Numerical Computation of Chow-Heegner Points

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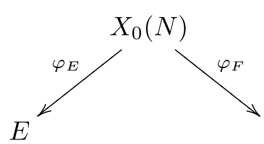
Abstract

In this note, we consider a special case of Chow-Heegner points that has a simple concrete description due to Shouwu Zhang. Given a pair E, F of nonisogenous elliptic curves, and a fixed choice of surjective morphisms $\varphi_E : X_0(N) \to E$ and $\varphi_F : X_0(N) \to F$ of curves over $\mathbb{Q}$, we associate a rational point $P \in E(\mathbb{Q})$. We describe a relatively elementary numerical approach to computing P, state some motivating results of Zhang et al. about the height of P, and present a table of data.

1 Introduction: Zhang's Construction

Consider a pair E, F of nonisogenous elliptic curves over $\mathbb{Q}$ and fix surjective morphisms from $X_0(N)$ to each curve. We do *not* assume that N is the conductor of either E or F, though N is necessarily a multiple of the conductor.



Let $(\varphi_E)_* : \mathrm{Div}(X_0(N)) \to \mathrm{Div}(E)$ and $\varphi_F^* : \mathrm{Div}(F) \to \mathrm{Div}(X_0(N))$ be the pushforward and pullback maps on divisors on algebraic curves. Let $Q \in F(\mathbb{C})$ be any point, and let

$$P_{\varphi_E,\varphi_F,Q} = \sum (\varphi_E)_* \varphi_F^*(Q) \in E(\mathbb{C}),$$

where $\sum$ means the sum of the points in the divisor using the group law on E, i.e., given a divisor $D = \sum n_i P_i \in \mathrm{Div}(E)$, we have $(\sum D) - \infty \sim D - \deg(D)\infty$, which uniquely determines $\sum D$.

Proposition 1.1. *The point $P_{\varphi_E,\varphi_F,Q}$ does not depend on the choice of Q.*

Proof. The composition $(\varphi_E)_* \circ \varphi_F^*$ induces a homomorphism of elliptic curves

$$\psi : \mathrm{Pic}^0(F) = \mathrm{Jac}(F) \to \mathrm{Jac}(E) = \mathrm{Pic}^0(E).$$

Our hypothesis that E and F are nonisogenous implies that $\psi = 0$. We denote by $[D]$ the linear equivalence class of a divisor in the Picard group. If $Q' \in F(\mathbb{C})$ is another point, then under the above composition of maps,

$$[Q - Q'] \mapsto [(\varphi_E)_* \varphi_F^*(Q) - (\varphi_E)_* \varphi_F^*(Q')] = [P_Q - P_{Q'}].$$

Thus the divisor $P_Q - P_{Q'}$ is linearly equivalent to 0. But F has genus 1, so there is no rational function on F of degree 1, hence $P_Q = P_{Q'}$, as claimed. □

We let $P_{\varphi_E,\varphi_F} = P_{\varphi_E,\varphi_F,Q} \in E(\mathbb{C})$, for any choice of Q. When there is a canonical choice of φ_E, φ_F, e.g., when E and F are both are optimal curves of the same conductor N, then we write $P_{E,F} = P_{\varphi_E,\varphi_F}$.

Corollary 1.2. *We have* $P_{\varphi_E,\varphi_F} \in E(\mathbb{Q})$.

Proof. Taking $Q = \mathcal{O} \in F(\mathbb{Q})$, we see that the divisor $(\varphi_E)_* \circ \varphi_F^*(Q)$ is rational, so its sum is also rational. □

1.1 Outline

In Section 2 we discuss in detail a point $P_{E,F}$, where E is the rank 1 curve of conductor 37. Section 3 is about a recent Gross-Zagier style formula of Yuan-Zhang-Zhang for the height of $P_{E,F}$ in terms of the derivative of an L-function, in some cases. In Section 4 we discuss the connection between the present paper and [DDLR11], which is about computing Chow-Heegner points using iterated integrals. The heart of the present paper is Section 5, which describes our numerical approach to approximating $P_{E,F}$. Finally, Section 6 presents extensive tables of points $P_{E,F}$.

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2 Example: $N = 37$

The smallest conductor curve of rank 1 is the curve E with Cremona label **37a** (see [Cre]). The beautiful paper [MSD74] discusses the modular curve $X_0(37)$ in detail. In particular, it presents the affine equation $y^2 = -x^6 - 9x^4 - 11x^2 + 37$ for $X_0(37)$, and describes explicitly how $X_0(37)$ is equipped with three independent involutions w, T, and S. The quotient of $X_0(37)$ by w is E, the quotient by T is an elliptic curve F with $F(\mathbb{Q}) \approx \mathbb{Z}/3\mathbb{Z}$ and Cremona label **37b**, and the quotient by S is the projective line $\mathbb{P}^1$.

$$\begin{array}{ccccc} & & X_0(37) & & \\ & \swarrow^{\varphi_E} & \downarrow{\scriptstyle \varphi_F} & \searrow & \\ E = X/w & & F = X/T & & \mathbb{P}^1 = X/S \end{array}$$

The maps φ_E and φ_F both have degree 2, since they are induced by quotienting out by an involution. As explained in [MSD74], the fiber over $Q = 0 \in F(\mathbb{Q})$ contains 2 points:

1. the cusp $[\infty] \in X_0(37)(\mathbb{Q})$, and
2. the noncuspidal affine rational point $(-1, -4) = T(\infty) \in X_0(37)(\mathbb{Q})$.

