so that it is characterized by the property

$$(4.4.1) \qquad \langle \eta_m, \eta \rangle = \int_m \eta.$$

The vector space $H_1(X^{\mathrm{an}}, \mathbf{C})$ is also equipped an intersection product, which is related to the Poincaré pairing by the formula

$$(4.4.2) \qquad m \cdot m_\eta = \frac{1}{2\pi i} \langle \eta_m, \eta \rangle.$$

The homology of X also admits a natural action of the Hecke algebra, compatible with the action on cohomology via Poincaré duality. For any normalized eigenform $f \in S_2(\Gamma_0(N))$ and any $m \in H_1(X^{\mathrm{an}}, \mathbf{C})$, write $m^f \in H_1(X^{\mathrm{an}}, \mathbf{C})[f]$ for the projection of m onto the f-isotypic component of homology. Similarly, for $\eta \in H^1_{\mathrm{dR}}(X/\mathbf{Q})$ write η^f for its projection onto the f-isotypic component.

We can assume that via the method described above a symplectic basis

$$\mathcal{S} = \{\omega_{f,1}, \ldots, \omega_{f,\ell}, \eta_{f,1}, \ldots, \eta_{f,\ell}\}$$

for $H^1_{\mathrm{dR}}(X/\mathbf{Q})[f]$ has already been computed, where $\langle \omega_{f,i}, \eta_{f,j} \rangle = \delta_{i,j}$.

Lemma 4.4.1. *Fix γ_1, $\gamma_2 \in \Gamma_0(N)$ and let m_1, $m_2 \in H_1(X^{\mathrm{an}}, \mathbf{Z})$ denote the corresponding homology classes on X. For any normalized eigenform $f \in S_2(\Gamma_0(N))$, we have*

$$m_1^f \cdot m_2^f = \frac{1}{2\pi i} \sum_{i=1}^{\ell} I(\omega_{f,i}; m_1) I(\eta_{f,i}; m_2) - I(\omega_{f,i}; m_2) I(\eta_{f,i}; m_1),$$

where $\omega_f = 2\pi i f(z) \mathrm{d}z$ is the 1-form corresponding to f.

Proof. Let $\eta_k = \eta_{m_k}$ and write $\eta_k^f = \sum c_i^{(k)} \omega_{f,i} + \sum d_i^{(k)} \eta_{f,i}$. Then we compute $m_1^f \cdot m_2^f = \frac{1}{2\pi i} \langle \eta_1^f, \eta_2^f \rangle = \sum_i \frac{1}{2\pi i} (c_i^{(1)} d_i^{(2)} - c_i^{(2)} d_i^{(1)}) = \frac{1}{2\pi i} \sum_i (I(\omega_{f,i}; \gamma_1) I(\eta_{f,i}, \gamma_2) - I(\eta_{f,i}; \gamma_1) I(\omega_{f,i}; \gamma_2))$. $\square$

Using (4.4.1), (4.4.2), and Lemma 4.4.1, we can compute the Poincaré dual γ_f of ω_f. Let $m_1, \ldots, m_{2\ell}$ be modular symbols giving rise to a basis of $H_1(X^{\mathrm{an}}, \mathbf{Z})[f]$, which can be computed using a modular symbols algorithm (*cf.* [Stn]). In particular, if f is new, then $\ell = 1$. Write A for the matrix $(m_i \cdot m_j)_{1 \leq i,j \leq 2\ell}$, which can be computed using Lemma 4.4.1, and let v be the column vector $(m_i \cdot m_{\omega_f})_{i=1}^{2\ell}$, which can be computed using (4.4.2) in conjunction with (4.4.1). Then the vector $A^{-1}v$ gives the coefficients expressing m_{ω_f} as a linear combination of $m_1, \ldots, m_{2\ell}$. These coefficients can then easily be converted to an expression for γ_f as a linear combination of $\{\gamma_0^{(j)}\}$.

4.5. Computing the adjustments $\int_{\gamma_f} \alpha$. Write the homology class γ_f Poincaré dual to ω_f as

$$\gamma_f = \sum \beta_j \gamma_0^{(j)}$$

for $\beta_j \in \mathbf{C}$ and homology classes $\gamma_0^{(j)}$ whose lifts $\gamma^{(j)}$ to $\Gamma_0(N)$ have small lower-left entries cN. Let ω and η be differentials of the second kind, at least one of which is regular at ∞. Using the methods described so far, we are already able to compute

$$z_{\omega,\eta} := \sum_j \beta_j \int_{\tau_0}^{\gamma^{(j)}\tau_0} \omega F_\eta.$$

We stress that the value of $z_{\omega,\eta}$ depends on τ_0 and the choices we made in representing γ_f. It is simply one part of the iterated integral $J_{\omega,\eta}(\gamma_f) = \int_{\gamma_f} \omega \cdot \eta - \alpha_{\omega,\eta}$, which is independent of these choices. In this section, we describe a method for computing $J_{\omega,\eta}(\gamma_f) - z_{\omega,\eta}$. This amounts to computing the q-expansion of $\alpha_{\omega,\eta}$, for then we have

$$J_{\omega,\eta}(\gamma_f) - z_{\omega,\eta} = -\sum_j \beta_j \int_{\tau_0}^{\gamma^{(j)}\tau_0} \alpha_{\omega,\eta}.$$

Recall that the defining property of $\alpha_{\omega,\eta}$ is that its principal part at ∞ agrees with that of ωF_η on $\tilde{X}$, modulo $\mathrm{d}q/q$; i.e., their difference has at worst logarithmic poles. However, note that since $\int_{\gamma_f} \lambda = 0$ for exact 1-forms λ, we may replace $\alpha_{\omega,\eta}$ by any cohomologous 1-form. The cohomology class of $\alpha_{\omega,\eta}$ is determined by the data $\langle \lambda_i, \alpha_{\omega,\eta} \rangle$, where $\lambda_1, \ldots, \lambda_{2t}$ (for t the genus of X) form a basis of $H^1_{\mathrm{dR}}(X/\mathbf{Q})$, so it suffices to compute these values of the Poincaré pairing.

