A Non-Gorenstein Eisenstein Descent

Frank Calegari William Stein

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1 Introduction

Let $X_0(N)$ be the modular curve parameterizing isomorphism classes of (generalized) elliptic curves together with a cyclic subgroup of order N. Let $J_0(N)$ be the Jacobian of $X_0(N)$, and $\mathbf{T}$ the ring of Hecke operators acting on $J_0(N)$. It is well known that we can associate to any maximal ideal $\mathfrak{m}$ of the Hecke algebra a self-dual two-dimensional residual Galois representation

$$\overline{\rho}_{\mathfrak{m}} : \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \to \mathrm{GL}_2(\mathbf{T}/\mathfrak{m})$$

that arises from the natural action of Galois on the division points of $J_0(N)$ (see, e.g., [9]). Assume that the representation $\overline{\rho}_{\mathfrak{m}}$ is reducible, in which case one says that $\mathfrak{m}$ is an *Eisenstein* prime. Mazur's analysis [3] of the "Eisenstein ideal" when N is prime allows for a thorough understanding of the arithmetic of the $\mathfrak{m}$-adic Tate module of $J_0(N)$, and through this the "Eisenstein quotients" of $J_0(N)$ that can naturally be associated to $\mathfrak{m}$. For example, when N is prime Mazur proved (see [3, II]) for Eisenstein primes $\mathfrak{m}$ that:

1. The maximal ideal $\mathfrak{m}$ satisfies multiplicity one; namely, $J_0(N)[\mathfrak{m}] = \mathbf{Z}/p\mathbf{Z} \oplus \mu_p$ if $p > 2$, and $J_0(N)[\mathfrak{m}] = D_{/\mathbf{Z}}$ if $p = 2$, where $D_{/\mathbf{Z}}$ is the unique nontrivial group scheme extension of $\mathbf{Z}/2\mathbf{Z}$ by μ_2 killed by 2.

2. $\mathbf{T}_{\mathfrak{m}}$ is a Gorenstein ring.

3. The $\mathfrak{m}$-adic Tate module
$$J^{(\mathfrak{m})} := \mathbf{T}_{\mathfrak{m}} \otimes_{\mathbf{T}} \varprojlim J[p^n]$$
is free of rank two as a $\mathbf{T}_{\mathfrak{m}}$-module.

There is a strong synergy between these results, and indeed their proofs. A notable consequence is the following theorem ([3, Thm. 12 p. 38 or Cor. 3.5, Prop. 3.6, p. 150]):

Theorem 1.1 (Mazur)**.** *Assume that N is prime, and let $\mathfrak{m}$ be an Eisenstein prime of $\mathbf{T}$. If M is the Mordell-Weil group of $J_0(N)$ over $\mathbf{Q}$ then $M \otimes_{\mathbf{T}_p} \mathbf{T}_{\mathfrak{m}}$ is finite; equivalently, the $\mathfrak{m}$-Eisenstein quotient J has finite Mordell-Weil group. Suppose moreover that the residue characteristic of $\mathfrak{m}$ is odd. Then $Ш(J) \otimes_{\mathbf{T}_p} T_{\mathfrak{m}} = 0$; equivalently, $Ш(J)[\mathfrak{m}] = 0$.*

If one tries to extend these results to non-prime level one quickly finds that the situation is more complicated (see, e.g., the list at [3, p. 39]). For example, there exist modular elliptic curves $E/\mathbf{Q}$ with nontrivial torsion and positive rank. Such curves are examples of Eisenstein quotients of nonzero rank. On the other hand, many of Mazur's arguments continue to apply in a more general situation. In this note, we concentrate on a particular Eisenstein quotient of non-prime level, namely a 15 dimensional quotient A of $J_0(559)$ (defined more precisely below), and an associated Eisenstein maximal ideal $\mathfrak{m}$ of residue characteristic 7. This example is particularly interesting in several respects. First, we already start to see pathologies not present in the case of prime conductor, namely, the failure of multiplicity one. On the other hand, we are still able to prove the following analogue of Theorem 1.1:

Theorem 1.2. *The Mordell–Weil group of A over $\mathbf{Q}$ is finite. Furthermore $Ш(A)[\mathfrak{m}] = 0$.*

In this paper, we do not strive for generality, but rather instead try by example to highlight the issues and difficulties one might expect in this more general context. A guiding and motivating problem for us is the following:

Question 1.3. Let A be a modular abelian variety attached to a newform $f_A \in S_2(\Gamma_0(N))$. Can one practically determine the following invariants of A?