We have $\varphi_E([\infty]) = 0 \in E(\mathbb{Q})$ and [MSD74, Prop. 3, pg. 30] explains that

$$\varphi_F((-1, -4)) = (6, 14) = -6(0, -1),$$

where $(0, -1)$ generates $E(\mathbb{Q})$. We conclude that

$$P_{E,F} = (6, 14) \qquad \text{and} \qquad [E(\mathbb{Q}) : \mathbb{Z}P_{E,F}] = 6.$$

On [MSD74, pg. 31], they remark: "It would be of the utmost interest to link this index [the index of 6 above] to something else in the theory."

This remark motivates our desire to compute many more examples. Unfortunately, it is very difficult to generalize the above approach directly, since it involves computations with $X_0(37)$ and its quotients that rely heavily on having simple explicit defining equations. Just as there are multiple complementary approaches to computing Heegner points, there are several approaches to computing $P_{E,F}$:

- Section 3: a Gross-Zagier style formula for the height of $P_{E,F}$,
- Section 4: explicit evaluation of iterated integrals, and
- Section 5: numerical approximation of the fiber in the upper half plane over a point on an elliptic curve using a polynomial approximation to φ_F.

The present paper is mainly about the last approach listed above.

3 The Formula of Yuan-Zhang-Zhang

Consider the triple product L-function of [GK92]

$$L(E, F, F, s) = L(E, s) \cdot L(E, \mathrm{Sym}^2(F), s), \tag{1}$$

where E and F are elliptic curves of the same conductor N. For simplicity, in this section all L-functions are normalized so that $1/2$ is the center of the critical strip. The following theorem is proved in [YZZ11]:

Theorem 3.1 (Yuan-Zhang-Zhang)**.** *Assume that the local root number of* $L(E, F, F, s)$ *at every prime of bad reduction is* $+1$ *and that the root number at infinity is* -1*. Then* $\hat{h}(P_{E,F}) = (*) \cdot L'(E, F, F, \frac{1}{2})$*, where* $(*)$ *is nonzero.*

Remark 3.2. The above formula resembles the Gross-Zagier formula

$$\hat{h}(P_K) = (*) \cdot (L(E/\mathbb{Q}, s) \cdot L(E^K/\mathbb{Q}, s))'|_{s=\frac{1}{2}},$$

where K is a quadratic imaginary field satisfying certain hypotheses.

Unfortunately, at present it appears that nobody has implemented a computer program to evaluate the formula of Theorem 3.1 numerically in any interesting cases yet. If one could evaluate $L'(E, F, F, \frac{1}{2})$, along with the factor $(*)$ in the theorem, this would yield an algorithm to compute $\pm P_{E,F} \pmod{E(\mathbb{Q})_{\mathrm{tor}}}$ in the special case when N is squarefree and all roots numbers for E are $+1$. We hope to carry out this approach (using [Dok04]) in future work.

We have the following proposition that we can apply in specific examples. This follows from [GK92, §1], which implies that the local root number of $L(E, F, F, s)$ at p is the same as the local root number of E at p when the level is square free (computing the local root number when the level is not square free is more complicated).

Proposition 3.3. *Assume that* E *and* F *have the same conductor* N*, that* N *is square free, that the local root numbers of* E *at primes* $p \mid N$ *are all* $+1$ *(equivalently, that we have* $a_p(E) = -1$*) and that* $r_{\mathrm{an}}(E/\mathbb{Q}) = 1$*. Then* $L(E, \mathrm{Sym}^2 F, \frac{1}{2}) \neq 0$ *if and only if* $\hat{h}(P_{E,F}) \neq 0$*.*

Proof. By hypothesis, we have $L(E, \frac{1}{2}) = 0$ and $L'(E, \frac{1}{2}) \neq 0$. Theorem 3.1 and the factorization (1) then imply that

$$\hat{h}(P_{E,F}) = (*) \cdot L'(E, \frac{1}{2}) \cdot L(E, \mathrm{Sym}^2 F, \frac{1}{2}),$$

from which the result follows. □

Section 6 contains numerous examples in which E has rank 1, F has rank 0, and yet $P_{E,F}$ is a torsion point. The first example is when E is **91b** and F is **91a**. Then $P_{E,F} = (1, 0)$ is a torsion point (of order 3). In this case, we cannot apply Proposition 3.3 since $\varepsilon_7 = \varepsilon_{13} = -1$ for E. Another example is when E is **99a** and F is **99c**, where we have $P_{E,F} = 0$, and $\varepsilon_3 = \varepsilon_{11} = +1$, but Proposition 3.3 does not apply since the level is not square free. Fortunately, there is an example with squarefree level $158 = 2 \cdot 79$: here E is **158b**, F is **158d**, we have $P_{E,F} = 0$ and $\varepsilon_2 = \varepsilon_{79} = +1$, so Proposition 3.3 implies that $L(E, \mathrm{Sym}^2 F, \frac{1}{2}) = 0$.

