THE EFFECTIVE COMPUTATION OF ITERATED INTEGRALS AND CHOW-HEEGNER POINTS ON TRIPLE PRODUCTS

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Abstract. FIXME

Introduction

The goal of this paper is to describe a complex-analytic algorithm for the computation of *triple Chow-Heegner points*. Fix cuspidal eigenforms $f, g \in S_2(\Gamma_0(N))$ and assume f is a newform with rational Fourier coefficients. To g is associated a Hecke correspondence $T_g \subset X_0(N) \times X_0(N)$, which gives rise to a rational point $P_g \in J_0(N)(\mathbf{Q})$. (A more precise definition of P_g is given below.) The triple Chow-Heegner point $P_{g,f}$ associated to the 3-tuple of modular forms (g, g, f) is the image of P_g in A_f, the elliptic curve quotient of $J_0(N)$ associated to the newform f.

It is shown in [DRS] that the rational point $P_{g,f}$ can be computed as an element of the analytic curve $A_f(\mathbf{C})$ in terms of a certain iterated path integral (in the sense of [Chen]). This formula is amenable to numerical computation, which we have implemented using the free software package `SAGE`. The results of [DRS] together with work of Yuan-Zhang-Zhang give a criterion (cf. [DRS, Theorem 1]) for when the points $P_{g,f}$ are non-torsion. This criterion implies that triple Chow-Heegner points comprise a collection of non-torsion points on many elliptic curves A_f of rank 1. Our algorithm makes these points readily computable in practice for many elliptic curves A_f of small conductor.

While the analytic formula is not the only way of computing the points $P_{g,f}$ (see the Appendix) our approach has a theoretical advantage: it requires knowing only the Hodge class ξ_g associated to the cycle T_g. In future work, we hope to adapt the algorithm described below to compute Chow-Heegner points [DRS2] associated to Hodge classes on "modular varieties" (related to Kuga-Sato varieties), such as classes ξ arising from modular forms with complex multiplication. The rationality of Chow-Heegner points computed in this manner could provide numerical evidence for certain open cases of the Hodge conjecture.

In §1 we recall necessary facts about itegrated integrals and related ingredients for our main algorithm. In §2 we specialize to the case of modular curves, define the points $P_{g,f}$ precisely, and write down an explicit analytic formula for them. In §3 we describe in detail an algorithm for evaluating this formula numerically. The algorithm is illustrated with numerical examples in §4. Some tables of triple Chow-Heegner points on elliptic curves of small conductor are presented in §5, along with some discussion of a few phenomena apparent from this data.

1. Preliminaries

1.1. Let F be a number field (we take $F = \mathbf{Q}$ in the sequel) and fix an embedding $\iota : F \hookrightarrow \mathbf{C}$. Let X be a smooth, complete algebraic curve of genus $g \geq 2$ over F, and let $Y = X \setminus \{\infty\}$ be the complement of a single point in $X(F)$. For a smooth variety $V_{/F}$ (such as X or Y) we denote by V^{an} the complex manifold $(V \otimes_{F,\iota} \mathbf{C})(\mathbf{C})$ with its analytic topology.

1.2. The *de Rham cohomology* $H^1_{\text{dR}}(X^{\text{an}}, \mathbf{C})$ is the cohomology of the de Rham complex of smooth $\mathbf{C}$-valued differential forms on X^{an}.

Because the Riemann surface X^{an} arises from an algebraic curve over F, we can identify $H^1_{\mathrm{dR}}(X^{\mathrm{an}}, \mathbf{C})$ with $H^1_{\mathrm{dR}}(X/F) \otimes \mathbf{C}$, where

$$H^1_{\mathrm{dR}}(X/F) := \mathbf{H}^1(0 \to \mathcal{O}_X \to \Omega^1_X \to 0) \tag{1.2.1}$$

is the *algebraic de Rham cohomology* of X/F, defined as the hypercohomology of the de Rham complex of sheaves of regular differential forms on X.

As is well known, the fact that X is a curve means that $H^1_{\mathrm{dR}}(X/F)$ has a particularly simple description in terms of $\Omega^{II}(X)$, the space of *differentials of the second kind* on X. By definition, these are rational 1-forms on X with vanishing residues at all points of X. By the residue formula we may identify $\Omega^{II}(X)$ with $\Omega^{II}(Y)$, the differentials of the second kind on Y. Thus we have a canonical isomorphism

$$H^1_{\mathrm{dR}}(X/F) = \Omega^{II}(Y)/\mathrm{d}F(X),$$

where $F(X)$ is the field of rational functions on X. By applying Riemann-Roch, this description can be simplified: it is not difficult to show that $\Omega^{II}(Y)/\mathrm{d}F(X) \cong \Omega^1(Y)/\mathrm{d}\Gamma(Y, \mathcal{O}_Y)$. So $H^1_{\mathrm{dR}}(X/F)$ can also be computed as the space of *regular* 1-forms on Y, modulo exact forms. For computational purposes, the latter description is the most useful: we will compute with classes in $H^1(Y)$ using rational 1-forms on X, regular away from the point ∞. These are amenable to computation via their Laurent expansions about ∞.

1.3. Fix a base point $o \in Y^{\mathrm{an}}$; let $\Gamma := \pi_1(Y^{\mathrm{an}}; o) = \pi_1(X^{\mathrm{an}}, o)$ denote the fundamental group of the Riemann surfaces Y^{an} and X^{an}. Let $\mathbf{Z}[\Gamma]$ be the integral group ring on Γ and write $I \subset \mathbf{Z}[\Gamma]$ for its augmentation ideal. Note that $H_1(X^{\mathrm{an}}, \mathbf{Z}) = H_1(Y^{\mathrm{an}}, \mathbf{Z}) \cong \Gamma^{\mathrm{ab}}$ is naturally identified with I/I^2.

The *path space* on Y based at o, denoted $\mathbf{P}(Y; o)$, is the set of piecewise-smooth paths

$$p : [0, 1] \longrightarrow Y^{\mathrm{an}}, \qquad \text{with } p(0) = o.$$

Let $\pi : \tilde{Y} \to Y^{\mathrm{an}}$ denote the universal covering space of Y^{an} corresponding to the choice of basepoint o, which can be regarded as the space of homotopy classes in $\mathbf{P}(Y; o)$. Likewise, denote by $\tilde{X}$ the universal cover of X corresonding to the same basepoint o. The group Γ acts on $\tilde{Y}$ transitively and without fixed points, and the map $p \mapsto p(1)$ identifies the quotient $\tilde{Y}/\Gamma$ with Y^{an}. Recall that if η is a closed C^∞ 1-form (resp. a meromorphic 1-form of the second kind) on X^{an}, then it admits a smooth (resp. meromorphic) *primitive function* $F_\eta : \tilde{X} \to \mathbf{C}$, defined by the rule

$$F_\eta(p) := \int_0^1 p^*\eta.$$

The *basic iterated integral* attached to a tuple of smooth 1-forms $\omega_1, \ldots, \omega_n$ on Y^{an}, evaluated along a path $p \in \mathbf{P}(Y; o)$, is defined to be

$$\int_p \omega_1 \cdot \omega_2 \cdot \ldots \cdot \omega_n := \int_{0 \le t_n \le t_{n-1} \le \cdots \le t_1 \le 1} p^*(\omega_1)(t_1) \cdots p^*(\omega_n)(t_n).$$

The integer n is called the *length* of this basic iterated integral. Note that when $n = 2$, the basic iterated integral attached to ω and η can be computed by the formula

$$\int_\gamma \omega \cdot \eta = \int_\gamma \omega F_\eta = \int_0^1 \gamma^*(\omega F_\eta).$$

An *iterated integral* is a linear combination of basic iterated integrals, perhaps of different lengths, viewed as a function on $\mathbf{P}(Y; o)$. The length of an iterated integral is then defined to be the maximum of the lengths of its constituent basic iterated integrals.

An iterated integral is said to be *homotopy invariant* if its value on any path p depends only on the homotopy class of p. The space $\mathrm{II}(Y)$ of homotopy invariant iterated integrals will be viewed as a subspace of the space of $\mathbf{C}$-valued functions on Γ. Extending $J \in \mathrm{II}(Y)$

to the group ring $\mathbf{C}[\Gamma]$ by $\mathbf{C}$-linearity, we regard $\mathrm{II}(Y)$ as a space of complex functionals on $\mathbf{C}[\Gamma]$ via the inclusion $\mathrm{II}(Y) \subset \mathrm{Hom}_{\mathbf{C}}(\mathbf{C}[\Gamma], \mathbf{C})$.