We can choose $\lambda_1, \ldots, \lambda_t$ to be holomorphic. In this case, we can compute

$$\langle \lambda_i, \alpha_{\omega,\eta} \rangle = \mathrm{res}_\infty(F_{\lambda_i} \cdot \alpha_{\omega,\eta}) = \mathrm{res}_{\tilde{\infty}}(F_{\lambda_i} \cdot F_\eta \cdot \omega),$$

where the second equality holds because $\mathrm{res}_\infty(F_{\lambda_i} \cdot \alpha_{\omega,\eta})$ depends only on

$$\mathrm{pp}_\infty(\alpha_{\omega,\eta}) \mod \frac{\mathrm{d}q}{q} = \mathrm{pp}_{\tilde{\infty}}(\omega F_\eta).$$

Lemma 4.5.1. *Let $\lambda_1, \ldots, \lambda_t \in H^0(X, \Omega^1_{X/\mathbf{Q}})$ be a basis of regular 1-forms on X. Then $\alpha \in H^1_{\mathrm{dR}}(X/\mathbf{Q})$ lies in the subspace $H^0(X, \Omega^1_{X/\mathbf{Q}})$ if and only if $\langle \lambda_i, \alpha \rangle = 0$ for all $1 \leq i \leq t$.*

Proof. The subspace $H^0(X, \Omega^1_{X/\mathbf{Q}}) \subseteq H^1_{\mathrm{dR}}(X)$ is isotropic for the Poincaré pairing because the pairing can be computed using residues. For dimension reasons, it is maximal isotropic, and the lemma follows. $\square$

By the lemma, if $\langle \lambda_i, \alpha \rangle = \langle \lambda_i, \alpha' \rangle$ for $i = 1, \ldots, t$, then $\alpha - \alpha'$ is cohomologous to a regular 1-form. Since $\alpha_{\omega,\eta}$ is only well-defined modulo $H^0(X, \Omega^1_{X/\mathbf{Q}})$, it follows that we can choose $\langle \lambda_i, \alpha_{\omega,\eta} \rangle$ for $i = t+1, \ldots, 2t$ arbitrarily. For convenience, we choose $\langle \lambda_i, \alpha_{\omega,\eta} \rangle = 0$ for $i = t+1, \ldots, 2t$. Define the matrix $B = (\langle \lambda_i, \lambda_j \rangle)_{i,j=1}^{2t}$ and the vector

$$w = (\langle \lambda_i, \alpha_{\omega,\eta} \rangle)_{i=1}^{2t} = (\mathrm{res}_{\tilde{\infty}}(F_{\lambda_1} F_\eta \omega), \cdots, \mathrm{res}_{\tilde{\infty}}(F_{\lambda_t} F_\eta \omega), \overbrace{0, \ldots, 0}^{t}).$$

It then follows by elementary linear algebra that the vector $B^{-1}w$ yields the coefficients of an expression for $\alpha_{\omega,\eta}$ as a linear combination of $\lambda_1, \ldots, \lambda_{2t}$.

4.6. Computing the coefficients c_{ij}. Using the methods described so far, we may compute all the integrals $\int_{\gamma_f}(\omega \cdot \eta - \alpha_{\omega,\eta})$ occuring in (3.3.2). The last input needed to evaluate (3.3.2) is the set of coefficients c_{ij}^n appearing in that formula. Using Lemma 3.3.1, these coefficients can be computed using the matrix A_n which expresses the right action of T_n on $H^1_{\mathrm{dR}}(X/\mathbf{Q})[g]$ with respect to the basis $\omega_{g,1}, \ldots, \omega_{g,2k}$, and the matrix $B = (\langle \omega_{g,i}, \omega_{g,j} \rangle)_{1 \leq i,j \leq 2k}$, which by our choice of basis is the standard symplectic matrix. In section 4.2, we gave a method for computing A_n using the action of T_n on q-expansions. However, this only works if $\gcd(n, N) =$

1^{11}, so we must resort to other methods. We exploit the fact that the action of T_n on $H_1(X^{\mathrm{an}}, \mathbf{C})[g]$ is readily computable using modular symbols; see [Stn] for details. Recall that we have a Hecke-equivariant duality

$$H^1_{\mathrm{dR}}(X^{\mathrm{an}}, \mathbf{C})[g] \times H_1(X^{\mathrm{an}}, \mathbf{C})[g] \to \mathbf{C}$$

given by the integration pairing $\langle \omega, \lambda \rangle = \int_\gamma \lambda$; here the Hecke-equivariance means that $\langle T_n \omega, \alpha \rangle = \langle \omega, T_n \alpha \rangle$. Using modular symbols and the techniques of [Stn] one can compute the matrix B_n of T_n acting on $H_1(X^{\mathrm{an}}, \mathbf{C})[g]$ on the left, with respect to a basis $m_1, \ldots, m_{2k}$. Write $Q = (\langle \omega_i, m_j \rangle)_{i,j=1}^{2k}$, which can be computed efficiently via the method explained in §4.1 (using an appropriate basis $\{m_j\}$ derived from the generators $\gamma_0^{(j)}$ for $H_1(X^{\mathrm{an}}, \mathbf{Z})$ discussed above). Then it is straightforward linear algebra to show that $A_n = QB_nQ^{-1}$.