4 Iterated Complex Path Integrals

The paper [DDLR11] contains a general approach using iterated path integrals to compute certain Chow-Heegner points, of which $P_{E,F}$ is a specific instance.

Comparing our data (Section 6) with theirs, we find that if E and F are optimal elliptic curves over $\mathbb{Q}$ of the same conductor $N \leq 100$, if $e, f \in S_2(\Gamma_0(N))$ are the corresponding newforms, and if $P_{f,e,1} \in E(\mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{Q}$ the associated Chow-Heegner point in the sense of [DDLR11], then $2P_{E,F} = P_{f,e,1}$. This is (presumably) a consequence of [DRS11].

5 A Numerical Approach to Computing $P_{E,F}$

The numerical approach to computing P that we describe in this section is easy to explain and implement and uses little abstract theory. It is inspired by work of Delaunay (see [Del02]) on computing the fiber of the map $X_0(389) \to E$, over rational points on the rank 2 curve E of conductor 389. We make no guarantees about how many digits of our approximation to $P_{E,F}$ are correct, instead viewing this as an algorithm to produce something useful for experimental mathematics.

Let $\mathfrak{h}$ be the upper half plane, and let $Y_0(N) = \Gamma_0(N)\backslash\mathfrak{h} \subset X_0(N)$ be the affine modular curve. Let E and F be nonisogenous optimal elliptic curve quotients of $X_0(N)$, with modular parametrization maps φ_E and φ_F, and assume both Manin constants are 1. Let Λ_E and Λ_F be the period lattices of E and F, so $E \cong \mathbb{C}/\Lambda_E$ and $F \cong \mathbb{C}/\Lambda_F$. Viewed as a map $[\tau] \mapsto \mathbb{C}/\Lambda_E$, we have (using square brackets to denote equivalence classes),

$$\varphi_E([\tau]) = \left[\sum_{n=1}^{\infty} \frac{a_n}{n} e^{2\pi i n\tau}\right],$$

and similarly for φ_F. This is explained in [Cre97, §2.10], which uses the oppositive convention for the sign. Here $a_n = a_n(E)$ are the L-series coefficients of E, so for good primes p, we have $a_p = p+1-\#E(\mathbb{F}_p)$. For any positive integer B, define the polynomial

$$\varphi_{E,B} = \sum_{n=1}^{B} \frac{a_n}{n} T^n \in \mathbb{Q}[T],$$

and similarly for $\varphi_{F,B}$.

To compute $P_{E,F}$, we proceed as follows. First we make some choices, and after making these choices we run the algorithm, which will either find a "probable" numerical approximation to $P_{E,F}$ or fail.

- $y \in \mathbb{R}_{>0}$ – minimum imaginary part of points in fiber in upper half plane.
- $d \in \mathbb{Z}_{>0}$ – degree of the first approximation to φ_F in Step 1.
- $r \in \mathbb{R}_{\neq 0}$ – real number specified to b bits of precision that defines $Q \in \mathbb{C}/\Lambda$.
- b' – bits of precision when dividing points into $\Gamma_0(N)$ orbits.
- n – number of trials before we give up and output FAIL.

We compute $P_{\varphi_E,\varphi_F,Q}$ using an approach that will always fail if Q is a ramification point. Our algorithm will also fail if any points in the fiber over

Q are cusps. This is why we do not allow $r = 0$. It is possible to modify the algorithm to usually work when Q is a torsion point, by using modular symbols and keeping track of images of cusps.

To increase our confidence that we have computed the right point $P_{E,F}$, we often carry out the complete computation with more than one choice of r.

1. **Low precision roots:** Compute all complex double precision roots of the polynomial $\varphi_{F,d} - r = 0$. One way to do this is to use "balanced QR reduction of the companion matrix", as implemented in GSL.[1] Record the roots that correspond to $\tau \in \mathfrak{h}$ with $\mathrm{Im}(\tau) \geq y$.

2. **High precision roots:** Compute an integer B such that if $\mathrm{Im}(\tau) \geq y$, then
$$\left| \sum_{n=B+1}^{\infty} \frac{a_n(F)}{n} \tau^n \right| < 2^{-b},$$
where b is the number of bits of precision of r. Explicitly, by summing the tail end of the series and using that $|a_n| \leq n$ (see [GJP$^+$09, Lem. 2.9]), we find that
$$B = \left\lceil \frac{\log(2^{-(b+1)} \cdot (1 - e^{-2\pi y_1}))}{-2\pi y} \right\rceil$$
works. Next, compute the polynomial $\varphi_{F,B} \in \mathbb{Q}[T]$, and use Newton iteration to refine all roots saved in Step 1 to roots α of $f = \varphi_{F,B} - r \in \mathbb{R}[T]$ to b bits of precision, i.e., so that $|f(\alpha)| < 2^{-b}$. Do this by iteratively replacing α by $\alpha - \frac{f(\alpha)}{f'(\alpha)}$, and save those roots that correspond to $\tau \in \mathfrak{h}$ with $\mathrm{Im}(\tau) \geq y$.