For each n, let $\mathrm{II}^{\leq n}(Y)$ denote the subspace of homotopy invariant iterated integrals of length $\leq n$. Observe that any element $J \in \mathrm{II}^{\leq n}(Y) \subset \mathrm{Hom}(\mathbf{Z}[\Gamma], \mathbf{C})$ vanishes on I^{n+1}, and hence gives rise to a well-defined element of $\mathrm{Hom}(I/I^{n+1}, \mathbf{C})$. The natural map $\mathrm{II}^{\leq n} \longrightarrow \mathrm{Hom}(I/I^{n+1}, \mathbf{C})$ is an isomorphism.[1]

We will be interested in numerically evaluating certain iterated integrals $J \in \mathrm{II}^{\leq 2}(Y)$. Specificaly, suppose we are given $\omega, \eta \in H^1_{\mathrm{dR}}(X/F)$, represented as differentials of the second kind, regular on Y. Recall that a differential on a Riemann surface is said to have a *logarithmic pole* at a point if its expansion in terms of a local parameter q at this point is of the form $\sum_{n=0}^{\infty} a_n q^n \frac{dq}{q}$. Let $\alpha_{\omega,\eta}$ be a meromorphic 1-form on X which is regular on Y and is such that the induced differential $\omega F_\eta - \alpha$ on $\tilde{X}$ has at worst a logarithmic pole at (any point lying over) ∞. This condition is well-posed because the principal part of ωF_η at at $\tilde{x} \in \tilde{X}$ depends only on the image x of $\tilde{x}$; see [DRS, §1]. The form $\alpha_{\omega,\eta}$ exists – and in fact can even be taken to be algebraic and defined over F – by Riemann-Roch.

Lemma 1.3.1. *The iterated integral $J_{\omega,\eta} := \int \omega \cdot \eta - \alpha_{\omega,\eta}$, viewed as a function on $\mathbf{P}(Y, o)$, is homotopy-invariant.*

Moreover, suppose that ω and η represent integral cohomology classes. Then when $\mathrm{II}^{\leq 2}(Y)$ is identified with $\mathrm{Hom}(I/I^3, \mathbf{C})$, the restriction of $J_{\omega,\eta}$ to I^2/I^3 is $\mathbf{Z}$-valued and can be identified with $\omega \otimes \eta$, viewed as an element of

$$H^1(X, \mathbf{Z}) \otimes H^1(X, \mathbf{Z}) \cong (H_1(X, \mathbf{Z}) \otimes H_1(X, \mathbf{Z}))^\vee = (I/I^2 \otimes I/I^2)^\vee = (I^2/I^3)^\vee,$$

where $A^\vee$ denotes the $\mathbf{Z}$-dual of an abelian group A.

Proof. The homotopy invariance of $J_{\omega,\eta}$ follows from the fact that $J_{\omega,\eta}(\gamma) = \int_\gamma \omega F_\eta - \alpha_{\omega,\eta}$, and the one form on $\tilde{X}$ in the integrand is holomorphic when restricted to $\tilde{Y}$. For the second claim, see the discussion at the beginning of §1 of [DRS], and *loc. cit.*, Lemma 1.1(2). □

Now consider an integral class $\xi = \sum \omega_i \otimes \eta_i \in H^1(X, \mathbf{Z}) \otimes H^1(X, \mathbf{Z})$. By the previous lemma, the iterated integral $J_\xi = \sum J_{\omega_i,\eta_i}$ is homotopy invariant and induces a homomorphism

$$J_\xi : H_1(X, \mathbf{Z}) = I/I^2 \to \mathbf{C}/\mathbf{Z}.$$

Fix an auxiliary holomorphic 1-form $\rho \in H^{1,0}(X_{\mathbf{C}}) \subset H^1(X^{\mathrm{an}}, \mathbf{C})$. Denote by Λ the period lattice $\langle \int_\gamma \rho : \gamma \in H_1(X^{\mathrm{an}}, \mathbf{Z}) \rangle$. The class $\gamma_\rho \in H_1(X^{\mathrm{an}}, \mathbf{C})$ which is Poincaré dual to ρ actually belongs to $H_1(X^{\mathrm{an}}, \mathbf{Z}) \otimes \Lambda$. Consequently $J_\xi(\gamma_\rho)$ is a well-defined element of $\mathbf{C}/\Lambda$.

1.4. Let X_1, X_2 denote copies of X, and X_{12} the diagonal copy of X in $X_1 \times X_2$. To a divisor $Z \subset X \times X = X_1 \times X_2$ (defined over F) we associate the point

$$P_Z = D_Z - \deg(D_Z) o \in \mathrm{Pic}^0(X),$$

where (recall) $o \in X(F)$ is a fixed base point and we set $D_Z = (Z \cap X_{12}) - (Z \cap X_1) - (Z \cap X_2)$.

We now state the iterated integral formula from [DRS] for the image of P_Z under the Abel-Jacobi map

$$\mathrm{AJ}_X : \mathrm{Pic}^0(X) \to \Omega^1(X^{\mathrm{an}})^\vee / H_1(X^{\mathrm{an}}, \mathbf{Z}).$$

Let ϵ_o be the projector on $\mathrm{Pic}(X \times X)$ defined by

$$\epsilon_o(Z) = Z - i_{1*}\pi_{1*} - i_{2*}\pi_{2*}$$

where $\pi_1, \pi_2 : X \times X \rightrightarrows X$ are the projections and $i_1, i_2 : X \rightrightarrows X \times X$ are the inclusions of "vertical and horizontal" copies of X over the basepoint o.

[1] FIXME: add reference? This is stated without proof in DRS.

Let

$$\mathrm{cl}(\epsilon_o -) : \mathrm{Pic}(X \times X) \to H^1_{\mathrm{dR}}(X^{\mathrm{an}}, \mathbf{Z}) \otimes H^1_{\mathrm{dR}}(X^{\mathrm{an}}, \mathbf{Z})$$

denote the composition of the cycle class map and the projector ϵ_o. (The effect of ϵ_0 is to annihilate $H^2 \otimes H^0$ and $H^0 \otimes H^2$ factors in the Künneth decomposition of $H^2(X \times X)$.) Suppose $\mathrm{cl}(\epsilon_o Z)$ is represented by $\sum \omega_i \otimes \eta_i$, where $\omega_i, \eta_i \in \Omega^1(Y)$ (one of each pair being regular at ∞, since $\mathrm{cl}(\epsilon_o Z)$ is a Hodge class).

Theorem 1.4.1 ([DRS], Corollary 3.6)**.** *The image $\mathrm{AJ}_X(P_Z)$ of P_Z under the Abel-Jacobi map is the element of $\Omega^1(X^{\mathrm{an}})^\vee / H_1(X^{\mathrm{an}}, \mathbf{Z})$ which maps $\rho \in \Omega^1(X^{\mathrm{an}})$ to*

$$\sum J_{\omega_i,\eta_i}(\gamma_\rho) = \sum \int_{\gamma_\rho} (\omega_i \cdot \eta_i - \alpha_{\omega_i,\eta_i}) \in \mathbf{C}/\Lambda_\rho.$$

Here $\gamma_\rho \in H_1(X^{\mathrm{an}}, \mathbf{C})$ is Poincaré dual to $\rho \in H^{1,0}(X^{\mathrm{an}}) \subset H^1_{\mathrm{dR}}(X^{\mathrm{an}}, \mathbf{C})$, and Λ_ρ is the period lattice of ρ. □

2. Triple Chow-Heegner points on modular curves

We now specialize the discussion of the preceding section to the case of classical modular curves X. We shall define certain rational points on an arbitrary elliptic curve $E_{/\mathbf{Q}}$ called *triple Chow-Heegner points*, such that the corresponding points of $E(\mathbf{C}) \cong \mathbf{C}/\Lambda_E$ can be computed using iterated path integrals via Theorem 1.4.1.

2.1. Let $N \geq 1$ be an integer and let $X = X_0(N)$ denote the canonical model over $\mathbf{Q}$ of the classical modular curve of level N; write $J_0(N)$ for the Jacobian of $X_0(N)$. With this choice of X we place ourselves in the setup of §1, taking the ground field F to be $\mathbf{Q}$ and the point $\infty \in X(\mathbf{Q})$ to be the usual cusp at infinity. Thus $Y := X_0(N) - \{\infty\}$. (Note that $Y \supsetneq Y_0(N) = X_0(N) - \{\mathrm{cusps}\}$.) We will be deliberately vague concerning our basepoint $o \in X^{\mathrm{an}}$ for topological constructions.[2]

We shall make use of the *Poincaré pairing* on $H^1(X)$, which is a symplectic form

$$\langle , \rangle : H^1_{\mathrm{dR}}(X/\mathbf{Q}) \times H^1_{\mathrm{dR}}(X/\mathbf{Q}) \to \mathbf{Q}.$$

If ω and η are smooth 1-forms on X, then $\langle \omega, \eta \rangle := \frac{1}{2\pi i} \int_X \omega \wedge \eta$. If ω and η are differentials of the second kind on X, holomorphic away from the cusp ∞, then the induced pairing on $H^1(Y)$ can also be computed as

$$\langle \omega, \eta \rangle = \mathrm{res}_\infty(F_\omega \cdot \eta) = -\mathrm{res}_\infty(\omega \cdot F_\eta),$$

where as above F_ν denotes the primitive function $\tilde{Y} \to \mathbf{C}$ of the differential ν. Given 1-forms ω, η of the second kind on X, regular on Y, the Poincaré pairing of their cohomology classes is thus computable from Laurent expansions of ω, η about ∞ by integrating formally.