4.7. Computing the denominator $d_{g,n}$. The final ingredient to be computed is the denominator $d_{g,n}$, or the smallest positive integer such that $d_{g,n}T_{g,n} \in \mathbf{T}$. This can be accomplished by computing a $\mathbf{Z}$-basis for the ($\mathbf{Z}$-finite free) Hecke algebra $\mathbf{T}$ as a subring of $M_{2t}(\mathbf{Q})$, where t is the genus of $X_0(N)$, by identifying $\mathbf{T}$ with an algebra of endomorphisms of the $(2t)$-dimensional $\mathbf{Q}$-vector space of cuspidal modular symbols of weight 2 and level N. As $\mathbf{T}$ is generated as an abelian group by T_i for $1 \le i \le \frac{m}{6} - \frac{m-1}{N}$, where $m = [\Gamma(1) : \Gamma_0(N)]$, this is a finite computation. For more details on modular symbols and generating the Hecke algebra, see [Stn]. Once $\mathbf{T}$ has been computed it is a simple matter to find the matrix representation of $T_{g,n}$ and compute the smallest $d_{g,n}$ such that $d_{g,n}T_{g,n} \in \mathbf{T}$.

4.8. Remarks on complexity. The complexity of the computations we have described is primarily determined by the *number n_D* of Fourier coefficients required to compute $J_{\omega,\alpha}(\gamma_f)$ to a given number D of digits of accuracy. In this subsection we sketch a method for obtaining a bound on n_D in terms of N.

4.8.1. Write the Fourier expansion of u as

$$u(\tau) = \sum_{n \ge -n_0} c_n q^n, \qquad q = e^{2\pi i \tau}, \tau \in \mathfrak{H}.$$

Let the principal part of u at ∞ be

$$\mathrm{pp}_\infty(u) = \sum_{-n_0 \le -m \le 0} \frac{d_m}{m} q^{-m}, \qquad d_m = mc_{-m}.$$

In [BO], Bringmann and Ono prove an exact formula for the Fourier coefficients of harmonic Maass forms, of which weakly holomorphic modular functions such as u are examples. To avoid introducing unnecessary notation, we state only the very special case of their result applicable to our situation. We remark that long ago Rademacher used the circle method to prove a similar exact formula for the coefficients of the j-function [R], and a modification his argument would probably yield a simpler and more direct proof of the special case we require. Using [BO, Thm. 1.1], one can express the coefficients $c_n, n > 0$ in terms of the coefficents d_m, the order-1 I-Bessel function $I_1(z)$, and the Kloosterman sum

$$K(-m, n, c) := \sum_{\substack{0 < v < c \\ (v,c)=1}} \exp\left(\frac{2\pi i}{c}(nv + m\bar{v}) \right),$$

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[11]FIXME: I'm still not sure why this doesn't work for primes dividing the level

where $\bar{v}$ denotes the multiplicative inverse of v modulo c. Namely, *loc. cit.* yields the formula

$$(4.8.1) \quad c_n = 2\pi \sum_{-n_0 \le -m \le 0} d_m \sum_{\substack{c>0 \\ c \equiv 0 \pmod N}} (\tfrac{m}{n})^{1/2} \frac{K_0(-m,n,c)}{c} I_1\left(\frac{4\pi\sqrt{|mn|}}{c}\right), \qquad n > 0.$$

By Remark 4.3.5, we have

$$(4.8.2) \qquad\qquad\qquad n_0 = -\mathrm{ord}_\infty(u) \le A_1 N^{2\sigma_0(N)+2},$$

where the constant A_1 is absolute and σ_0 denotes the divisor function. We can trivially bound the numbers d_m as follows. Let $\xi_r(x) = re^{2\pi i x}$ for $0 < r < 1$ and set $y = -\frac{1}{2\pi}\log r > 0$. The Cauchy integral formula applied to the meromorphic function $U(q) = \sum c_n q^n$ of q in the unit disk gives

$$\frac{d_m}{m} = \frac{1}{2\pi i}\int_{\xi_r} \frac{U(q)}{q^{m+1}}\,\mathrm{d}q = e^{2\pi m y}\int_0^1 u(x+iy)e^{-2\pi i n x}\,\mathrm{d}x.$$

Taking $y = 1$, say, we thus have

$$|d_m| \le me^{2\pi m}\int_0^1 |u(x+i)|\,\mathrm{d}x = me^{2\pi i m}\int_0^1 \prod_{d|N} |\eta(dx+id)|^{r_d}\,\mathrm{d}x,$$

where η is the Dedekind eta function and $(r_d)_{d|N}$ is the vector giving rise to u. Recall that $\eta(z)$ is a nonvanishing holomorphic function on $\mathfrak{H}$, and satisfies $|\eta(z+1)| = |\eta(z)|$. Thus there is an absolute upper bound B for $\max(|\eta(z)|, |\eta(z)|^{-1})$ on the strip $\{1 - \epsilon \le \mathrm{Im}(z) \le N + \epsilon\} \subset \mathfrak{H}$. It follows that

$$|d_m| \le me^{2\pi m}B^{\sum |r_d|}.$$

To control this quantity we must bound both B and $\sum |r_d|$ in terms of N.

As regards B, note that since $\eta(z)$ is holomorphic on $\mathfrak{H}^*$, the upper bound for $|\eta(z)|$ on the strip of interest poses no problem; it is a constant independent of N. To find a lower bound for $|\eta(z)|$ on our strip, we can make use of the Euler formula

$$\eta(z) = e^{\pi i z/12}\left(1 + \sum_{m \ge 1} q^{\frac{3m^2-m}{2}} + q^{\frac{3m^2+m}{2}}\right), \qquad q = e^{2\pi i z}.$$

From this it is straightforward to deduce a bound $|\eta(z)|^{-1} = O(e^{\pi N/12})$, for z in the strip in question.