3. $\Gamma_0(N)$**-orbits:** Divide the τ's from Step 2 into $\Gamma_0(N)$-equivalence classes, testing equivalence to some bit precision $b' \leq b$. To test $\Gamma_0(N)$ equivalence of two points τ_1 and τ_2, we first decide whether or not they are $\mathrm{SL}_2(\mathbb{Z})$ equivalent, using the standard fundamental domain, then if they are equivalent, we check whether an explicit transformation matrix is in $\Gamma_0(N)$ – see Section 5.1 below for details. It is easy to efficiently compute the modular degree $m_F = \deg(\varphi_F)$ (see [Wat02]). If we find m_F distinct $\Gamma_0(N)$ classes of points, we suspect that we have found the fiber over $[r]$, so we map each element of the fiber to E using φ_E and sum, then apply the elliptic exponential to obtain $P_{E,F}$ to some precision, then output this approximation and terminate. If we find more than m_F distinct classes, that indicates an error in the choices of precision in our computation, so we increase b or possibly decrease b'.

4. **Try again:** We did not find enough points in the fiber. Systematically replace r by $r + m\Omega_F$, for $m = 1, -1, 2, -2, \ldots$ and Ω_F the least real period

[1] GSL is the the GNU scientific library, which is part of Sage [S$^+$11]. Rough timings of GSL for this computation: it takes less than a half second for degree 500, about 5 seconds for degree 1000, about 45 seconds for degree 2000, and several minutes for degree 3000.

of F, then try again going to Step 1 and including the new points found. If upon trying n choices $r + m\Omega_F$ we find no new points at all, we output FAIL and terminate the algorithm.

5.1 Determining $\Gamma_0(N)$ equivalency

To determine numerically if two points z_1 and z_2 in the upper half plane $\mathfrak{h}$ are equivalent modulo the action of $\Gamma_0(N)$, we first determine whether or not z_1 and z_2 are equivalent modulo $\mathrm{SL}_2(\mathbb{Z})$ using the standard fundamental domain, as explained in [Cre97, §2.14]. If z_1 and z_2 are not $\mathrm{SL}_2(\mathbb{Z})$ equivalent, then they are not $\Gamma_0(N)$ equivalent and we are done.

If z_1 and z_2 are $\mathrm{SL}_2(\mathbb{Z})$ equivalent, then the algorithm mentioned above also produces explicit elements $g_1, g_2 \in \mathrm{SL}_2(\mathbb{Z})$ such that $g_1(z_1) = g_2(z_2)$ is in the standard fundamental domain. Let $g = g_2^{-1} g_1$, so $g(z_1) = z_2$. Suppose h is any other matrix in $\mathrm{SL}_2(\mathbb{Z})$ such that $h(z_1) = z_2$ as well. Then

$$(h^{-1}g)(z_1) = h^{-1}(g(z_1)) = h^{-1}(z_2) = z_1,$$

so $h^{-1}g$ fixes z_1. Assume that $k = h^{-1}g \neq 1$, viewed as elements of $\mathrm{PSL}_2(\mathbb{Z})$. Then k has a fixed point in the upper half plane. The only elements of $\mathrm{PSL}_2(\mathbb{Z})$ with a fixed point in the upper half plane are $\mathrm{Stab}(z)$, where

- $z = i$, so $\mathrm{Stab}(z)$ is generated by $S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$.
- $z = \rho = \exp(2\pi i/3)$ so $\mathrm{Stab}(z)$ is generated by ST, where $T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$.
- $z = -\overline{\rho} = \exp(\pi i/3)$, so $\mathrm{Stab}(z)$ is generated by TS.

Assume that none of the 3 above are the case. Then $g = h$, so there is a matrix in $\Gamma_0(N)$ that sends z_1 to z_2 if and only if $g \in \Gamma_0(N)$, since g is the unique matrix in $\mathrm{SL}_2(\mathbb{Z})$ that sends z_1 to z_2. In the other cases, we check the following:

- $z = i$: check that neither of g, gS are in $\Gamma_0(N)$
- $z = \rho$: check that neither of g, gST, $g(ST)^2$ are in $\Gamma_0(N)$
- $z = -\overline{\rho}$: check that neither of g, gTS, $g(TS)^2$ are in $\Gamma_0(N)$.

6 Data

The columns of the tables in the rest of this section are as follows. The columns labeled E and F contain Cremona labels for elliptic curves, and those labeled r_E and r_F gives the corresponding ranks. The column labeled $E(\mathbb{Q})$ gives a choice of generators $P_1, P_2, \ldots$ for the Mordell-Weil group as explicit points, with r_E points of infinite order listed first, then 0,1, or 2 torsion points listed with a subscript of their order. The column labeled $P_{E,F}$ contains a near rational point (see below) to the numerically computed Chow-Heegner point, represented in terms of the generators P_i from the column labeled $E(\mathbb{Q})$, where P_1 is the first generator, P_2, the second, and so on. The columns labeled m_E and m_F give the modular degrees of E and F. The column labeled ε's contains the local root numbers of $L(E, s)$ at each bad prime. The notes column refers to the notes after the table, which give information about the input parameters needed to compute $P_{E,F}$.