2.2. Now let $E_{/\mathbf{Q}}$ be an elliptic curve of conductor N whose isogeny class corresponds to a newform $f \in S_2(\Gamma_0(N))$ with rational Fourier coefficients. In particular there is a *modular parametrization* $\pi_E : J_0(N) \to E$, a homomorphism of abelian varieties defined over $\mathbf{Q}$. We will assume for the remainder of this paper that E is optimal, so $\ker \pi_E$ is connected. In this case the Néron lattice of E coincides with the period lattice Λ_f of the differential $\omega_f = 2\pi i f(z) \mathrm{d}z \in \Omega^1(X^{\mathrm{an}})$ corresponding to f. The map π_E can be computed on complex points explicitly, using the Abel-Jacobi isomorphism $\mathrm{AJ}_X : J_0(N)(\mathbf{C}) \cong \Omega^1(X^{\mathrm{an}})/H_1(X^{\mathrm{an}}, \mathbf{Z})$ and the Weierstrass uniformization $W : \mathbf{C}/\Lambda_f \cong E(\mathbf{C})$. Namely, for $P_{\mathbf{C}} \in J_0(N)(\mathbf{C})$ we have $\pi_{E,\mathbf{C}}(P_{\mathbf{C}}) = W(\mathrm{AJ}_X(P_{\mathbf{C}})(\omega_f))$.

Let $\mathbf{T}_0 = \mathbf{Z}[\{T_n\}_{n \nmid N}]$ be the "anemic" Hecke algebra. Then $\mathbf{T}_0 \otimes \mathbf{Q}$ factors as a product $\prod_{g'} K_{g'}$ where g' runs over newforms of all levels M dividing N and $K_{g'} = \mathbf{Q}(\{a_n(g')\}_{n \geq 1})$ is the number field generated by the Hecke eigenvalues of g'. For any divisor M of N and a

[2] But see §3 for the relevance of the choice of o when performing explicit computations.

newform $g \in S_2(\Gamma_0(M))$, denote by $T_g \in \mathbf{T}_0 \otimes \mathbf{Q} \cong \prod_{g'} K_{g'}$ the idempotent with 1 in the K_g component and 0 elsewhere. This gives rise to a correspondence in $\mathrm{Pic}(X \times X) \otimes \mathbf{Q}$, which by abuse of notation will also be denoted by T_g.

Definition 2.2.1. The *triple Chow-Heegner point* $P_{g,f}$ corresponding to the 3-tuple (g, g, f) of modular forms is the element $\pi_E(P_{T_g}) \in E(\mathbf{Q}) \otimes \mathbf{Q}$.

For generalizations of this definition, see for example [BDP2].

Remark 2.2.2. In this definition, P_{T_g} is defined as in §1 taking $Z = T_g$. However note that T_g might not literally be a divisor on $X \times X$; the correspondence T_g is merely a $\mathbf{Q}$-linear combination of such divisors. Thus P_{T_g} belongs to $\mathrm{Pic}^0(X) \otimes \mathbf{Q} = J_0(N)(\mathbf{Q}) \otimes \mathbf{Q}$. If we define the "denominator" d_g of $T_g \in \mathbf{T}_0 \otimes \mathbf{Q}$ to be the smallest positive integer such that $d_g T_g$ lies in the image of $\mathbf{T}_0$ under the inclusion $\mathbf{T}_0 \hookrightarrow \mathbf{T}_0 \otimes \mathbf{Q}$, then $d_g P_{g,f} \in E(\mathbf{Q})$ is rational. See §3 for more details.

2.3. To obtain from Theorem 1.4.1 an explicit formula for a triple Chow-Heegner point $P_{g,f}$ in terms of iterated integrals, we must know the components of $\mathrm{cl}(\epsilon_o T_g) \in H^1_{\mathrm{dR}}(X/\mathbf{Q})^{\otimes 2}$ when this class is decomposed as a sum of pure tensors.

The action of the Hecke algebra $\mathbf{T}_0$ on modular forms extends to an action on the de Rham cohomology of X. Under this action, we have

$$H^1_{\mathrm{dR}}(X/\mathbf{Q}) \cong H^1_{\mathrm{dR}}(X/\mathbf{Q})[g_1] \oplus \cdots \oplus H^1_{\mathrm{dR}}(X/\mathbf{Q})[g_n],$$

indexed by Galois conjugacy classes of newforms of all levels M dividing N.

Lemma 2.3.1. *Let $M|N$ and let $g \in S_2(\Gamma_0(M))$ be a newform. Let $\{\omega_{g,1}, \ldots, \omega_{g,k}, \eta_{g,1}, \ldots, \eta_{g,k}\}$ be a collection of differentials of the second kind on X representing a basis for $H^1_{\mathrm{dR}}(X/\mathbf{Q})[g]$ that is symplectic with respect to the Poincaré pairing; i.e. assume $\langle \omega_{g,i}, \eta_{g,j} \rangle = \delta_{i,j}$ and $\langle \omega_{g,i}, \omega_{g,j} \rangle = \langle \eta_{g,i}, \eta_{g,j} \rangle = 0$. Then $\mathrm{cl}(\epsilon T_g) = \sum_{i=1}^k \omega_{g,i} \otimes \eta_{g,i} - \eta_{g,i} \otimes \omega_{g,i}$.*

Proof. By definition T_g is a correspondence which acts on $H^1(X)$ as the idempotent projector onto $H^1(X)[g]$. The $H^2 \otimes H^0$ and $H^0 \otimes H^2$ Künneth components of $\mathrm{cl}(T_g)$ act trivially on $H^1(X)$. (See [BL, 11.5.1], for example.) Thus $\mathrm{cl}(\epsilon_o T_g) \in H^1(X) \otimes H^1(X)$ also acts by projecting onto $H^1(X)[g]$.

The action on $H^1(X)$ of a correspondence $Z \subset X \times X$ whose cycle class is in $H^1(X) \otimes H^1(X)$ can be written in terms of the Poincaré pairing. Using that $\mathrm{cl}(\epsilon_o T_g)$ is a projector on $H^1(X)[g]$, one finds that $\mathrm{cl}(\epsilon_o T_g) = \sum_{i,j} \langle b_i, b_j \rangle b_i \otimes b_j$ for *any* basis $\{b_1, \ldots, b_{2k}\}$ of $H^1(X)[g]$. From the the claim follows immediately. ☐

Combining the previous results, we obtain the following formula for $P_{g,f}$. Let γ_f be the Poincaré dual of ω_f and let $\omega_{g,1}, \ldots, \omega_{g,k}, \eta_{g,1}, \ldots, \eta_{g,k}$ be (differentials of the second kind which give rise to) a symplectic basis for the g-isotypic $\mathbf{Q}$-subspace $H^1(X/\mathbf{Q})[g] \subset H^1(X/\mathbf{Q})$. Then, recalling that W denotes the Weierstrass uniformization of $E(\mathbf{C})$, we find that $P_{g,f} \in E(\mathbf{Q}) \otimes \mathbf{Q} \subset E(\mathbf{C})$ can be computed as[3]

$$P_{g,f} = W\left(\sum_{i=1}^{k}\left(\int_{\gamma_f} \omega_{g,i} \cdot \eta_{g,i} - \eta_{g,i} \cdot \omega_{g,i} - 2\alpha_{\omega_{g,i},\eta_{g,i}}\right)\right). \tag{2.3.1}$$

2.4. Recall that in the above definition of iterated integral, everything depends on the choice of a base point o. Likewise, the projector ϵ_o depends on o, and hence *a priori* so does the point $P_{g,f}$. However we have the following.

Lemma 2.4.1. *The point $P_{g,f}$ is independent of o.*

[3]FIXME: This actually lives in $E(\mathbf{C}) \otimes \mathbf{Q}$, right?

Proof. Changing the basepoint from o to o' amounts to conjugating the representative path γ_f for the homology class Poincaré dual to ω_f by a path β from o to o'. This manifestly does not affect the value of the integral of the meromorphic 1-form $2\alpha_{\omega_{g,i},\eta_{g,i}}$. Thus the issue is whether we have an identity

$$\int_{\gamma_f} \omega_{g,i}\cdot\eta_{g,i} - \eta_{g,i}\cdot\omega_{g,i} \stackrel{?}{=} \int_{\beta\gamma_f\beta^{-1}} \omega_{g,i}\cdot\eta_{g,i} - \eta_{g,i}\cdot\omega_{g,i}. \tag{2.4.1}$$

But by [H1, Exer. 8], for any 1-forms ω,η, loop γ, and path β, we have

$$\int_{\beta\gamma\beta^{-1}} \omega\cdot\eta = \int_\gamma \omega\cdot\eta + \begin{vmatrix} \int_\gamma \omega & \int_\gamma \eta \\ \int_\beta \omega & \int_\beta \eta \end{vmatrix}.$$

In our situation, the determinants expressing the difference between the two sides of (2.4.1) vanish. Indeed, $\int_{\gamma_f}\omega_{g,i} = \langle\omega_{g,i},\omega_f\rangle = 0 = \langle\eta_{g,i},\omega_f\rangle = \int_{\gamma_f}\eta_{g,i}$, since the decomposition into isotypic components for the action of the Hecke algebra is orthogonal with respect to the Poincaré pairing. □

2.5. We record a fundamental property of the points $P_{g,f}$.

Theorem 2.5.1 ([DRS], Theorem 1)**.** *The point $P_{g,f}\in E(\mathbf{Q})\otimes\mathbf{Q}$ is nonzero (or equivalently, the point $\pi_E(d_g P_{T_g})\in E(\mathbf{Q})$ is non-torsion) if and only if the following three conditions hold:*

i. $L(f,1)=0$,
ii. $L'(f,1)\neq 0$, *and*
iii. $L(f\otimes \mathrm{Sym}^2(g),2)\neq 0$.
iv. *A condition on the local ϵ-factors.*[4] □

3. Algorithm for effective computation of triple Chow-Heegner points

We now turn to the question of numerically evaluating formula (2.3.1) for a triple Chow-Heegner point $P_{g,f}\in E(\mathbf{Q})\otimes\mathbf{Q}$ for an optimal elliptic curve $E=E_f$. We retain all the notation from §§1-2.