To handle $\sum |r_d|$, we recall some results from numerical analysis. Let A be an invertible $\sigma \times \sigma$ matrix of real numbers. We denote by $||A||_2$ the quantity $\sup_{x \in \mathbf{R}^\sigma, |x|=1} |Ax|$ and by $||A||_{\max}$ the maximum of the absolute values of the matrix entries a_{ij}. A result in numerical linear algebra [M, Thm. 2.1] asserts that when A has *integer* entries, one can control its so-called "condition number" and obtain the estimate

$$||A^{-1}||_2 \le \sigma^{2+\sigma/2}||A||_{\max}^{\sigma-1}.$$

We apply this result to the matrix A_N from Definition 4.3.1 with $\sigma = \sigma_0(N)$. By Newman and Ligozat's result, the vector $\mathbf{r} = (r_d)_{d|N}$ and the vector $\mathbf{n} = (24\mathrm{ord}_{a/d}(u))_{d|N}$ (where a/d is any cusp of $X_0(N)$ of denominator d) satisfy $\mathbf{r} = A_N^{-1}\mathbf{n}$. Moreover since u has a pole of order n_0 at infinity and no other poles, the fact that $\div(u)$ has degree 0 implies $|\mathbf{n}| = O(n_0\sqrt{\sigma_0(N)})$. Finally, the formula for the entry of A_N indexed by (d,d') entails $||A_N||_{\max} \le N^3$. We thus have, using (4.8.2),

$$\sqrt{\sum_{d|N} |r_d|} \le \sqrt{\sum_{d|N} |r_d|^2} = |\mathbf{r}| \le ||A_N^{-1}||_2|\mathbf{n}| \le \sigma_0(N)^{\frac{\sigma_0(N)+5}{2}} n_0 N^{3(\sigma_0(N)-1)} = O(N^{\frac{11\sigma_0(N)+1}{2}}).$$

Consequently, for an absolute constant C we have

$$\begin{aligned}
|d_m| &\ll me^{2\pi m}e^{\pi N\sum|r_d|/12} \leq me^{2\pi m}e^{CN^{11\sigma_0(N)+2}} \\
&\leq n_0 e^{2\pi n_0}e^{CN^{11\sigma_0(N)+2}} \\
&= O(n_0\exp(2\pi N^{2\sigma_0(N)+2}+CN^{11\sigma_0(N)+2}))
\end{aligned}$$

From this we deduce

$$(4.8.3)\qquad\qquad |d_m| = O(n_0\exp(A_0 N^{11\sigma_0(N)+2})),$$

for an absolute constant A_0.

From (4.8.1) and (4.8.3), standard estimates for Kloosterman sums, and asymptotics for the I-Bessel function, one obtains by the method of [BrPh, §§5.1-2] the estimate

$$c_n = O(N^{5/4}n_0^{7/4}n^{-3/4}\exp(A_0 N^{11\sigma_0(N)+2}+N^{-1}4\pi\sqrt{nn_0})).$$

In light of (4.8.2) this yields

$$(4.8.4)\qquad\qquad c_n = O(n^{-3/4}\exp(A_2 N^{11\sigma_0(N)+2}+\tfrac{4\pi\sqrt{A_1}}{N}N^{\sigma_0(N)+1}\sqrt{n})),$$

for absolute constants A_1, A_2.

4.8.2. The coefficients c_n determine the Fourier coefficients of the 1-forms occuring in the formula (3.3.2) for $P_{g,f,n}$. Unfortunately the relationship is indirect, as the construction of the 1-forms $\omega_{g,i}$ and $\alpha_{\omega_{g,i},\omega_{g,j}}$ involves multiplying u against a basis of cusp forms for $\Gamma_0(N)$ and then performing a lot of linear algebra. By Deligne's proof of the Ramanujan-Petersson conjecture, the cusp forms have nth coefficient of size $O_\delta(n^{\frac{1}{2}+\delta})$. It follows that nth Fourier coefficient of an element of the basis $\mathcal{B}$ for $H^1_{\mathrm{dR}}(X/\mathbf{Q})$ computed in §4.2 has size at most

$$O(P(n)\exp(A_2 N^{11\sigma_0(N)+2}+\tfrac{4\pi\sqrt{A_1}}{N}N^{\sigma_0(N)+1}\sqrt{n})),$$

for absolute constants A_1, A_2 and a universal polynomial $P(n)$. To compute the 1-forms $\omega_{g,i}$ and $\alpha_{\omega_{g,i},\omega_{g,j}}$, linear algebra operations are performed on this basis, which spans a vector space of dimension $\mathrm{genus}(X) = O(N\log\log N)$. It thus seems likely that a careful analysis of the linear algebra operations performed would yield a bound

$$(4.8.5)\qquad\qquad O(Q(n)\exp(A_3 N^{11\sigma_0(N)+2}+\tfrac{4\pi\sqrt{A_1}}{N}N^{\sigma_0(N)+1}\sqrt{n})),$$

for the nth Fourier coefficient of *any* 1-form integrated in the course of computing (3.3.2). Here A_1, A_3 are absolute constants and $Q(X)$ is a universal polynomial independent of N.

Suppose λ is such a 1-form (on X or $\tilde{X}$), and consider the problem of integrating the pullback of λ to $\mathfrak{H}^*$ along a path from τ_1 to τ_2. By the method explained in §4.1, we can assume that $\mathrm{im}\tau_1, \mathrm{im}\tau_2 \geq (c_N^* \cdot N)^{-1}$, where $c_N^* \cdot N$ is the largest of the lower-left entries of the elements $\gamma^{(j)} \in \Gamma_0(N)$ introduced at the beginning of §4.4. Recall that these consisted of a collection of elements that span $H_1(X^{\mathrm{an}}, \mathbf{Z})$ and have lower-left entries as small as possible. We do not know how to bound c_N^* in terms of N, although in practice it seems to be very small.