We believe that the values of $P_{E,F}$ are "likely" to be correct, but we emphasize that **they are not proven correct**. In the table we give an exact point, but the algorithm computes a numerical approximation $\tilde{P}_{E,F}$ to $P_{E,F} \in E(\mathbb{Q})$. We find the exact point by running through several hundred points in $E(\mathbb{Q})$ and finding the one closest to $\tilde{P}_{E,F}$.

The table contains *every* pair E, F of nonisogenous optimal elliptic curves of the same conductor $N \leq 184$ with $r_E = 1$, and most curves for $N \leq 250$. It also contains a few additional miscellaneous examples, e.g., with $r_E = 0$ and some of larger conductor with $r_F = 2$. Most rows took only a few seconds to compute, though ones with m_F large in some cases took much longer; the total CPU time to compute the entire table was less than 10 hours. Unless otherwise noted, we used $y = 10^{-4}$, $d = 500$, $b' = 20$, and $r = 0.1$ with 53 bits of precision, as in Section 5. We also repeated all computations with at least one additional value of $r \neq 0.1$, and in every case got the same result (usually we used $r = 0.2$).

6.1 Discussion

One numerical observation in the table is that in every case $2 \mid [E(\mathbb{Q})_{/\,\mathrm{tor}} : \mathbb{Z}P_{E,F}]$. This can likely be proved in some cases by using that $r_{\mathrm{an}}(E) = 1$ implies that the sign in the functional equation for $L(E, s)$ is -1, so at least one nontrivial Atkin-Lehner involution W_q acts as $+1$ on E, which means that the map $X_0(N) \to E$ factors through $X_0(N) \to X_0(N)/W_q$.

There are four cases in which the index $[E(\mathbb{Q})/_{\mathrm{tor}} : \mathbb{Z}P_{E,F}]$ is divisible by a prime $\ell \geq 5$. They are (**106b**, **106c**, $\ell = 11$), (**118a**, **118d**, $\ell = 7$), (**121b**, **121d**, $\ell = 7$), and (**158b**, **158c**, $\ell = 7$). In each case, the prime divisor ℓ of the index does not appear to have anything to do with the invariants of E and F, individually.

E	ε_p's	r_E	$E(\mathbb{Q})$	m_E	F	r_F	m_F	$P_{E,F}$	Notes
37a	$+$	1	$(0,-1)$	2	**37b**	0	2	$-6P_1$	
37b	$-$	0	$(8,18)_3$	2	**37a**	1	2	P_1	
57a	$++$	1	$(2,1)$	4	**57c**	0	12	$8P_1$	
57a	$++$	1	$(2,1)$	4	**57b**	0	3	$-8P_1$	
57b	$-+$	0	$(7/4,-11/8)_2,(1,-1)_2$	3	**57a**	1	4	0	
57b	$-+$	0	$(7/4,-11/8)_2,(1,-1)_2$	3	**57c**	0	12	0	
57c	$-+$	0	$(2,4)_5$	12	**57a**	1	4	$3P_1$	
57c	$-+$	0	$(2,4)_5$	12	**57b**	0	3	P_1	
58a	$++$	1	$(0,-1)$	4	**58b**	0	4	$8P_1$	
58b	$-+$	0	$(-1,2)_5$	4	**58a**	1	4	$3P_1$	
77a	$++$	1	$(2,3)$	4	**77b**	0	20	$24P_1$	(1)
77a	$++$	1	$(2,3)$	4	**77c**	0	6	$-4P_1$	
89a	$+$	1	$(0,-1)$	2	**89b**	0	5	$4P_1$	
91a	$++$	1	$(0,0)$	4	**91b**	1	4	$4P_1$	
91b	$--$	1	$(-1,3),(1,0)_3$	4	**91a**	1	4	P_2	
92b	$--$	1	$(1,1)$	6	**92a**	0	2	0	
99a	$++$	1	$(2,0),(-1,0)_2$	4	**99b**	0	12	$-4P_1$	
99a	$++$	1	$(2,0),(-1,0)_2$	4	**99c**	0	12	0	
99a	$++$	1	$(2,0),(-1,0)_2$	4	**99d**	0	6	$2P_1$	
102a	$+++$	1	$(2,-4),(0,0)_2$	8	**102b**	0	16	$-8P_1$	(1)
102a	$+++$	1	$(2,-4),(0,0)_2$	8	**102c**	0	24	$32P_1$	
106b	$++$	1	$(2,1)$	8	**106a**	0	6	$-4P_1$	
106b	$++$	1	$(2,1)$	8	**106c**	0	48	$-88P_1$	
106b	$++$	1	$(2,1)$	8	**106d**	0	10	$12P_1$	
112a	$++$	1	$(0,-2),(-2,0)_2$	8	**112b**	0	4	0	
112a	$++$	1	$(0,-2),(-2,0)_2$	8	**112c**	0	8	0	
118a	$++$	1	$(0,-1)$	4	**118b**	0	12	$-8P_1$	(1)
118a	$++$	1	$(0,-1)$	4	**118c**	0	6	$4P_1$	
118a	$++$	1	$(0,-1)$	4	**118d**	0	38	$-28P_1$	
121b	$+$	1	$(4,5)$	4	**121a**	0	6	$4P_1$	
121b	$+$	1	$(4,5)$	4	**121c**	0	6	$4P_1$	
121b	$+$	1	$(4,5)$	4	**121d**	0	24	$-28P_1$	(2)
123a	$--$	1	$(-4,1),(-1,4)_5$	20	**123b**	1	4	0	
123b	$++$	1	$(1,0)$	4	**123a**	1	20	$4P_1$	
124a	$--$	1	$(-2,1),(0,1)_3$	6	**124b**	0	6	0	
128a	$+$	1	$(0,1),(-1,0)_2$	4	**128b**	0	8	0	
128a	$+$	1	$(0,1),(-1,0)_2$	4	**128c**	0	4	0	
128a	$+$	1	$(0,1),(-1,0)_2$	4	**128d**	0	8	0	
129a	$++$	1	$(1,-5)$	8	**129b**	0	15	$-8P_1$	
130a	$+--$	1	$(-6,10),(-1,10)_6$	24	**130b**	0	8	0	
130a	$+--$	1	$(-6,10),(-1,10)_6$	24	**130c**	0	80	0	
135a	$++$	1	$(4,-8)$	12	**135b**	0	36	0	(1)
136a	$--$	1	$(-2,2),(0,0)_2$	8	**136b**	0	8	0	
138a	$+++$	1	$(1,-2),(-2,1)_2$	8	**138b**	0	16	$-16P_1$	(1)
138a	$+++$	1	$(1,-2),(-2,1)_2$	8	**138c**	0	8	$-8P_1$	
141a	$--$	1	$(-3,-5)$	28	**141b**	0	12	0	
141a	$--$	1	$(-3,-5)$	28	**141c**	0	6	0	
141a	$--$	1	$(-3,-5)$	28	**141d**	1	4	0	