The following ingredients occur in the formula (2.3.1) for $P_{g,f}$.

1. The Poincaré dual $\gamma_f\in H_1(X,\mathbf{C})$ of $\omega_f\in H^1_{\mathrm{dR}}(X^{\mathrm{an}},\mathbf{C})$.
2. A collection of rational differentials of the second kind $\omega_{g,1},\dots,\omega_{g,k},\eta_{g,1},\dots,\eta_{g,k}$ on X, regular away from ∞, whose images in $H^1_{\mathrm{dR}}(X/\mathbf{Q})$ are a symplectic basis for the g-isotypic component $H^1_{\mathrm{dR}}(X/\mathbf{Q})[g]$.
3. Meromorphic differentials $\alpha_{\omega_{g,i},\eta_{g,i}}$ on X, regular on Y, such that $\omega_{g,i}F_{\eta_{g,i}} - \alpha_{\omega_{g,i},\eta_{g,i}}$ has at worst a logarithmic pole at (any point lying over) ∞.

These data must be "known" in a sufficiently concrete form to evaluate the iterated integrals occuring in (2.3.1). It is also desirable to know

4. the denominator d_g of the projector onto the g-isotypic component of the chomology of X.

This last item will allow for the computation of a point in $E(\mathbf{Q})$, as opposed to one in $E(\mathbf{Q})\otimes\mathbf{Q}$. This section is devoted to methods of computing these four ingredients.

3.1. Evaluating iterated integrals. Let $J=\sum\omega_i\cdot\eta_i-\alpha_i\in \mathrm{II}^{\leq 2}(Y)$ be a homotopy-invariant iterated integral of length ≤ 2 on Y, expressed in terms of differentials of the second kind on X, regular on Y. The first homology group $H_1(X^{\mathrm{an}},\mathbf{Z})$ is the abelianization of the quotient $\pi_1(X^{\mathrm{an}},o)=\bar\Gamma_0(N)$ of $\Gamma_0(N)$ by the smallest normal subgroup containing the elliptic and parabolic elements. To evaluate $J(\gamma_0)$ for $\gamma_0\in H_1(X^{\mathrm{an}},\mathbf{Z})$, we lift γ_0 arbitrarily to a path $\tilde\gamma$ in $\tilde X$ based at o. If we choose the basepoint o away from the divisor of cusps on X,

[4] FIXME

then o can be lifted to an element τ_0 in the upper half-plane $\mathfrak{H}$, regarded as a cover[5] of $Y_0(N)$. The path $\tilde{\gamma}$ can then be viewed as a path in $\mathfrak{H}$ from τ_0 to $\gamma\tau_0$, where $\gamma \in \Gamma_0(N)$ is a lift of γ_0.

Lemma 3.1.1. *Suppose γ_0 is Poincaré-dual to ρ. As an element of $\mathbf{C}/\Lambda_\rho$, we have*

$$J(\gamma_0) = \sum \int_{\tau_0}^{\gamma\tau_0} \omega_i F_{\eta_i} - \alpha_{\omega_i,\eta_i}$$

where we conflate 1-forms on X with their pullbacks to $\mathfrak{H}^* = \mathfrak{H} \cup \{\infty\}$. Moreover, F_{η_i} has Laurent expansion about $i\infty \in \mathfrak{h}^*$ given by formally integrating the Laurent expansion of η_i about the cusp $\infty \in X$.

Proof. Clear. □

Given any differential form η of the second kind on X, and any $\gamma \in \Gamma_0(N)$, let

$$I(\eta;\gamma) := \int_{\tau_0}^{\gamma\tau_0} \eta.$$

This expression is independent of the choice of path on the upper half-plane $\mathfrak{H}$ from τ_0 to $\gamma\tau_0$ since η is of the second kind. The $\Gamma_0(N)$-invariance of η also shows that this expression is independent of a choice of base point τ_0 on $\mathfrak{H}$, which justifies suppressing τ_0 from the notation. If η is merely defined on $\tilde{X}$ then this integral still makes sense but depends on the basepoint τ_0 lifting o. However we will only be interested in evaluating such integrals in the context of (2.3.1), for which the choice of basepoint is irrelevant, essentially because the Poincaré dual of the homology class of γ, for any choice of basepoint, is "orthogonal" to the 1-forms in the iterated integral giving rise to the path integral we seek to evaluate. Thus we will (abusively) use the same notation $I(\eta;\gamma)$ in this case as well.

Since η is meromorphic on X (or $\tilde{X}$) the integrals $I(\eta;\gamma)$ can be computed by integrating term-wise a Laurent expansion for η using the fundamental theorem of calculus. Thus, in practice, one computes the Laurent expansion for the primitive F_η about $\infty \in X$ (or a choice of $\tilde{\infty} \in \tilde{X}$ lying over ∞), regarded as function given by a convergent power series in $q = e^{2\pi i\tau}$ on $\mathfrak{h}$, and evaluates it at τ_0 and $\tau_0' = \gamma\tau_0$. The larger the imaginary part of τ_0, τ_0' is, the faster this series converges and the fewer coefficients of the Laurent series of η are necessary to approximate $I(\eta;\gamma)$ to a give degree of accuracy. Writing $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, it is well-known that the best compromise between $\mathrm{Im}(\tau_0)$ and $\mathrm{Im}(\tau_0')$ is achieved when we choose $\tau_0 = -\frac{d}{c} + \frac{1}{|c|}i$.

With this remark in mind, we take the following approach to computing $J(\gamma_0)$ as in the lemma above. First compute Laurent expansions for the differentials $\omega_i, \eta_i, \alpha_{\omega_i,\eta_i}$. Then find an "optimal" expression for the homology class $\gamma_0 \in H_1(X^{\mathrm{an}}, \mathbf{C})$, writing it as a $\mathbf{C}$-linear combination of classes $\gamma_0^{(j)} \in H_1(X^{\mathrm{an}}, \mathbf{Z})$ which lift to elements $\gamma^{(j)} \in \Gamma_0(N)$ with small lower-left entries cN. Finally, calculate approximations to the integrals $I(\omega_i F_{\eta_i}; \gamma^{(j)})$ and $I(\alpha_{\omega_i,\eta_i}; \gamma^{(j)})$ using the optimal choice of basepoint for each. [6] The appropriate linear combination of these integrals is an (approximate) representative for the coset $J(\gamma_0) \in \mathbf{C}/\Lambda_\rho$.)

In the rest of this section we outline how to implement the strategy just sketched.

3.2. Calculating a symplectic basis for $H^1_{\mathrm{dR}}(X/\mathbf{Q})[g]$. The calculation of a basis for the deRham cohomology can be carried out by first writing down a modular function u – that is, a rational function on $X = X_0(N)$ – which is regular away from ∞. Such a function exists by Riemann-Roch and a q-expansion for one such function can be computed explicitly using the Dedekind eta-function, as explained in the next subsection.

[5]FIXME: this cover is ramified at the elliptic points, so some additional care in required to make this step rigorous.

[6]FIXME: why is it OK to use different basepoints for different $\gamma^{(j)}$ integrals, which is what we actually do in the code?

Using a modular symbol algorithm, one can compute q-expansions for a basis of $S_2(\Gamma_0(N), \mathbf{Q})$; cf. [S2], for example. Write $\omega_1, \ldots, \omega_t$ for the corresponding holomorphic 1-forms on X, where for convenience we set $t = p_a(X) = \dim S_2(\Gamma_0(N))$.

Define $\eta_1 = u\omega_i$, which is a differential of the second kind by the residue theorem, and let $\mathcal{B} = \{\omega_1, ..., \omega_t, \eta_1, ..., \eta_t\} \subset H^1_{\mathrm{dR}}(X/\mathbf{Q})$ be the corresponding set of cohomology classes. A simple application of Riemann-Roch shows the following.

Lemma 3.2.1. *The set $\mathcal{B}$ is basis for $H^1_{\mathrm{dR}}(X/\mathbf{Q})$ whenever ∞ is not a Weierstrass point on X and u has a pole of order $t+1$ (i.e., the smallest possible) at ∞.* □

Proof. Since ∞ is not a Weierstrass point on X, we may assume that $\mathrm{ord}_\infty(\omega_i) = i-1$, and thus $\mathrm{ord}_\infty(\eta_i) = i-t-2$. For any differential of the second kind ω', we can find a linear combination of $\eta_1, \ldots, \eta_t$ and dh for an appropriate rational function h having the same principal part as ω'. Thus the difference is holomorphic, and lies in the span of $\{\omega_1, \ldots, \omega_t\}$. □

Remark 3.2.2. By a result of Ogg [O], the cusp ∞ is not a Weierstrass point when the level N is prime, or more generally when $N = pM$ for prime p and an integer $M \geq 1$ such that $X_0(M)$ has genus zero and $p \nmid M$. Even if u has a pole of order $> g(X) + 1$, the set $\mathcal{B}$ may still be a basis of $\mathrm{H}^1_{\mathrm{dR}}(X/\mathbf{Q})$. This can be checked by computing the matrix for the Poincare pairing, and in every example we have computed this is the case.