If the Laurent expansion for λ about ∞ (or a lift of ∞ to $\tilde{X}$) is $\lambda = \sum a_\lambda(n)\frac{dq}{q}$, then setting $\tau_j = x_j + iy_j$ for $j = 1,2$ (where $y_j \geq (c_N^* N)^{-1}$), we have

$$\int_{\tau_1}^{\tau_2}\lambda = \sum_{n\gg-\infty}\frac{a_\lambda(n)}{n}(e^{2\pi inx_2}e^{-2\pi ny_2}-e^{2\pi inx_1}e^{-2\pi ny_1}).$$

Our problem is to determine n_D such that the tails of these sums are bounded by the requisite precision, say 10^{-D}. It clearly suffices to take for n_D any m such that

$$S(m) := \sum_{n\geq m}n^{-1}|a_\lambda(n)|e^{-2\pi ny} \leq 10^{-D}.$$

Granting (4.8.5), we have

$$S(m) \ll \sum_{n \geq m} n^{-1} Q(n) \exp(A_3 N^{11\sigma_0(N)+2} + \tfrac{4\pi\sqrt{A_1}}{N} N^{\sigma_0(N)+1}\sqrt{n} - \tfrac{2\pi n}{Nc_N^*}).$$

For a suitable constant A_4, we can assume $n^{-1}Q(n)\exp(\tfrac{4\pi\sqrt{A_1}}{N}N^{\sigma_0(N)+1}\sqrt{n}) \leq \exp(\tfrac{A_4}{N}N^{\sigma_0(N)+1}\sqrt{n})$. Thus

$$\begin{aligned}
S(m) &\ll \exp(A_3 N^{11\sigma_0(N)+2}) \sum_{n \geq m} \exp(\tfrac{1}{Nc_N^*}(A_4 c_N^* N^{\sigma_0(N)+1}\sqrt{n} - 2\pi n)) \\
&\leq \exp(A_3 N^{11\sigma_0(N)+2}) \sum_{n \geq m} e^{-n/Nc_N^*} = \exp(A_3 N^{11\sigma_0(N)+2})\frac{e^{-m/Nc_N^*}}{1 - e^{-1/Nc_N^*}},
\end{aligned}$$

provided m is large enough that $A_4 c_N^* N^{\sigma_0(N)+1}\sqrt{n} - 2\pi n \leq -n$ for $n \geq m$. The latter can be ensured by requiring $m \geq A_5 c_N^{*2} N^{2\sigma_0(N)+2}$ for an absolute constant A_5. It follows with a bit of algebra that provided (4.8.5) holds, we have the following estimate for n_D in terms of D and N:

$$(4.8.6) \qquad n_D = O(\max\{Nc_N^*(D + N^{11\sigma_0(N)+2}), c_N^{*2}N^{2\sigma_0(N)+2}\}),$$

where the implied constant is absolute, as always.

5. NUMERICAL EXAMPLES

5.1. Example: 37a1. Take $N = 37$ in the setup of our algorithm. In this setting, the space of regular differentials on $X = X_0(37)$ is spanned by ω_f and ω_g, which are associated to elliptic curves over $\mathbf{Q}$ (labeled **37a1** and **37b1** in Cremona's database) of ranks 1 and 0, respectively.

By computing the periods attached to ω_f and ω_g, it can be checked that the classes of the matrices

$$\gamma_1 = \begin{pmatrix} 2 & -1 \\ 37 & -18 \end{pmatrix}, \gamma_2 = \begin{pmatrix} 3 & -1 \\ 37 & -12 \end{pmatrix}, \gamma_3 = \begin{pmatrix} 5 & 2 \\ 37 & 15 \end{pmatrix}, \gamma_4 = \begin{pmatrix} 14 & 3 \\ 37 & 8 \end{pmatrix}$$

generate the rational homology of X. These are a "nice" basis for the homology in the sense of the first paragraph of §4.4; that is, the lower left entries are exactly 37 (rather than $37c$ for $|c| > 1$), so the integral $\int_\tau^{\gamma_i \tau} \lambda$ can be evaluated efficiently for any meromorphic differential 1-form λ on $X_0(37)$ or its universal cover regular away form ∞, by the method of §4.1.

To obtain differentials of the second kind representing classes in the deRham cohomology, we consider the elements of the form

$$\eta_1 = u \cdot \omega_f, \qquad \eta_2 = u \cdot \omega_g, \qquad u = \eta(q)^2\eta(q^{37})^{-2} = q^{-3}\prod_{n=1}^{\infty}(1 - q^n)^2(1 - q^{37n})^{-2},$$

where $\eta(q)$ is the Dedekind eta function. The modular function u is an example of an eta product with its only pole at ∞, as considered in §4.3. It is not hard to check directly (by calculating the Poincaré pairing on all pairs of elements) that the classes of $\omega_f, \omega_g, \eta_1$ and η_2 generate the deRham cohomology of X; alternatively one could apply Lemma 4.2.1.

After computing the matrix M of the Hecke operator T_2 acting on $H^1_{\mathrm{dR}}(X_0(37))$ with respect to the basis $\omega_f, \omega_g, \eta_1, \eta_2$, and then determining the eigenspaces of M, one finds that

$$\begin{aligned}
\eta_f &= \frac{1}{4}(-37\omega_g + 4\eta_1 - 8\eta_2), \\
\eta_g &= \frac{1}{4}(37\omega_f - 6\eta_1 + 10\eta_2)
\end{aligned}$$

are in the f and g isotypic components of the deRham cohomology respectively. In addition these linear combinations of 1-forms have been chosen so that $\{\omega_f, \eta_f\}$ and $\{\omega_g, \eta_g\}$ form symplectic bases for the components with respect to the Poincaré pairing.