E	ε_p's	r_E	$E(\mathbb{Q})$	m_E	F	r_F	m_F	$P_{E,F}$	Notes
141a	- -	1	$(-3,-5)$	28	**141e**	0	12	0	
141d	++	1	$(0,-1)$	4	**141a**	1	28	$-12P_1$	
141d	++	1	$(0,-1)$	4	**141b**	0	12	$4P_1$	
141d	++	1	$(0,-1)$	4	**141c**	0	6	$4P_1$	
141d	++	1	$(0,-1)$	4	**141e**	0	12	$4P_1$	
142a	- -	1	$(1,1)$	36	**142b**	1	4	0	
142a	- -	1	$(1,1)$	36	**142c**	0	9	0	
142a	−−	1	$(1,1)$	36	**142d**	0	4	0	
142a	−−	1	$(1,1)$	36	**142e**	0	324	0	(2)
142b	++	1	$(-1,0)$	4	**142a**	1	36	$4P_1$	(1)
142b	++	1	$(-1,0)$	4	**142c**	0	9	$-4P_1$	
142b	++	1	$(-1,0)$	4	**142d**	0	4	$4P_1$	
142b	++	1	$(-1,0)$	4	**142e**	0	324	$8P_1$	(2)
152a	++	1	$(-1,-2)$	8	**152b**	0	8	0	
153a	++	1	$(0,1)$	8	**153b**	1	16	$8P_1$	
153a	++	1	$(0,1)$	8	**153c**	0	8	$8P_1$	
153a	++	1	$(0,1)$	8	**153d**	0	24	0	
153b	−−	1	$(5,-14)$	16	**153a**	1	8	0	
153b	−−	1	$(5,-14)$	16	**153d**	0	24	0	
154a	+++	1	$(5,3),(-6,3)_2$	24	**154b**	0	24	$-24P_1$	
154a	+++	1	$(5,3),(-6,3)_2$	24	**154c**	0	16	$16P_1$	
155a	−−	1	$(5/4,31/8),(0,2)_5$	20	**155b**	0	8	0	
155a	−−	1	$(5/4,31/8),(0,2)_5$	20	**155c**	1	4	0	
155c	++	1	$(1,-1)$	4	**155a**	1	20	$-12P_1$	
155c	++	1	$(1,-1)$	4	**155b**	0	8	$4P_1$	
156a	−+−	1	$(1,1),(2,0)_2$	12	**156b**	0	12	0	(1)
158a	−−	1	$(-1,-4)$	32	**158b**	1	8	0	
158a	−−	1	$(-1,-4)$	32	**158c**	0	48	0	(1)
158a	−−	1	$(-1,-4)$	32	**158d**	0	40	0	
158a	−−	1	$(-1,-4)$	32	**158e**	0	6	0	
158b	++	1	$(0,-1)$	8	**158a**	1	32	$-8P_1$	
158b	++	1	$(0,-1)$	8	**158c**	0	48	$-56P_1$	(1)
158b	++	1	$(0,-1)$	8	**158d**	0	40	0	
158b	++	1	$(0,-1)$	8	**158e**	0	6	$-8P_1$	
160a	++	1	$(2,-2),(1,0)_2$	8	**160b**	0	8	0	
162a	++	1	$(-2,4),(1,1)_3$	12	**162b**	0	6	0	
162a	++	1	$(-2,4),(1,1)_3$	12	**162c**	0	6	0	
162a	++	1	$(-2,4),(1,1)_3$	12	**162d**	0	12	0	
170a	+−−	1	$(0,2),(1,-1)_2$	16	**170d**	0	12	0	
170a	+−−	1	$(0,2),(1,-1)_2$	16	**170e**	0	20	0	
171b	−−	1	$(2,-5)$	8	**171a**	0	12	0	
171b	−−	1	$(2,-5)$	8	**171c**	0	96	0	(1)
171b	−−	1	$(2,-5)$	8	**171d**	0	32	0	
175a	−−	1	$(2,-3)$	8	**175b**	1	16	0	(1)
175a	−−	1	$(2,-3)$	8	**175c**	0	40	0	(1)
175b	++	1	$(-3,12)$	16	**175a**	1	8	$16P_1$	
175b	++	1	$(-3,12)$	16	**175c**	0	40	$16P_1$	(1)
176c	−−	1	$(1,-2)$	8	**176b**	0	8	0	(1)