When ∞ is a Weierstrass point, there is a rational function with a single pole at ∞ of order $\leq g(X)$. When u is taken to be such a function, then the set $\mathcal{B}$ will never be a basis. Indeed, since ∞ is a Weierstrass point, there exists a holomorphic differential form ω with order of vanishing $\geq g(X)$ at ∞. Then $u\omega$ is still holomorphic, and thus lies in the span of $\{\omega_1, \ldots, \omega_t\}$. But $u\omega$ also is in the span of $\{\eta_1, \ldots, \eta_t\}$ by definition of the η_i, giving rise to a linear dependence relation. Hence, in order for $\mathcal{B}$ to be a basis, it is necessary for u to have a pole at ∞ of order greater than the order of vanishing at ∞ of any holomorphic differential.

Given one basis $\mathcal{B}$ for $H^1_{\mathrm{dR}}(X/\mathbf{Q})$ – for example, one computed as above – it is then a matter of linear algebra to produce a better basis which is adapted to the action of the Hecke algebra. Note that the usual formula for the action of $\mathbf{T}$ on holomorphic modular forms in terms of q-expansions extends to weakly holomorphic modular forms, such as 1-forms of the second kind on X, as well.[7] Thus, using q-series for the elements of the basis $\mathcal{B}$, we can write down the matrix $[T_p] \in \mathrm{Mat}_{2t\times 2t}(\mathbf{Q})$ which describes the action of any $T_p \in \mathbf{T}$ on de Rham cohomology. By finding the eigenspaces of finitely many such matrices[8] we can write down $\mathbf{Q}$-bases for each isotypic component of H^1. Using these it is elementary to produce the desired symplectic bases $\{\omega_{g,i}, \eta_{g,j}\}$ for each isotypic component $H^1_{\mathrm{dR}}(X/\mathbf{Q})[g]$.

3.3. Modular units and η-products. The preceding discussion raises the question of how to compute the rational function u used to write down an initial choice of basis $\mathcal{B}$ for $H^1_{\mathrm{dR}}(X/\mathbf{Q})$. To "compute u" means to compute its Laurent expansion about ∞.

Recall that the *modular units* U (for $\Gamma_0(N)$) are the multiplicative group of modular functions $u \in \mathbf{C}(X)^\times$ with divisor supported on the cusps of $X = X_0(N)$.

Definition 3.3.1. The *eta group* U_η is the group of rational functions $u \in \mathbf{Q}(X)$ of the form

$$u(q) = \prod_{0<d|N} \eta(q^d)^{r_d},$$

[7] FIXME: add reference?

[8] FIXME: Add a reference to the bound on the number of generators of $\mathbf{T}$ acting on modular forms of level N; must explain why a similar *effective* bound holds also for all of H^1. Remark that in practice only very small Hecke operators are required when N is small, but that using our strategy to write down an essentially "random" basis in terms of ωs and $u\omega$s, the rational numbers showing up even in the matrix of T_2 seem to grow complicated exponentially fast (as a function of N).

where $\eta(q) = q^{1/24}\prod_{n>0}(1-q^n)$ is the classical eta function, and $\{r_d\}_{d|N}$ is a collection of integers satisfying the following conditions.

i. $\sum_{d|N} r_d = 0$,
ii. $\prod_{d|N} d^{r_d} \in \mathbf{Q}^\times$ is a square,
iii. $(n_d) := A_N \cdot (r_d)$ is a vector (indexed by divisors d of N) of integers divisible by 24, where A_N is the $\sigma(N)\times\sigma(N)$-matrix whose entry indexed by (d,d') is $\frac{N\cdot(d,d')^2}{dd'(d',N/d')}$.

Work of Newman and Ligozat shows that such functions are indeed modular units on X; that is, $U_\eta \subset U$. In fact more is true:

Proposition 3.3.2. *$\mathbf{Q}\otimes U_\eta = \mathbf{Q}\otimes U$.*

Proof. It easy to see that the set $\{\frac{a}{d} : d \mid N, a \in (\mathbf{Z}/(d,N/d)\mathbf{Z})^\times\} \subset \mathbf{P}^1(\mathbf{Q})$ is a complete set of representatives of the cusps of X. The subspace $\mathbf{Q}\otimes U_\eta \subset \mathbf{Q}\otimes U$ coincides with $\mathbf{Q}\otimes U'$, where $U' \subset U$ consists of modular units which have the same valuation at any two cusps a/d, a'/d with the same denominator; cf. [G, Prop. 2]. This implies the proposition in light of the next lemma, since an element $u \in U \subset \mathbf{Q}(X)$ has the same valuation at any two Galois-conjugate cusps. □

Lemma 3.3.3. *Let $d|N$. Then the cusp $1/d$ is rational if and only if $(d,N/d) = 1$. More generally, the Galois orbit of the cusp $1/d$ is the set of cusps a/d with a relatively prime to $(d,N/d)$.*

Proof of the lemma. We prove the first statement using the results of [St, §1.3]. Namely, it is known that the cusps of X are rational over $\mathbf{Q}(\zeta_N)$, and the Galois action can be described explicitly as follows [St, Thm. 1.3.1]: if τ is the automorphism of $\mathbf{Q}(\zeta_N)$ which sends $\zeta_N \mapsto \zeta_N^n$ and $n' \in \mathbf{Z}$ is chosen so that $nn' \equiv 1 \pmod N$ then τ sends the cusp x/y to $x/n'y$. In particular, it follows straightforwardly that $[1/d]^\tau = [n'/d]$. Thus it suffices to prove that the cusps n'/d and $1/d$ coincide for all n' relatively prime to N, if and only if $(d,N/d) = 1$. This fact can be shown by an elementary argument using the conditions for the integer matrix sending $1/d$ to n'/d to be in $\Gamma_0(N)$. The second statement is proved similarly[9]. □

By the Riemann-Roch theorem, there exist nonconstant rational functions on X which are regular away from ∞. The proposition implies that an integer power of such a function belongs to the subgroup $U_\eta \subset U$, which yields the following.

Corollary 3.3.4. *There exists an eta product $u \in U_\eta$ which is regular away from ∞.* □

It is thus possible to compute the rational function u required in the compation of a basis for $H^1_{\mathrm{dR}}(X)$ as an eta product. A practical approach to finding the vector $(r_d)_{d|N}$ giving rise to the u we seek is to apply a mixed-integer linear programming algorithm: one minimizes the pole order $-n_N$ of u at ∞ subject to the criteria of Newman-Ligozat in Definition 3.3.1 and the condition that the orders n_d of u at other cusps are non-negative.

Remark 3.3.5. By minimizing the pole order of u at ∞, we may compute using the method of the previous subsection a basis $\mathcal{B}$ for the de Rham cohomology of X to a desired degree of precision using as few Fourier coefficients as possible for the cusp forms $\omega_1,\ldots,\omega_t$. It is desirable that this minimal pole order $-n_N$ equal $t+1$. This condition is relevant for the computation of $\alpha_{\omega_{g,i},\eta_{g,i}}$ (cf. §3.5.1), as well as to apply Lemma 3.2.1 from the previous subsection. Unfortunately it does not always hold; see the discussion in §3.5.1.

[9]FIXME: double check this!

3.4. Computing the Poincaré dual γ_f of ω_f. Assume that $\{\sigma_j\}$ is a collection of elements of $\Gamma_0(N)$ with small lower-left entries cN, whose homology classes s_j generate $H_1(X^{\mathrm{an}}, \mathbf{Z})$. By a brute-force search it is straightforward to find such elements σ_j in practice. (For small N, often one need take c no greater than 2 or 3.)

For any $m \in H_1(X^{\mathrm{an}}, \mathbf{C})$, write η_m for the Poincaré dual of c; conversely, for any differential η of the second kind on X, let $m_\eta \in H_1(X^{\mathrm{an}}, \mathbf{C})$ denote the Poincaré dual of its cohomology class. We normalize the Poincaré duality isomorphism so that it is characterized by the property

$$\langle \eta_m, \eta \rangle = \int_m \eta. \tag{3.4.1}$$

The vector space $H_1(X^{\mathrm{an}}, \mathbf{C})$ is also equipped an intersection product, which is related to the Poincaré pairing by the formula

$$m \cdot m_\eta = \frac{1}{2\pi i} \langle \eta_m, \eta \rangle.$$

The homology of X also admits a natural action of the Hecke algebra, compatible with the action on cohomology via Poincaré duality. For any normalized eigenform $f \in S_2(\Gamma_0(N))$ and any $m \in H_1(X^{\mathrm{an}}, \mathbf{C})$, write $m^f \in H_1(X^{\mathrm{an}}, \mathbf{C})[f]$ for the projection of f onto the f-isotypic component of homology. Simiarly, for $\eta \in H^1_{\mathrm{dR}}(X/\mathbf{Q})$ write η^f for its projection onto the f-isotypic component.