When one computes the Poincaré dual γ_f of ω_f as in §4.4, one finds (with our normalization) that it is

$$\gamma_f = \frac{1}{2\pi i}\left(A\left([\gamma_2] - [\gamma_3] + [\gamma_4]\right) - B\left(-[\gamma_1] + 2[\gamma_2]\right)\right).$$

Here

$$A \approx (2.4513893\ldots)i, \qquad B \approx 2.9934586\ldots$$

are certain linear combinations of the periods of f against a basis of $H_1(X)[f]$; see §4.4 for a more exact description.

The method of §4.5 can be used to compute $\alpha_{\omega_g, \eta_g}$. However, in this case, it is easy to find $\alpha_{\omega_g, \eta_g}$ by inspection. Working with principal parts, one finds that $\mathrm{pp}_\infty(\omega_g F_{\eta_g}) \equiv \mathrm{pp}_\infty(\frac{1}{4}(\eta_1 - \eta_2)) \mod \frac{\mathrm{d}q}{q}$. Thus we may take $\alpha_{\omega_g, \eta_g} = \frac{1}{4}(\eta_1 - \eta_2)$. Integrating this over γ_f yields the rational number $\int_{\gamma_f} \alpha_{\omega_g, \eta_g} = -\frac{1}{2}$ (to many digits of precision).

Since g is a rational newform, then by Remark 3.2.2, we can find all the points $P_{g,f,n}$ by only computing $P_{g,f,1}$. According to Remark 3.3.2, this amounts to computing the complex number $z_{g,f,1} := \int_{\gamma_f}(\omega_g \cdot \eta_g - \eta_g \cdot \omega_g - 2\alpha_{\omega_g, \eta_g})$. The method in §4.1, coupled with the previous paragraph, yields

$$z_{g,f,1} = -0.4093610\ldots + (1.2256946\ldots)i.$$

Let W be the Weierstrass uniformization of E. Then the point $W(z_{g,f,1}) \in E(\mathbf{C})$ does not necessarily lie in $E(\mathbf{Q})$. This is because $T_{g,1}$ is a *rational* combination of cycles, and so $W(z_{g,f,1})$ is a $\mathbf{Q}$-linear combination of points in $E(\mathbf{Q})$. Thus, the image of $W(z_{g,f,1})$ in $E(\mathbf{C}) \otimes \mathbf{Q}$ lies in the subspace $E(\mathbf{Q}) \otimes \mathbf{Q}$. So in order to write $P_{g,f,1}$ as an element of this space, we must compute the "denominator" of $T_{g,1}$. Let $\mathbf{T}$ denote the Hecke algebra generated over $\mathbf{Z}$ by the Hecke operators T_n. Then, as in §4.7, one can compute using the first few Fourier coefficients of f and g that the idempotent $e = (0,1) \in \mathbf{Q} \times \mathbf{Q} \simeq \mathbf{T} \otimes \mathbf{Q}$ ($\star$) does not belong to $\mathbf{T} \subset \mathbf{T} \otimes \mathbf{Q}$ but $2e$ does. Here, the identification ($\star$) associates $T_n \otimes 1 \in \mathbf{T} \otimes \mathbf{Q}$ to $(a_n(f), a_n(g)) \in \mathbf{Q} \times \mathbf{Q}$. By definition, $T_{g,1}$ corresponds to e as an element of the Hecke algebra, so its denominator is 2. Thus, we can write $P_{g,f,1} = W(2z_{g,f,1}) \otimes \frac{1}{2} \in E(\mathbf{Q}) \otimes \mathbf{Q}$. One finds that $W(2z_{g,f,1}) = (\frac{1357}{841} : \frac{28888}{24389} : 1)$ to within 13 digits of accuracy using 350 Fourier coefficients, so

$$P_{g,f,1} = \left(\frac{1357}{841} : \frac{28888}{24389} : 1\right) \otimes \frac{1}{2} \in E(\mathbf{Q}) \otimes \mathbf{Q}$$

5.2. Example: 43a1. Let $N = 43$ and let E be the elliptic curve labeled **43a1** in Cremona's database. The modular curve $X = X_0(43)$ has genus 3. There are two isotypic components of $H^1_{\mathrm{dR}}(X)$, one of dimension 2 corresponding to the modular form f that parametrized E, and another of dimension 4 corresponding to a newform g with Fourier coefficients in $\mathbf{Q}(\sqrt{2})$, associated to an abelian surface quotient of $J_0(43)$.

In this case, a linear programming algorithm identifies the eta-quotient u that is modular for $\Gamma_0(43)$ of weight 0, holomorphic away from the cusp ∞, and with minimal pole order at ∞, as

$$u = \frac{\eta(q)^4}{\eta(q^{43})^4} = q^{-7} - 4q^{-6} + 2q^{-5} + 8q^{-4} - 5q^{-3} - 4q^{-2} - 10q^{-1} + 8 + 9q + 14q^3 + O(q^4).$$

Computing the residue pairing shows that for a basis of cuspforms with rational Fourier coefficients, corresponding to holomorphic 1-forms $\omega_f, \omega_{g,1}, \omega_{g,2}$ on X, the collection

$$\omega_f, \omega_{g,1}, \omega_{g,2}, u\omega_f, u\omega_{g,1}, u\omega_{g,2}$$

forms a basis for $H^1_{\mathrm{dR}}(X/\mathbf{Q})$. By finding the matrices of a few Hecke operators with respect to this basis, one can as in the case $N = 37$ produce symplectic bases

$$\omega_f, \eta_f, \qquad \text{and} \qquad \omega_{g,1}, \omega_{g,2}, \eta_{g,1}, \eta_{g,2}$$

for $H^1_{\mathrm{dR}}(X)[f]$ and $H^1_{\mathrm{dR}}(X)[g]$ respectively.