E	ε_p's	r_E	$E(\mathbb{Q})$	m_E	F	r_F	m_F	$P_{E,F}$	Notes
176c	$--$	1	$(1,-2)$	8	**176a**	0	16	0	
176c	$--$	1	$(1,-2)$	8	**176b**	0	8	0	(1)
184a	$--$	1	$(0,1)$	8	**184c**	0	12	0	
184a	$--$	1	$(0,1)$	8	**184d**	0	24	0	
184b	$++$	1	$(2,-1)$	8	**184a**	1	8	0	
184b	$++$	1	$(2,-1)$	8	**184c**	0	12	0	
184b	$++$	1	$(2,-1)$	8	**184d**	0	24	0	
185a	$++$	1	$(4,-13)$	48	**185b**	1	8	$8P_1$	
185a	$++$	1	$(4,-13)$	48	**185c**	1	6	$24P_1$	
$\cdots$								$\cdots$	
185b	$--$	1	$(0,2)$	8	**185c**	1	6	0	
185c	$++$	1	$(-5/4,3/8),(-1,0)_2$	6	**185b**	1	8	$2P_1$	
189a	$++$	1	$(-1,-2)$	12	**189b**	1	12	$-12P_1$	
189a	$++$	1	$(-1,-2)$	12	**189c**	0	12	$12P_1$	
189b	$--$	1	$(-3,9),(3,0)_3$	12	**189a**	1	12	0	
189b	$--$	1	$(-3,9),(3,0)_3$	12	**189c**	0	12	0	
190a	$-+-$	1	$(13,-47)$	88	**190b**	1	8	0	
190a	$-+-$	1	$(13,-47)$	88	**190c**	0	24	0	(1)
190b	$+++$	1	$(1,2)$	8	**190c**	0	24	$16P_1$	(1)
192a	$++$	1	$(3,2),(-1,0)_2$	8	**192b**	0	8	0	
192a	$++$	1	$(3,2),(-1,0)_2$	8	**192c**	0	8	0	
192a	$++$	1	$(3,2),(-1,0)_2$	8	**192d**	0	8	0	
196a	$--$	1	$(0,-1)$	6	**196b**	0	42	0	(1)
198a	$+--$	1	$(-1,-4),(-4,2)_2$	32	**198b**	0	32	0	(1)
198a	$+--$	1	$(-1,-4),(-4,2)_2$	32	**198c**	0	32	0	
198a	$+--$	1	$(-1,-4),(-4,2)_2$	32	**198d**	0	32	0	(1)
198a	$+--$	1	$(-1,-4),(-4,2)_2$	32	**198e**	0	160	0	(1)
200b	$--$	1	$(-1,1),(-2,0)_2$	8	**200c**	0	24	0	
200b	$--$	1	$(-1,1),(-2,0)_2$	8	**200d**	0	40	0	(1)
200b	$--$	1	$(-1,1),(-2,0)_2$	8	**200e**	0	24	0	
201a	$++$	1	$(1,-2)$	12	**201b**	1	12	$4P_1$	
201b	$--$	1	$(-1,2)$	12	**201a**	1	12	0	
201c	$++$	1	$(16,-7)$	60	**201a**	1	12	$-24P_1$	
201c	$++$	1	$(16,-7)$	60	**201b**	1	12	$8P_1$	
203b	$--$	1	$(2,-5)$	8	**203a**	0	48	0	
203b	$--$	1	$(2,-5)$	8	**203c**	0	12	0	
205a	$--$	1	$(-1,8),(2,1)_4$	12	**205b**	0	16	0	
205a	$--$	1	$(-1,8),(2,1)_4$	12	**205c**	0	8	0	
208a	$--$	1	$(4,-8)$	16	**208c**	0	12	0	
208a	$--$	1	$(4,-8)$	16	**208d**	0	48	0	(1)
208b	$++$	1	$(4,4)$	16	**208a**	1	16	0	(1)
208b	$++$	1	$(4,4)$	16	**208c**	0	12	0	
208b	$++$	1	$(4,4)$	16	**208d**	0	48	0	(1)
212a	$--$	1	$(2,2)$	12	**212b**	0	21	0	
214a	$--$	1	$(0,-4)$	28	**214b**	1	12	0	(1)
214a	$--$	1	$(0,-4)$	28	**214d**	0	12	0	
214b	$++$	1	$(0,0)$	12	**214a**	1	28	$-8P_1$	(1)
214b	$++$	1	$(0,0)$	12	**214d**	0	12	$-4P_1$	