We can assume that via the method described above a symplectic basis

$$\mathcal{S} = \{\omega_{f,1}, \ldots, \omega_{f,n}, \eta_{f,1}, \ldots, \eta_{f,n}\}$$

for $H^1_{\mathrm{dR}}(X/\mathbf{Q})[f]$ has already been computed.

Lemma 3.4.1. *Fix γ_1, $\gamma_2 \in \Gamma_0(N)$ and let m_1, $m_2 \in H_1(X^{\mathrm{an}}, \mathbf{Z})$ denote the corresponding homology classes on X.For any normalized eigenform $f \in S_2(\Gamma_0(N))$, we have*

$$m_1^f \cdot m_2^f = \frac{1}{2\pi i} \sum_{i=1}^{n} I(\omega_{f,i}; m_1) I(\eta_{f,i}; m_2) - I(\omega_{f,i}; m_2) I(\eta_{f,i}; m_1),$$

where $\omega_f = 2\pi i f(z) \mathrm{d}z$ is the 1-form corresponding to f.

Proof. Let $\eta_k = \eta_{m_k}$ and write $\eta_k^f = \sum c_i^{(k)} \omega_{f,i} + \sum d_i^{(k)} \eta_{f,i}$. Then we compute $m_1^f \cdot m_2^f = \frac{1}{2\pi i} \langle \eta_1^f, \eta_2^f \rangle = \sum_i \frac{1}{2\pi i} (c_i^{(1)} d_i^{(2)} - c_i^{(2)} d_i^{(1)}) = \frac{1}{2\pi i} \sum_i (I(\omega_{f,i}; \gamma_1) I(\eta_{f,i}, \gamma_2) - I(\eta_{f,i}; \gamma_1) I(\omega_{f,i}; \gamma_2))$. □

Now assume f is the newform with rational Fourier coefficients which parametrizes the elliptic curve E, and as above denote by ω_f the corresponding holomorphic 1-form. Then the f-isotypic components of the homology and cohomology of X are two-dimensional. Write η_f for the "complementary" form of the second kind such that $\{\omega_f, \eta_f\}$ is a symplectic basis for $H^1_{\mathrm{dR}}(X/\mathbf{Q})$; in particular, $\langle \omega_f, \eta_f \rangle = 1$. Let γ_f^+ (resp. γ_f^-) be a generator of the plus (resp. minus) eigenspace of the f-isotypic component of $H_1(X^{\mathrm{an}}, \mathbf{Z})$ under the action of complex conjugation; these are unique up to sign. Since the splitting into plus and minus subspaces only takes place over $\mathbf{Z}[\frac{1}{2}]$, we have $\gamma_f^+ \cdot \gamma_f^- \in \pm 2^{\mathbf{N}}$; after adjusting the signs if necessary, we can assume that $\gamma_f^+ \cdot \gamma_f^- = 2^n$ for some $n \geq 0$. This determines the pair (γ_f^+, γ_f^-) up to a sign.

Let $\Omega_f^\pm = I(\omega_f, \gamma_f^\pm)$. Note that Ω_f^+ and Ω_f^- generate a lattice $\mathbf{Z}\Omega^+ + \mathbf{Z}\Omega^- \subset \mathbf{C}$ which is independent of the choice of the sign above, and is commensurable with the Néron lattice Λ_f of E. Finally, let

$$\gamma_f := \frac{1}{2^{n+1}\pi i} (\Omega_f^+ \gamma_f^- - \Omega_f^- \gamma_f^+) \quad \in H_1(X^{\mathrm{an}}, \mathbf{C})[f].$$

Lemma 3.4.2. *The homology class γ_f is Poincaré dual to the cohomology class of ω_f.*

Proof. We only need to check (3.4.1) for $\eta = \eta_f$, and $I(\eta_f; \gamma_f) = 1 = \langle \omega_f, \eta_f \rangle$ by 3.4.1. □

The preceding discussion reduces the computation of γ_f to finding the homology classes $\gamma_f^{\pm}$, from which the periods $\Omega_f^{\pm}$ are readily obtained by integrating. The classes $\gamma_f^{\pm}$ can be calculated using modular symbols, for which we refer to [S2].

3.5. Computing the adjustments $\int_{\gamma_f} \alpha$. At this point we are already able to compute

$$z_{g,f} := \sum_{i=1}^{k} \int_{\gamma_f} (\omega_{g,i} \cdot \eta_{g,i} - \eta_{g,i} \cdot \omega_{g,i}),$$

part of the righthand side of formula (2.3.1).

We describe two approaches for computing the difference $\Delta_{g,f} = P_{g,f} - z_{g,f}$.

3.5.1. The direct approach is to compute q-expansions for each 1-form $\alpha = \alpha_{\omega_{g,i},\eta_{g,i}}$ explicitly. We do not know how to do this in general, but the method we now describe works under the following hypothesis:

(†) The point ∞ is not a Weierstrass point, and the optimized eta product u of §3.3 has a pole of order $t + 1 = p_a(X) + 1$ at ∞.

Assume (†) holds. Then the meromorphic differentials $u\omega$ as $\omega \in H^0(X, \Omega^1_X)$ varies are all of the second kind, regular on Y, and have poles of all orders $2, 3, \ldots, t+1$ at ∞.

Recall that the defining property of α is that its principal part at ∞ agrees with that of $\xi = \omega_{g,i} F_{\eta_{g,i}}$ on $\tilde{X}$, modulo $\mathrm{d}q/q$; i.e., $\xi - \alpha$ has at worst logarithmic poles. The symplectic basis $\{\omega_{g,i}, \eta_{g,i}\}$ for $H^1_{\mathrm{dR}}(X/\mathbf{Q})[g]$ is obtained by applying linear operations to a basis $\mathcal{B}$ consisting of 1-forms which are either holomorphic or of the type $u\omega$ for holomorphic ω. In particular, ξ has poles of order $\leq t$ at the points lying over ∞. Thus there exists holomorphic ω such that $u\omega$ has exactly the same principal part as ξ, modulo $\mathrm{d}q/q$. So assuming (†), α can be computed explicitly knowing only Fourier expansions of u and of a basis for $S_2(\Gamma_0(N))$. The resulting explicit Laurent expansions for α can then be integrated over γ_f using the same approach discussed above for computing $z_{g,f}$, to find an approximation to the rational number $\Delta_{g,f} = P_{g,f} - z_{g,f}$. See §4.1 for an example computation of this sort.

Unfortunately, (†) is not always satisfied, but it does hold in some situations. For example, using the results of Newman and Lizogat combined with the formula for the genus of $X_0(N)$, it is an easy exercise to show that (†) holds if $N = p$ for a prime $p \equiv 1 \pmod{12}$, $N = 2q$ for a prime $q \equiv 1 \pmod 4$, or $N = 3r$ for a prime $r \equiv 1 \pmod 3$. On the other hand, it never holds if $N = p$ for a prime $p \not\equiv 1 \pmod{12}$.

3.5.2. When (†) fails, an alternative approach is required. One such method is to make an educated guess as to the value of $\Delta_{g,f}$. This also has the advantage of avoiding computationally expensive iterated integral evaluations. Note that $\Delta_{g,f} = \sum_i \langle \omega_f, \alpha_{\omega_{g,i},\eta_{g,i}} \rangle$ is a rational number because ω_f and each α are algebraic and defined over $\mathbf{Q}$. The method of lattice reduction is well-suited to guessing the value of an unknown rational number.

We now sketch how this might be done. First compute the elliptic logarithm $Q \in \mathbf{C}/\Lambda_f$ of a generator of the Mordell-Weil group $E(\mathbf{Q})$. (This can be done in various ways. We remark that the interest of computing the Chow-Heegner points $P_{g,f}$ is not as a tool for computing $E(\mathbf{Q})$, so appealing to an independent algorithm to obtain Q is not circular reasoning.) Next find a basis $\{b_1, b_2\}$ for Λ_f. Since $d_g P_{g,f}$ corresponds a rational point of E, some integer multiple of it must be a $\mathbf{Z}$-linear combination of b_1, b_2, and Q. Using LLL or another lattice reduction algorithm, find an approximate dependence relation

$$D d_g z_{g,f} + A_1 b_1 + A_2 b_2 + NQ + M = 0, \qquad A_1, A_2, D, M, N \in \mathbf{Z}.$$

There will be such a relation with $M/Dd_g = \Delta_{g,f}$, indicating that up to D-torsion in E, $d_g P_{g,f}$ maps to N times the chosen generator of $E(\mathbf{Q})$.

Unfortunately it is not easy in practice to compute $\Delta_{g,f}$ in this manner. The problem is that a prohibitively large degree of accuracy is usually necessary to identify the "correct" dependence relation as above, since in general the rational number $\Delta_{g,f}$ may have fairly large height. See §4.2 for an example.

3.6. Computing the denominator d_g. The final ingredient to be computed is the denominator d_g, or the smallest positive integer such that $d_g T_g \in \mathbf{T}_0$. This can be accomplished by computing the Hecke algebra in `SAGE`. Under the inclusion $\mathbf{T}_0 \hookrightarrow \mathbf{T}_0 \otimes \mathbf{Q} \cong \prod_g K_g$, the operator T_n is sent to the vector $(a_n(g))_g$. Since each eigenvalue is an algebraic integer, then the image lies in $\prod_g \mathcal{O}_{K_g}$. Therefore, $\mathbf{T}_0$ can be embedded (as a $\mathbf{Z}$-module) in $\mathbf{Z}^t$, where t is the genus of $X_0(N)$. The image of any Hecke operator T_n in $\mathbf{Z}^t$ can be computed by finding the image of $a_n(g)$ in $\mathcal{O}_{K_g}$ with respect to an integral basis.