We can compute the Poincaré dual γ_f and the iterated integrals

$$\int_{\gamma_f} (\omega_{g,i} \cdot \omega_{g,j} - \alpha_{\omega_{g,i},\omega_{g,j}}), \int_{\gamma_f} (\omega_{g,i} \cdot \eta_{g,j} - \alpha_{\omega_{g,i},\eta_{g,j}}), \int_{\gamma_f} (\eta_{g,i} \cdot \omega_{g,j} - \alpha_{\eta_{g,i},\omega_{g,j}})$$

in the same manner as in the case $N = 37$ with one exception. One simply cannot find $\alpha_{\omega_{g,i},\eta_{g,j}}$ by inspection. No linear combination of our chosen basis has the same principal part as $\omega_{g,i} F_{\eta_{g,j}}$, however some linear combination is cohomologous to such a form. The techniques from §4.5 can be used to find such a form.

Each $T_{g,n}$ gives rise to an element of $\mathrm{End}(H^1_{\mathrm{dR}}(X)[g]) \otimes \mathbf{Q}$. The collection of elements arising from $T_{g,n}$, $n \geq 1$ generate a subspace of dimension 2, generated by $T_{g,1}$ and $T_{g,2}$. Thus, we can effectively compute $P_{g,f,n}$ for all n simply by computing $P_{g,f,1}$ and $P_{g,f,2}$. The formula for $P_{g,f,1}$ is the one given in Remark 3.3.2, so we have

$$z_{g,f,1} = \int_{\gamma_f} (\omega_{g,1} \cdot \eta_{g,1} - \eta_{g,1} \cdot \omega_{g,1} - 2\alpha_{\omega_{g,1},\eta_{g,1}} + \omega_{g,2} \cdot \eta_{g,2} - \eta_{g,2} \cdot \omega_{g,2} - 2\alpha_{\omega_{g,2},\eta_{g,2}})$$

$$= -2.0768300\ldots + (2.7263648\ldots)i$$

The Hecke algebra $\mathbf{T} \otimes \mathbf{Q}$ can be identified with $\mathbf{Q} \times \mathbf{Q}(\sqrt{2})$ via $T_n \otimes 1 \mapsto (a_n(f), a_n(g))$. Under this identification, $T_{g,1}$ corresponds to $e_1 = (0,1)$, and an examination of the Fourier coefficients of f and g shows that e_1 does not lie in the image of $\mathbf{T}$, but $2e_1$ does. So, we have

$$P_{g,f,1} = W(2z_{g,f,1}) \otimes \frac{1}{2} = \left(\frac{11}{49} : -\frac{363}{343} : 1\right) \otimes \frac{1}{2} \in E(\mathbf{Q}) \otimes \mathbf{Q}.$$

Finding $P_{g,f,2}$ is a little more involved, as we must compute the matrix of T_2 acting on $\omega_{g,1}, \omega_{g,2}, \eta_{g,1}, \eta_{g,2}$. Two methods for doing this were discussed in §4.2 and §4.6, and either shows that $T_2\omega_{g,1} = 2\omega_{g,2}$, $T_2\omega_{g,2} = \omega_{g,1}$, $T_2\eta_{g,1} = -\frac{97997}{132319}\omega_{g,2} + \eta_{g,2}$ and $T_2\eta_{g,2} = \frac{97997}{132319}\omega_{g,1} + 2\eta_{g,1}$. So the matrix A_2 is given by

$$\begin{pmatrix} 0 & 2 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & -\frac{97997}{132319} & 0 & 1 \\ \frac{97997}{132319} & 0 & 2 & 0 \end{pmatrix}.$$

Combining this with Lemma 3.3.1 and remembering that $\alpha_{\omega_{g,i},\omega_{g,j}} = 0$ by Remark 2.2.6, we find that

$$z_{g,f,2} = -\frac{97997}{132319} \int_{\gamma_f} (\omega_{g,1} \cdot \omega_{g,2} - \omega_{g,2} \cdot \omega_{g,1}) + \int_{\gamma_f} (\omega_{g,1} \cdot \eta_{g,2} - \eta_{g,2} \cdot \omega_{g,1} - 2\alpha_{\omega_{g,1},\eta_{g,2}})$$

$$+ 2\int_{\gamma_f} (\omega_{g,2} \cdot \eta_{g,1} - \eta_{g,1} \cdot \omega_{g,2} - 2\alpha_{\omega_{g,2},\eta_{g,1}})$$

$$= 2.4055874\ldots - (1.0710898\ldots)i.$$

The cycle $T_{g,2}$ corresponds to the element $e_2 = (0, \sqrt{2})$ in $\mathbf{Q} \times \mathbf{Q}(\sqrt{2})$, which belongs to the image of $\mathbf{T}$ by inspection of the Fourier coefficients of f and g. Thus, we have

$$P_{g,f,2} = W(z_{g,f,2}) \otimes 1 = (-1 : 0 : 1) \otimes 1 \in E(\mathbf{Q}) \otimes \mathbf{Q}.$$

5.3. Table. In Table 1 we report some Chow Heegner points that lie on elliptic curves over $\mathbf{Q}$ of rank 1 and conductor < 100. The format of the table is as follows. We list the strong Weil curve E in each isogeny class of rank Let N be the conductor of the curve E in question, and $f \in S_2(\Gamma_0(N))$ the newform corresponding to E. The first two columns of the table list the label for E in Cremona's database and a choice of generator for $E(\mathbf{Q})$, using the canonical minimal Weierstrass equation for E with invariants $a_1, a_3 \in \{0, 1\}$ and $a_2 \in \{-1, 0, 1\}$.