1. The rational torsion subgroup,
2. The endomorphism ring,
3. The minimal degree of a polarization $A \to A^\vee$,

A positive answer to this question is relevant to computing some of the invariants of A that appear in the BSD conjecture, and to the problem of systematically enumerating up to isomorphism all simple abelian variety quotients of $J_0(N)$. Although there exist various algorithms for computing the torsion subgroup of an abelian variety, they usually require a concrete description of A in terms of equations. Our hold on the abelian varieties A is decidedly more fleeting: we have, concretely, a q-expansion f_A that is canonically associated to A. However, the form f_A only determines A up to *isogeny*, which is not enough to determine the torsion subgroup. In practise one can often use f_A to determine a multiple of the order of the torsion subgroup, and then hope to physically realize this subgroup by some geometric construction, say by considering the cuspidal subgroup (see [AS05, §3.5–3.6]). We have not yet found a way to make this approach effective in general, however. Initial computations performed by Tseno Tselkov using the algorithm of Section 4.7 suggested that there always existed a polarization $A \to A^\vee$ of degree some power of 2. We produce an explicit example to show that this is false. Namely, suppose that A is the 15 dimensional quotient of $J_0(559)$ mentioned above. Then we have the following:

Theorem 1.4. *Any polarization $A \to A^\vee$ has degree divisible by 7. [[Todo: Where exactly do we prove this?]]*

We suspect[[Todo: have?]] that A is the newform abelian variety of smallest level that exhibits this phenomenon (see Section 4.8).

2 The Abelian Variety A

Let $\mathbf{T}$ be the Hecke algebra of level $N = 559 = 13 \cdot 43$ associated to $J = J_0(559)$, and let $\mathscr{I}$ be the ideal generated by $T_\ell - 1 - \ell$ for all $\ell \neq 13, 43$, and by $U_{13} + 1$ and $U_{43} - 1$. We call $\mathscr{I}$ the *Eisenstein ideal*, but note that a choice has been made, and that there are two "Eisenstein ideals" of this level; the other choice is to replace $U_{13} + 1$ and $U_{43} - 1$ with $U_{13} - 1$ and $U_{43} + 1$. (The two other choices of sign corresponds to factors of J where the L-function is forced to vanish because the sign in the functional equation is -1.) We find (see Section 4.1) that $\mathbf{T}/\mathscr{I} \simeq \mathbf{Z}/(2 \cdot 7)\mathbf{Z}$. Let $\mathfrak{m} = (7) + \mathscr{I}$, and note that $\mathfrak{m}$ is maximal since $\mathbf{T}/((7) + \mathfrak{I}) \cong \mathbf{Z}/7\mathbf{Z}$. Let

$$\mathfrak{I} = \bigcap_{k \geq 0} (7, \mathfrak{m}^k)$$

and let $\mathbf{T}_7 = \mathbf{T} \otimes_{\mathbf{Z}} \mathbf{Z}_7$ be the 7-adic completion of $\mathbf{T}$.

Let A be the new simple abelian subvariety at J such that $A[\mathfrak{m}]$ is nontrivial. This determines A uniquely. It is a 15 dimensional geometrically simple abelian variety, and is associated to a normalized Hecke eigenform $f = \sum a_n q^n \in S_2(\Gamma_0(559), \overline{\mathbf{Q}})$ (see Section 4.2).

There are two natural degeneracy maps $\alpha, \beta : J_0(43) \to J$. Since $X_0(p\ell) \to X_0(p)$ is totally ramified [7], it follows that both maps are injections.[[Todo: I couldn't find the totally ramified assertion in [7]; if it is there we should give a precise reference. Also, why this implies injective is a general fact, but I still think we should give a reference for that. If [7] doesn't work out as a reference, Ribet's level raising paper discusses this sort of thing and certainly has enough to get injectivity (since he proves that the kernel of the sum of both maps is the anti-diagonal Shimura subgroup).]] Let J_{old} be the sum of the images of α and β. The kernel of $J_0(43) \times J_0(43) \to J_{\text{old}}$ is isomorphic to the Shimura subgroup Σ anti-diagonally embedded in the product (see [8, Thm. 4.3]).