It is well known that $\mathbf{T}_0$ is generated as a $\mathbf{Z}$-module by $\{T_n\}_{1\le n\le r,(n,N)=1}$, where $r = \lceil \frac{N}{6}\prod_{p|N}(1+\frac{1}{p})\rceil$; see for instance [S1]. Hence, the image of $\mathbf{T}_0$ as a submodule of $\mathbf{Z}^t$ can be computed by taking the submodule generated by the images of a finite number of Hecke operators. It is then a simple matter to find d_g for each newform g.

3.7. Remark on complexity. The complexity of the computations we have described is primarily determined by the *number* n of fourier coefficients required to compute $z_{g,f}$ (and also the correction $\Delta_{g,f}$, if using the method of §3.5.1) to a given number D of digits of accuracy. [10]

4. Numerical examples

4.1. Example: `37a1`. Take $N = 37$ in the setup of our algorithm. In this setting, the space of regular differentials on $X = X_0(37)$ is spanned by ω_f and ω_g, which are associated to elliptic curves over $\mathbf{Q}$ (labeled `37a1` and `37b1` in Cremona's database) of ranks 1 and 0, respectively.

By computing the periods attached to ω_f and ω_g, it can be checked that the classes of the matrices

$$\gamma_1 = \begin{pmatrix} 2 & -1 \\ 37 & -18 \end{pmatrix}, \gamma_2 = \begin{pmatrix} 3 & -1 \\ 37 & -12 \end{pmatrix}, \gamma_3 = \begin{pmatrix} 5 & 2 \\ 37 & 15 \end{pmatrix}, \gamma_4 = \begin{pmatrix} 14 & 3 \\ 37 & 8 \end{pmatrix}$$

generate the rational homology of X. These are a "nice" basis for the homology in the sense that the lower left entries are exactly 37 (rather than $37c$ for $|c| > 1$), so the integral $\int_\tau^{\gamma_i\tau}\lambda$ can be evaluated efficiently for any meromorphic differential 1-form on $\mathbf{H}$ or $X_0(37)$ which is regular away form ∞, using relatively few Fourier coefficients of λ.

If we denote by $[\gamma]$ the homology class attached to the group element γ, then

$$\begin{aligned} \gamma_f^+ &= \frac{-1}{2}[\gamma_2] + \frac{1}{2}[\gamma_3] - \frac{1}{2}[\gamma_4] \\ \gamma_f^- &= [\gamma_1] - 2[\gamma_2] \end{aligned}$$

generate the f-isotypic part of the integral homology of X. The superscripts indicate the eigenvalue of complex conjugation acting on the homology class.

To obtain differentials of the second kind representing classes in the deRham cohomology, we consider the elements of the form

$$\eta_1 = u \cdot \omega_f, \qquad \eta_2 = u \cdot \omega_g, \qquad \text{where } u = q^{-3}\prod_{n=1}^{\infty}(1-q^n)^2(1-q^{37n})^{-2}.$$

[10]FIXME: I still haven't figured out how to bound effectively the coefficients of u; waiting to hear back from Rob Rhoades about this. –S.L.

It is not hard to check (by calculating the periods along $\gamma_f^\pm$ and $\gamma_g^\pm$) that the classes of $\omega_f, \omega_g, \eta_1$ and η_2 generate the deRham cohomology of X. Furthermore, by finding the matrix M of the Hecke operator T_2 acting on $H^1_{\mathrm{dR}}(X_0(37))$ with respect to the basis $\omega_f, \omega_g, \eta_1, \eta_2$, and then determining the eigenspaces of M, one finds that

$$\begin{aligned}\eta_f &= \frac{1}{4}(-37\omega_g + 4\eta_1 - 8\eta_2),\\ \eta_g &= \frac{1}{4}(37\omega_f - 6\eta_1 + 10\eta_2)\end{aligned}$$

are in the f and g isotypic parts of the deRham cohomology respectively, and

$$\langle \omega_f, \eta_f\rangle = \langle \omega_g, \eta_g\rangle = 1.$$

When one computes the Poincaré dual γ_f of ω_f, one finds (with our normalization) that it is

$$\gamma_f = \frac{1}{2\pi i}\left(\Omega_E^-([\gamma_2] - [\gamma_3] + [\gamma_4]) - \Omega_E^+(-[\gamma_1] + 2[\gamma_2])\right).$$

Here

$$\Omega_E^+ \approx 2.993458646..., \qquad \Omega_E^- \approx (2.45138938...)i$$

are the real and imaginary periods of the elliptic curve E corresponding to f (labeled '37a1' in Cremona's database).

By computing with principal parts, one finds that $\mathrm{pral}_\infty(2\omega_g F_{\eta_g}) = \mathrm{pral}_\infty(\frac{1}{2}(\eta_1 - \eta_2))$ mod $\frac{dq}{q}$. Thus $\frac{1}{2}(\eta_1 - \eta_2) = \alpha_{\omega_g;\eta_g}$ and integrating this over γ_f yields -1 (to many digits of precision).

Computing the iterated integral $\int_{\gamma_f}(\omega_g \cdot \eta_g - \eta_g \cdot \omega_g)$ via the algorithm we have described yields the "raw" point

$$z_{g,f} = -1.40936100075... + (1.22569469099...)i.$$

Thus $P_{g,f} = z_{g,f} - \int_\gamma \alpha_{\omega_g;\eta_g} = 0.4093610075... + (1.22569469099...)i$.

Now $E(\mathbf{Q})$ is generated by the point $p = (0 : -1 : 1) \in \mathbf{P}^2(\mathbf{Q})$. The elliptic logarithm of this point in $\mathbf{C}/\Lambda_E$ is $P \approx 2.06386593094656... + (1.22569469099340...)i$.

By a lattice reduction algorithm one easily finds the linear dependence relation

$$2P_{g,f} - 8\Omega_E^+ - 7\Omega_E^- + 12P \approx 0$$

holds to at least 15 digits of accuracy. (In this example, all of the iterated integrals have been computed using 350 Fourier coefficients, and on a laptop comuter the entire computation finished in a matter of seconds.) This says that the image of $P_{g,f}$ in $E(\mathbf{C})$ is equal to $-6p = (6 : 14 : 1)$ modulo an irrational 2-torsion point. To explain the denominator "2" that has occured, let $\mathbf{T}_0$ denote the "anemic" Hecke algebra generated over $\mathbf{Z}$ by the Hecke operators T_p for $p \neq 37$. Then one can compute using the first few Fourier coefficients of f and g that the idempotent $e = (0, 1) \in \mathbf{Q} \times \mathbf{Q} \overset{(\star)}{\cong} \mathbf{T} \otimes \mathbf{Q}$ does not belong to $\mathbf{T} \subset \mathbf{T} \otimes \mathbf{Q}$ but $2e$ does so. (The identification $(\star)$ associates $T_p \otimes 1 \in \mathbf{T} \otimes \mathbf{Q}$ to $(a_p(f), a_p(g)) \in \mathbf{Q} \times \mathbf{Q}$.) Thus in fact the point we have denoted $P_{g,f}$ is the wrong thing; the true triple Chow-Heegner point on E associated to (g, g, f) is $2P_{g,f}$ which is the rational point $-12P = (1357/841 : 28888/24389 : 1)$.

4.2. Example: `43a1`. Let $N = 43$ and let E be the elliptic curve labeled `43a1` in Cremona's database. The modular curve $X = X_0(43)$ has genus 3. There are two isotypic components of $H^1_{\mathrm{dR}}(X)$, one of dimension 2 corresponding to the modular form f which parametrized E, and another of dimension 4 corresponding to a newform g with Fourier coefficients in $\mathbf{Q}(\sqrt{2})$, associated to an abelian surface quotient of $J_0(43)$.

In this case, a linear programming algorithm identifies the eta-quotient u which is modular for $\Gamma_0(47)$ of weight 0, holomorphic away from the cusp ∞, and with minimal pole order at ∞, as

$$u = \frac{\eta(q)^4}{\eta(q^{43})^4} = q^{-7} - 4q^{-6} + 2q^{-5} + 8q^{-4} - 5q^{-3} - 4q^{-2} - 10q^{-1} + 8 + 9q + 14q^3 + O(q^4).$$

Since this has a pole of order $7 > 3 + 1 = 4$ at ∞, u is not optimal for the purpose of our computations.

Nonetheless, computing the residue pairing shows that for a basis of cuspforms with rational Fourier coefficients, corresponding to holomorphic 1-forms $\omega_f, \omega_{g,1}, \omega_{g,2}$ on X, the collection

$$\omega_f, \omega_{g,1}, \omega_{g,2}, u\omega_f, u\omega_{g,1}, u\omega_{g,2}$$

forms a basis for $H^1_{\mathrm{dR}}(X/\mathbf{Q})$. By finding the matrices of a few Hecke operators with respect to this basis, one can as in the case $N = 37$ produce symplectic bases

$$\omega_f, \eta_f, \qquad \text{and} \qquad \omega_{g,1}, \omega_{g,2}, \eta_{g,1}, \eta_{g,2}$$

for $H^1_{\mathrm{dR}}(X)[f]$ and $H^1_{\mathrm{dR}}(X)[g]$ respectively.