The next column indicates a Galois orbit of normalized Hecke eigenforms $g \in S_2(\Gamma_0(N))$. The column gives the index of the orbit in question when the orbits are ordered (starting with index 0) by[12] The case $g = f$ is omitted (often f has index 0).

The fourth column gives an integer $n \geq 1$ corresponding to a Hecke operator T_n. The table includes $[K_g : \mathbf{Q}]$ values of n such that $T_g \cdot T_n$ form a $\mathbf{Q}$-basis for for the direct factor K_g of $\mathbf{T} \otimes \mathbf{Q}$. The fifth column lists $P_{g,f,n}$ as an element of $E(\mathbf{Q}) \otimes_{\mathbf{Z}} \mathbf{Q}$, in terms of the generator P in the second column. The sixth column gives the denominator $d_{g,n}$ defined above, and the seventh column gives a point of $E(\mathbf{Q})$ (with respect to the same Weierstrass model as above) which represents the coset $d_{g,n}P_{g,f,n} \in E(\mathbf{Q})/E(\mathbf{Q})^{\text{tor}}$. (The symbol $*$ indicates that such a coset representative would take up too much space to be included in the table. The symbol O indicates that $d_{g,n}P_{g,f,n}$ is 0 modulo torsion.

FIXME: Should we include the index $I_{g,f}$ of the subgroup of $E(\mathbf{Q})/E(\mathbf{Q})^{\text{tor}}$ generated by $\pi_E(P_Z)$ for all $Z \in \mathbf{T} \cap K_g \subset \mathbf{T} \otimes \mathbf{Q}$? If so, what is the best way to compute this index?

5.4. Discussion. [13]

6. Extensions of the method

6.1. Allowing f to be an oldform. [14]

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[12]FIXME: Say how. Lexicographic by T_p-traces for g new, but when g is old the scheme is more complicated...

[13]FIXME: Check when possible the conditions in the theorem saying when $P_{g,f}$ is non-torsion,

Explain why quadratric twists switch the sign of the points.

[14]FIXME: to be added!

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TABLE 1. Chow Heegner points on curves of rank 1 and conductor < 100

Curve $E = E_f$	generator $P \in E(\mathbf{Q})$	g	n	$P_{g,f,n}$	$d_{g,n}$
37a1	$(0,-1)$	1	1	$-6P$	2
43a1	$(0,-1)$	1	1	$4P$	2
			2	$2P$	1
53a1	$(0,-1)$	1	1	$-2P$	2
			2	$-8P$	2
			3	$4P$	2
57a1	$(2,1)$	1	1	$\frac{4}{3}P$	12
		2	1	$-\frac{16}{3}P$	3
		3	1	$-4P$	2
			3	$-4P$	2
58a1	$(0,-1)$	1	1	$4P$	4
		2	1	0	2
			2	$4P$	2
			3	$4P$	2
			4	$4P$	2
61a1	$(1,-1)$	1	1	$-2P$	2
			2	$4P$	2
			3	$-4P$	1
65a1	$(-1,1)$	1	1	P	2
			2	P	2
		2	1	P	2
			2	$3P$	2
77a1	$(2,3)$	1	1	$\frac{12}{5}P$	20
		2	1	$-\frac{4}{3}P$	6
		3	1	$\frac{4}{3}P$	6
			7	$\frac{44}{3}P$	6
		4	1	$-\frac{12}{5}P$	10
			2	$-4P$	2
79a1	$(0,0)$	1	1	$-4P$	2
			2	$-4P$	2
			3	$-4P$	2
			4	0	2
			5	0	2
82a1	$(0,0)$	1	1	0	4
			3	$2P$	2
		2	1	$2P$	2
			2	0	2
			3	$2P$	2
			4	0	2
			5	$4P$	2
			6	$2P$	2
83a1	$(0,0)$	1	1	0	2
			2	$2P$	2
			3	$4P$	2
			4	$-4P$	2
			5	$-4P$	1
			7	0	2

Curve $E = E_f$	generator $P \in E(\mathbf{Q})$	g	n	$P_{g,f,n}$	$d_{g,n}$
88a1	$(2,-2)$	1	1	0	16
			3	0	16
		2	1	0	8
			2	$8P$	2
		3	1	0	2
			2	$8P$	2
			4	$16P$	1
			8	$16P$	1
89a1	$(0,-1)$	1	1	$\frac{8}{5}P$	5
		2	1	$\frac{2}{5}P$	10
			2	$\frac{22}{5}P$	10
			3	$-\frac{16}{5}P$	10
			4	$-\frac{2}{5}P$	10
			6	$\frac{24}{5}P$	10
91a1	$(0,0)$	1	1	$2P$	4
		2	1	$2P$	4
			2	$-2P$	2
		3	1	$4P$	4
			2	$2P$	2
			3	$6P$	2
91b1	$(-1,3)$	0	1	0	4
		2	1	0	4
			2	0	2
		3	1	0	4
			2	0	2
			3	0	2
92b1	$(1,1)$	1	1	0	2
		2	1	0	15
			2	0	5
		3	1	0	20
			2	0	5
			3	0	4
			4	0	6
			6	0	1
			8	0	5
99a1	$(2,0)$	1	1	$-\frac{2}{3}P$	12
		2	1	0	12
		3	1	$\frac{2}{3}P$	6
		4	1	$\frac{2}{3}P$	12
			3	$\frac{8}{3}P$	3
		5	1	$-\frac{2}{3}P$	6
			3	$-\frac{2}{3}P$	3
			9	$-\frac{22}{3}P$	3