Let $\mathcal{O}$ be the ring generated over $\mathbf{Z}$ by the coefficients a_n of f. The surjection $\mathbf{T} \to \mathcal{O}$ gives rise to a map $\mathbf{T}_{\mathfrak{m}} \to \mathcal{O}_{\mathfrak{m}}$. The ring $\mathcal{O}_{\mathfrak{m}}$ has index 7 inside its integral closure (see Section 4.3). One usually says in this situation that 7 is a prime of "self-fusion", since it implies the existence of a congruence between f and a Galois conjugate f^σ, modulo some prime ideal above 7. Congruences $f \equiv g$ between non-conjugate eigenforms give rise to primes of fusion. The prime 7 is also a prime of fusion for A, since f is congruent modulo $\mathfrak{m}$ to g, where g is one of the two old forms associated to the unique cusp form of level 43. These simultaneous pathologies (Eisenstein, fusion, self-fusion) occur for levels less than 559 only at the prime 2 (see Section 4.6). We shall also see that 7 divides the degree of the modular polarization $A \to A^\vee$ (see Section 4.4). We start by proving the following:

Theorem 2.1. *We have the following equalities and identifications, where all dimensions are over* $\mathbf{F}_7 \cong \mathbf{T}/\mathfrak{m}$*:*

(1) $\dim A[\mathfrak{m}] = 3$.

(2) $\mathfrak{I}\mathbf{T}_7 = (\mathfrak{m}^k, 7)\mathbf{T}_7$ *for* $k \geq 2$.

(3) $\dim A[(\mathfrak{m}^k, 7)] = 4$, *for* $k \geq 2$.

(4) *If* $A_{\text{old}} := A \cap J_{\text{old}}$, *then*

   (a) $\dim A_{\text{old}}[\mathfrak{m}] = 2$,

   (b) $\mathbf{Z}/7\mathbf{Z} \subset A_{\text{old}}[\mathfrak{m}]$, *where* $\mathbf{Z}/7\mathbf{Z}$ *is the intersection of* A *with the image of the cuspidal subgroup of* J_{old}.

Proof. The identification of $\mathfrak{I}$ in part (2.1) follows by a consideration of q-expansions. All the other results can essentially be proved by computations over the complex numbers.[[Todo: as William explains]] The only arithmetic input is to understand the cuspidal subgroup of J_{old}, but this is completely described by [3]. For part (2.1) it suffices to consider finitely many Hecke operators, by the Sturm bound (see [2, App.]). □

Part (2.1) of Theorem 2.1 shows that A does not satisfy "multiplicity one" at the Eisenstein prime $\mathfrak{m}$. Calculation (2.1) reflects the fact that $\mathcal{O}_{\mathfrak{m}}$ has rank two over $\mathbf{Z}_7$. Calculation (2.1) uses exact "intersection" algorithms inside J (see Section 4.5).

Let us consider further the Hecke algebra structure of $\mathbf{T}$ on A.

Definition 2.2. Let $\mathcal{O}$ be the image of $\mathbf{T}$ in $\mathrm{End}(A)$. Let $\mathcal{O}_{\mathfrak{m}}$ denote the localization of $\mathcal{O}$ at the image of $\mathfrak{m}$.

Lemma 2.3. *The ring* $\mathcal{O}_{\mathfrak{m}}$ *is the unique index* 7 *ring inside the Witt vectors* $W(\mathbf{F}_{49})$. *There is an isomorphism*

$$\mathcal{O}_{\mathfrak{m}}/(7) \simeq \mathbf{F}_7[\epsilon]/\epsilon^2.$$

Proof. The Hecke algebra is finitely generated, and an explicit list of generators may be found from [2, App.]. The lemma follows by calculation.[[Todo: A calc.]] □

These rings are *explicitly computable*, i.e., one may find a Hecke operator $\epsilon \in \mathbf{T}$ that gives rise to the isomorphism of Lemma 2.3. This will be important below. One way to construct an element ϵ is to consider the Hecke operators

$$\eta_\ell = 1 + \ell - T_\ell$$

for various primes $\ell \neq 13, 43$. They will *a priori* land in $\mathfrak{m}$, but by the Cebotarev density theorem will land in $\mathfrak{I}$ infinitely often. If $\mathcal{O}_{\mathfrak{m}}/7$ is generated by η_ℓ over $\mathbf{F}_7$ then we say that ℓ is a *good prime*. Note that $\mathfrak{m}\mathcal{O}_{\mathfrak{m}}$ is equal to $(7, \eta_\ell)$ for a good prime ℓ, but $\mathfrak{m}\mathcal{O}_{\mathfrak{m}}$ is not generated by η_ℓ; indeed, the maximal ideal $\mathfrak{m}\mathcal{O}_{\mathfrak{m}}$ cannot be generated by a single element since $\mathcal{O}_{\mathfrak{m}}$ is not a discrete valuation ring.