While we can compute the Poincaré dual γ_f and the "raw" point

$$z_{g,f} = \sum_{i=1}^{2} \int_{\gamma_f} (\omega_{g,i} \cdot \eta_{g,i} - \eta_{g,i} \cdot \omega_{g,i}) \approx -1.1460154... + (2.726364836...)i,$$

the fact that u has such a large pole at ∞ prevents us from being able to find the 1-forms $\alpha_{\omega_{g,i};\eta_{g,i}}$ on $\tilde{Y}$. Nonetheless, we can use lattice reduction to try to see whether $z_{g,f}$ differs from (the elliptic logarithm of) a rational point of E by an adjustment factor in $\mathbf{Q}$. Actually, we should first scale $z_{g,f}$ by the "denominator" of the idempotent $e \in \mathbf{T} \otimes \mathbf{Q}$ which projects onto the g-isotypic component when viewed as an operator on $H^1(X)$; as in the case $N = 37$ a computation with Hecke eigenvalues of g and f reveals that the denominator of e is 2.

The elliptic curve E again has rank 1, generated by the point $p = (0 : -1 : 1)$, with elliptic logarithm $P \approx 1.53155105...$. Lattice reduction reveals the linear dependence (to at least 58 digits of accuracy, the previous computations having been done with 1200 Fourier coefficients):

$$(2z_{g,f}) + 5\Omega_E^+ - 4\Omega_E^- - 8P + \frac{1847467}{1984785} \approx 0.$$

The fact that so many Fourier coefficients were necessary to obtain this relation reflects the fairly large prime factor occuring in the denominator

$$1984785 = 3 \times 5 \times 11 \times 23 \times 523.$$

When the differentials $\eta_{g,i}$ are expanded with respect to a basis for $H^1_{\mathrm{dR}}(X/\mathbf{Q})$ consisting of differentials holomorphic away from ∞ and with integral Laurent expansions about ∞ – which thus have integral period over γ_f – the prime factors above arise in the denominators of the coefficients. One thus expects these primes to occur in the denominators of the forms $\alpha_{\omega_{g_i},\eta_{g_i}}$.

5. Table

Here we report the triple Chow heegner points which lie on a number of strong Weil curves of conductor < 200. With the exception of the curves labeled `43a1` and `65a1` in Cremona's database, we report only in those cases where the differnetials $\alpha_{\omega_{g,i},\eta_{g,i}}$ can be computed explicity and integrated using the method explained in §3.5.1. The exceptional cases were computed using the method of §3.5.2.

The format of the table is as follows. Let N be the conductor of the curve E in question. We report the points as elements of $E(\mathbf{Q}) \otimes \mathbf{Q}$. They are ordered according to the ordering of the isotypic components of the space $\mathbf{S}_2(\Gamma_0(N))$ of cuspidal modular symbols for $\Gamma_0(N)$, as listed via the command `ModularSymbols(N).cuspidal_subspace().decomposition()` in `SAGE`. In this table, the newform f which parametrizes E usually corresponds to the 0th isotypic component in this list; [11] the Chow-Heegner points $P_{g,f}$ correspond to the remaining components in the order listed below (via g is an eigenform in the corresponding component). The $-\otimes\mathbf{Q}$ factor accounts for the denominator of the projector onto the corresponding isotypic component of the anemic Hecke algebra, as explained above. All the curves E in the list are optimal and have rank 1; they are listed by their Cremona label. The point P is the generator of $E(\mathbf{Q})$ output by `SAGE`. The hecke index is the denominator d_g of the projector onto the corresponding isotypic component of the Hecke algebra $/\mathbf{Q}$.

The last two columns indicate the number of Fourier coefficients used in the computation and a lower bound for number of decimal digits of accuracy to which each of the points we have computed (after adjusting by the periods of the $\alpha_{\omega_{g,i},\eta_{g,i}}$) agree (modulo Λ_E) with the indicated elements of $E(\mathbf{Q})\otimes\mathbf{Q}$.

5.1. Discussion. Remark on the low accuracy for 88a1 (this is because on the homology generators has 1/176 rather than 1/88 in the lower left).

Explain conditions for the points to be torsion, check them when possible.

Explain why quadratric twists switch the sign of the points.

If we guess that clearing denominators of the CH-points exactly accounts for the true Hecke denominators, then the CH-points generate subgroups of index

$$6, 4, 4, 4, 2, 1, 2, \infty, \infty, 2, 1, ?, \infty, 1, \infty, 4, \infty, 2.$$

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[11]Exceptions: for E = `153b1`, E corresponds to the 1st isotypic component, and the listed CH points correspond to components labels 0,2,3,4,5,6,7; similarly, for E = `171b1`, E corresponds to the 1st isotypic component, and the listed CH points corresond to components labeled 0,2,3,4,5,6,7,8. FIXME: What about `92b1`, `106b1`, `176c1`, `184b1`?

Table 1. Some elliptic curves with non-torsion triple Chow Heegner points

Curve $E = E_f$	generator $P \in E(\mathbf{Q})$	$P_{g,f}$ (mod torsion)	d_g	$[E(\mathbf{Q}) : \mathbf{Z} d_g P_{g,f}]$	accuracy and # coeff.	
37a1	(0,-1)	$-6P$	2	12	54	800
43a1	(0,-1)	$4P$	2	8	58	1200
57a1	(2,1)	$\frac{4}{3}P$	12	16	31	800
		$-\frac{16}{3}P$	3	16		
		$-4P$	2	8		
58a1	(0,-1)	$4P$	4	16	32	800
		0	2	∞		
61a1	(1,-1)	$-2P$	2	4	32	800
65a1	(-1,1)	P	2	2	55	3200
		P	2	2		
82a1	(0,0)	0	4	∞	33	1200
		$2P$	2	4		
99a1	(2,0)	$-\frac{2}{3}P$	12	8	37	1600
		0	12	∞		
		$\frac{2}{3}P$	6	4		
		$\frac{2}{3}P$	12	8		
		$-\frac{2}{3}P$	6	4		
106b1	(2,1)	$\frac{12}{5}P$	10	24	38	1800
		$-\frac{4}{3}P$	6	8		
		$-\frac{11}{3}P$	48	176		
		P	16	16		
		$\frac{28}{5}P$	10	56		
122a1	(1,-3)	$-\frac{16}{13}P$	26	32	28	1600
		$-P$	16	16		
		P	16	16		
		$-\frac{36}{13}P$	26	72		
129a1	(1,-5)	$-\frac{16}{15}P$	15	16	28	1600
		$-\frac{4}{3}P$	12	16		
		$-\frac{20}{7}P$	14	40		
		$-\frac{8}{5}P$	40	64		
		$-\frac{8}{7}P$	14	16		
153a1	(0,1)	P	48	48	24	1800
		$2P$	24	48		
		0	24	∞		
		$-P$	48	48		
		$-P$	16	16		
		$-2P$	24	48		
		P	16	16		

Table 2. Some elliptic curves with only torsion Chow-Heegner points

Curve $E = E_f$	generator $P \in E(\mathbf{Q})$	$P_{g,f}$	d_g	$E(\mathbf{Q})_{\text{tor}}$	accuracy and # coeff.		reason
33a1	rank 0	0	3	$\mathbf{Z}/2 \oplus \mathbf{Z}/2$	53	800	OK
34a1	rank 0	$(2, -3)$[a]	2	$\mathbf{Z}/6$	56	800	OK
35a1	rank 0	$(1, -4)$[a]	2	$\mathbf{Z}/3$	54	800	OK
88a1	(2,-2)	3 points, all zero	-[b]	0	19	1600	()
92b1	(1,1)	3 points, all zero	-[b]	0	30	1200	()
112a1	(0,-2)	4 points, all zero $\frac{1}{2}\cdot$(irrational 2-torsion point)[c]	-[b] 16	$\mathbf{Z}/2$	23	2400	()
117a1	$(-\frac{1}{4}, \frac{15}{8})$	4 points, all zero	-[b]	$\mathbf{Z}/4$	23	1200	()
124a1	(-2,1)	4 points, all zero	-[b]	0	30	1600	()
136a1	(2,-2)	6 points, all zero	-[b]	$\mathbf{Z}/2$	29	3600	()
148a1	(-1,2)	5 points, all zero	-[b]	0	25	1600	()
152a1	(-1,-2)	6 points, all zero	-[b]	0	10	1200	
153b1	(5,-14)	7 points, all zero	-[b]	0	25	1800	
171b1	(2, -5)	8 points, all zero	-[b]	0	15	1200	
172a1	(-3,6)	5 points, all zero	-[b]	$\mathbf{Z}/3$	18	1400	()
176c1	(1,-2)	7 points, all zero	-[b]	0	9	1600	
184a1	(0,1)	8 points, all zero	-[b]	0	30	2400	

[a] 3-torsion point
[b] various Hecke indices d_g depending on g
[c] Namely, "$(1, 1)$" if we identify $\mathbf{E}[2](\mathbf{C}) \cong \mathbf{Z}/2 \oplus \mathbf{Z}/2$.

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