E	ε_p's	r_E	$E(\mathbb{Q})$	m_E	F	r_F	m_F	$P_{E,F}$	Notes
214c	$++$	1	$(11,10)$	60	**214a**	1	28	$-4P_1$	(1)
214c	$++$	1	$(11,10)$	60	**214d**	0	12	$16P_1$	
214c	$++$	1	$(11,10)$	60	**214b**	1	12	$12P_1$	(1)
216a	$++$	1	$(-2,-6)$	24	**216b**	0	24	0	
219a	$++$	1	$(2,-1)$	12	**219c**	1	60	$-12P_1$	(1)
219a	$++$	1	$(2,-1)$	12	**219b**	1	12	$-4P_1$	
216a	$++$	1	$(-2,-6)$	24	**216d**	0	72	0	
219b	$--$	1	$(-3/4,-1/8),(0,1)_3$	12	**219a**	1	12	0	
219b	$--$	1	$(-3/4,-1/8),(0,1)_3$	12	**219c**	1	60	0	(1)
219c	$++$	1	$(-6,7),(10,-5)_2$	60	**219a**	1	12	$-12P_1$	
219c	$++$	1	$(-6,7),(10,-5)_2$	60	**219b**	1	12	$4P_1$	
220a	$--+$	1	$(-7,11),(15,55)_6$	36	**220b**	0	12	0	
224a	$++$	1	$(1,2),(0,0)_2$	8	**224b**	0	8	0	
225a	$++$	1	$(1,1)$	8	**225b**	0	40	0	(1)
225e	$--$	1	$(-5,22)$	48	**225a**	1	8	0	(1)
225e	$--$	1	$(-5,22)$	48	**225b**	0	40	0	(1)
228b	$-+-$	1	$(3,6)$	24	**228a**	0	18	0	
232a	$++$	1	$(2,-4)$	16	**232b**	0	16	0	
234c	$+++$	1	$(1,-2),(-2,1)_2$	16	**234b**	0	48	0	(1)
234c	$+++$	1	$(1,-2),(-2,1)_2$	16	**234e**	0	20	0	(1)
235a	$--$	1	$(-2,3)$	12	**235c**	0	18	0	(1)
236a	$--$	1	$(1,-1)$	6	**236b**	0	14	0	
238a	$--+$	1	$(24,100),(-8,4)_2$	112	**238b**	1	8	0	(1)
238a	$--+$	1	$(24,100),(-8,4)_2$	112	**238c**	0	16	0	(1)
238a	$--+$	1	$(24,100),(-8,4)_2$	112	**238d**	0	16	0	(1)
238b	$+++$	1	$(1,1),(0,0)_2$	8	**238a**	1	112	$12P_1$	(1)
238b	$+++$	1	$(1,1),(0,0)_2$	8	**238c**	0	16	$-4P_1$	(1)
238b	$+++$	1	$(1,1),(0,0)_2$	8	**238d**	0	16	$4P_1$	(1)
240c	$+++$	1	$(1,2),(0,0)_2$	16	**240a**	0	16	0	
240c	$+++$	1	$(1,2),(0,0)_2$	16	**240d**	0	16	0	(1)
243a	$+$	1	$(1,0)$	6	**243b**	0	9	0	(1)
245a	$--$	1	$(7,17)$	48	**245c**	1	32	0	
246d	$+++$	1	$(3,-6),(4,-2)_2$	48	**246a**	0	84	$24P_1$	(1)
446a	$++$	1	$(4,-6)$	24	**446d**	2	88	0	(2)
446b	$--$	1	$(5,-10)$	56	**446d**	2	88	0	(2)
681a	$++$	1	$(4,4)$	32	**681c**	2	96	$-24P_1$	(2)
446d	$+-$	2	-	88	**446a**	1	12	0	(1)
446d	$+-$	2	-	88	**446b**	1	56	0	(1)

Notes:

(1) We used $y = 10^{-5}$, $d = 1500$, which typically takes about 4 minutes.
(2) We used $y = 10^{-5}/2$, $d = 3000$, which takes up to 2 hours.

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