Definition 2.4. Consider the Jordan–Hölder factors of a Galois module G. Let $\delta(G)$ denote the number of times $\mathbf{Z}/7\mathbf{Z}$ occurs as a factor minus the number of occurrences of μ_7. For an abelian variety B, let $\delta_B(n)$ denote $\delta(B[7^n])$.

Lemma 2.5. *We have* $\delta_A(n) = \delta_{A^\vee}(n) = 0$, *and* $\delta(A[\mathfrak{I}]) = \delta(A[\mathfrak{m}^2, 7]) = 0$.

Proof. Note that δ is additive in exact sequences. Since $A[7^n]$ has a filtration by copies of $A[7]$ is follows that $\delta_A(n) = n\delta_A(1)$. Since $A^\vee$ is dual to A it follows that $\delta_{A^\vee}(n) = -\delta_A(n)$. Thus

$$\delta_A(n) - \delta_{A^\vee}(n) = 2n\delta_A(1).$$

Yet A is isogenous to $A^\vee$, and thus $\delta_A(n) - \delta_{A^\vee}(n) = O(1)$. Thus $\delta_A(1) = 0$, so $\delta_A(n) = \delta_{A^\vee}(n)$; but $\delta_A(n) = -\delta_{A^\vee}(n)$, so this proves the first part of the lemma. For the second it suffices to note that $\mathfrak{m}$ is the unique Eisenstein prime (of A) with residue characteristic 7.[[Todo: Why? I do not see this.]] □

It follows[[Todo: I do not see this.]] from Lemma 2.5 that $A[\mathfrak{I}]$ has a filtration by two copies of $\mathbf{Z}/7$ and two copies of μ_7. We may also control the constant and multiplicative submodules of $A[\mathfrak{I}]$:

Lemma 2.6. *The maximal constant Galois submodule of $A[\mathfrak{I}]$ is contained in $A[\mathfrak{m}]$. The maximal μ-type subgroup of $A[\mathfrak{I}]$ is also contained in $A[\mathfrak{m}]$.*

Proof. It suffices to prove that both constant Galois modules and μ-type Galois modules are killed by the Eisenstein ideal. Let M be a constant Galois module, and suppose that that $M \subseteq A[\mathfrak{I}]$. Let $\ell \neq 7, 13, 43$. By the Eichler–Shimura relations the Frobenius element σ_ℓ acting on M satisfies the polynomial $x^2 - T_\ell x + \ell = 0$. Thus if M is constant it follows that $\eta_\ell = 1 + \ell - T_\ell$ kills M. Choosing ℓ to be a good prime it follows that M is killed by $(7, \eta_\ell) = \mathfrak{m}$, since $7 \in \mathfrak{I}$. Yet for good primes ℓ we have that $(7, \eta_\ell)\mathcal{O}_\mathfrak{m} = \mathfrak{m}\mathcal{O}_\mathfrak{m}$, and thus M is killed by $\mathfrak{m}$, and $M \subseteq A[\mathfrak{m}]$. An identical argument works for μ-type subgroups. □

2.1 Determining the Jordan–Hölder factors of $A[\mathfrak{m}]$

As in [3] (§14, p.114), we see that the Galois module $A[\mathfrak{m}]$ has a filtration by modules of the form $\mathbf{Z}/7\mathbf{Z}$ and μ_7, and correspondingly $A[\mathfrak{m}]$ considered as a group scheme over $\mathrm{Spec}(\mathbf{Z}_7)$ or $\mathrm{Spec}(\mathbf{F}_7)$ also has such a filtration (as one sees from [5]). By Theorem 2.1, we see that there is at least one copy of $\mathbf{Z}/7\mathbf{Z}$ that includes into $A[\mathfrak{m}]$. A general argument of Mazur [3] shows that all the other filtered pieces of μ_7 must be isomorphic to μ_7.

