Chow Heegner Thoughts

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Abstract

1 Algorithm

Let E and F be elliptic curves of conductor N.

For $B > 0$, let $\varphi_{F,B} = \sum_{n=1}^{B} \frac{a_n}{n} q^n$ be the degree B approximation to the modular parametrization attached to F, where we have $\varphi_F : \mathfrak{h} \to \mathbb{C}$ induces by composition with $\mathbb{C} \to \mathbb{C}/\Lambda_F = F(\mathbb{C})$ the map $X_0(N)(\mathbb{C}) \to F(\mathbb{C})$.

Given a point $P \in F(K)$, with K a quadratic extension of $\mathbb{Q}$, our task it to numerically approximate, to any requested precision, *some* $z \in \mathfrak{h}$ such that $\varphi_F(z) = P$. We are not required to compute representative orbits for *all* such z modulo the action of $\Gamma_0(N)$, though of course that would be interesting. (There may be a trick to do that via translation and finding a generic fiber?)

Write $z = x + iy$. We have

$$\begin{aligned}\varphi_{F,B}(z) &= \sum_{n=1}^{B} \frac{a_n}{n} e^{2\pi i n(x+iy)} \\ &= \sum_{n=1}^{B} \frac{a_n}{n} e^{-2\pi y n} e^{2\pi i x}\end{aligned}$$

Thus using that $|a_n| \leq n$, and letting $c = |e^{-2\pi y}|$, we have

$$\begin{aligned}|\varphi_{F,B}(z) - \varphi_F(z)| &= \left| \sum_{n=B+1}^{\infty} \frac{a_n}{n} e^{-2\pi y n} e^{2\pi i x} \right| \\ &\leq \sum_{n=B+1}^{\infty} |e^{-2\pi y n}| = \sum_{n=B+1}^{\infty} c^n = c^{B+1} \sum_{n=0}^{\infty} c^n = \frac{c^{B+1}}{1-c}.\end{aligned}$$

Let $B = B(y, \epsilon)$, be an integer such that $|\varphi_{F,B}(z) - \varphi_F(z)| < \epsilon$.

Proposition 1.1. *We may take*

$$B(y, \epsilon) = \left\lceil \frac{\log(\epsilon \cdot (1 - e^{-2\pi y}))}{-2\pi y} \right\rceil.$$

Proof. We solve for B in the above equation. If

$$\frac{c^{B+1}}{1-c} = \epsilon,$$

then

$$(B+1)\log(c) - \log(1-c) = \log(\epsilon),$$

so

$$B = \frac{\log(\epsilon) + \log(1-c)}{\log(c)} - 1.$$

□

Strategy to find root.

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1. Compute $\varphi_{F,B}$ with B small, find all roots using a double precision numerical root finder, and choose the best root, i.e., one corresponding to point in upper half plane with largest imaginary part.
2. Refine this root to one of higher precision of $\varphi_{F,B}$ with B larger, using improved bound on B coming from knowing imaginary part y of root.

Using Newton-Raphson one can refine a low precision point to a high precision point. Pretty insane, but this works really, really well in practice. I can't believe how well.