Lemma 2.7. *Let N be an integer, let $p \nmid 2N$ be a prime, and let $\mathfrak{m}$ be a maximal ideal of the Hecke algebra of residue characteristic p. Let $J[\mathfrak{m}]^{\text{ét}}_{/\mathbf{F}_p}$ be the étale part of the group scheme $J_0(N)[\mathfrak{m}]_{/\mathbf{F}_p}$. Then $J[\mathfrak{m}]^{\text{ét}}_{/\mathbf{F}_p}$ is one dimensional.*

Proof. The argument of Mazur ([3], Cor 14.8, p.119) applies *mutatis mutandis*. □

Note that this result does not imply that the p-torsion subgroup of $J_0(N)$ is cyclic, for $p \nmid N$, since different rational cyclic subgroups may be killed by different maximal Eisenstein ideals. The point is that the argument requires the q-expansion principal, which is only valid if one works with the full Hecke algebra (i.e. specifying the action of U_ℓ for bad primes ℓ). Thus we infer that $A[\mathfrak{m}]/(\mathbf{Z}/7\mathbf{Z})$ has a filtration by two copies of μ_7. We show that this second piece is a direct summand.

Lemma 2.8. *Let N be an integer, let $p \nmid 2N$, and let $\mathfrak{m}$ be a maximal ideal of the Hecke algebra of residue characteristic p. Let V be a Galois module subquotient of $J[\mathfrak{m}]$ whose filtration consists either entirely of $(\mathbf{Z}/p\mathbf{Z})$'s or μ_p's. Then V is a direct summand.*

Proof. We apply the same argument as in the proof of [3], Lemma 16.7, p. 126. Since V is a subquotient of $J[\mathfrak{m}]$, by the Eichler-Shimura relations any Frobenius element σ_ℓ satisfies the minimal polynomial $(x-1)\cdot(x-\ell)=0$. Choose a prime $\ell \not\equiv 1 \bmod p$, possible since $p \neq 2$. Suppose that V has a filtration by constant pieces. Then $(\sigma_\ell - \ell)$ is an isomorphism on each filtered piece and thus has no kernel on V. It follows that $\sigma_\ell = 1$ on V. By the Cebotarev density theorem it follows that V is constant, so it is clearly a direct sum of constant modules. If V has a filtration by μ_p's, then the same argument applied to $V^\vee$ shows that $V^\vee$ is also a direct summand of constant modules and thus V is a direct summand of multiplicative Galois modules. □

Thus we have the following:

Lemma 2.9. *1. $\delta(A[\mathfrak{m}]) = -1$, and $A[\mathfrak{m}]$ sits inside an exact sequence of the form:*

$$0 \to \mathbf{Z}/7\mathbf{Z} \to A[\mathfrak{m}] \to \mu_7 \oplus \mu_7 \to 0.$$

2. $A[\mathfrak{I}]/A[\mathfrak{m}] \simeq \mathbf{Z}/7\mathbf{Z}$.

Proof. The second claim follows from Lemma 2.5 which implies that

$$0 = \delta(A[\mathfrak{I}]) = \delta(A[\mathfrak{I}]/A[\mathfrak{m}]) + \delta(A[\mathfrak{m}]).$$

□

2.2 Determining $A[\mathfrak{m}]$

We show now that the extension (1) of Lemma 2.9 is nontrivial.

Lemma 2.10. *Suppose that $A[\mathfrak{m}] \simeq \mathbf{Z}/7\mathbf{Z} \oplus (\mu_7)^2$. Then*

$$\eta_\ell = T_\ell - 1 - \ell \notin \mathfrak{I}$$

(equivalently, ℓ is a good prime) if and only if both conditions are satisfied:

1. $\ell \not\equiv 1 \bmod 7$

2. ℓ is not a 7th power modulo 43.

Proof. Choosing an appropriate basis for $A[\mathfrak{I}]$ as a four dimensional $\mathbf{F}_7$ vector space we may write the action of $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ as

$$\rho = \begin{pmatrix} 1 & 0 & 0 & a \\ 0 & \chi & 0 & b \\ 0 & 0 & \chi & c \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where χ is the cyclotomic character, and a, b and c are functions on $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$. We compute that

$$(\rho(\sigma)-1)(\rho(\sigma)-\chi(\sigma)) = (\chi(\sigma)-1)\begin{pmatrix} 0 & 0 & 0 & a(\sigma) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Since $\mathfrak{I} \neq \mathfrak{m}$ it follows that a is nontrivial. Moreover, we see that a prime ℓ is good if and only if $\ell \not\equiv 1 \bmod 7$ and $a(\mathrm{Frob}_\ell)$ is nontrivial. The function a defines an extension class inside $\mathrm{Ext}^1(\mathbf{Z}/7\mathbf{Z}, \mathbf{Z}/7\mathbf{Z})$. These extensions are taking place inside the category of finite flat group schemes over $\mathbf{Z}[1/559]$. Such extensions are étale, and thus the generic fibre defines a degree 7 field unramified over $\mathbf{Q}$ outside 559. The only such extension is the degree 7 extension inside $\mathbf{Q}(\zeta_{43})$, since $7 \nmid 13-1$. Thus if $\sigma = \mathrm{Frob}_\ell$, then $a(\sigma) = 0$ if and only if ℓ splits completely in this degree 7 field. This is equivalent to ℓ being a 7th power modulo 43. This proves one implication. Now assume that either $\ell \equiv 1 \bmod 7$ or ℓ is a 7th power modulo 43. Then if $\sigma = \mathrm{Frob}_\ell$ and $x = \rho(\sigma)$, then $(x-1)(x-\ell) = 0$. Thus

$$0 = x^2 - T_\ell x + \ell = (x-1)(x-\ell) + (T_\ell - 1 - \ell)x = \eta_\ell x.$$

Since $x = \rho(\sigma)$ is invertible, this implies that $\eta_\ell = 0$, and thus ℓ is not a good prime. This proves the lemma. □

It is easy to compute that $\eta_2 \in \mathfrak{I}$ and thus 2 is not a good prime. By the previous lemma this implies that $A[\mathfrak{m}] \neq \mathbf{Z}/7\mathbf{Z} \oplus \mu_7 \oplus \mu_7$.

To sum up what we know so far, the action of Galois on $A[\mathfrak{I}] = A[7, \mathfrak{m}^2]$ can be written as follows:

$$\rho(\sigma) = \begin{pmatrix} 1 & x & y & a \\ 0 & \chi & 0 & b \\ 0 & 0 & \chi & c \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

Where x, y, a, b and c are all continuous functions of $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$, and x and y are not both zero. We may also assume that the first 2×2 matrix represents the action of Galois on $A_{\mathrm{old}}[\mathfrak{m}]$. Since $A^\vee[\mathfrak{I}]$ is the Cartier dual to $A[\mathfrak{I}]$ we also understand the Galois action on $A^\vee[\mathfrak{I}]$. Explicitly, it is given by $\chi\rho^{-1}$ which is

$$\rho^\vee(\sigma) = \begin{pmatrix} \chi & -c & -b & a' \\ 0 & 1 & 0 & -y \\ 0 & 0 & 1 & -x \\ 0 & 0 & 0 & \chi \end{pmatrix},$$

Lemma 2.11. *At least one of b and c is nonzero, and thus there is no surjection $A[\mathfrak{I}] \to \mu_7 \oplus \mu_7$ and no inclusion $\mathbf{Z}/7\mathbf{Z} \oplus \mathbf{Z}/7\mathbf{Z} \to A^\vee[\mathfrak{I}]$.*

Proof. If both b and c are zero then one finds that any $\ell \equiv 1 \bmod 7$ cannot be a good prime (that is, η_ℓ must kill $A[\mathfrak{I}]$). Yet this contradicts the fact that 29 is a good prime (as one can compute).[[Todo: A calc.]] □

It follows that the torsion subgroup of $A^\vee[\mathfrak{I}]$ has order at most 7.

Lemma 2.12. *We have $c \neq 0$, and $b = 0$.*

Proof. There is an injective map $A[\mathfrak{I}]/A_{\mathrm{old}}[\mathfrak{m}] \to A^\vee[\mathfrak{m}]$.[[Todo: This is this a calculation for William.]] If $c = 0$ then the image contains a rational 7-torsion point. Yet the intersection of the image with the cuspidal torsion group of $A^\vee$ is zero,[[Todo: A calc.]] and thus $A^\vee[\mathfrak{I}]$ would contain two linearly disjoint rational subgroups of order 7, contradicting Lemma 2.11. Thus $c \neq 0$. On the other hand, there is a rational torsion point of order 7 in $A^\vee[\mathfrak{m}]$,[[Todo: a calc.]] and thus (changing bases slightly if necessary) we have that $b = 0$. □

Let us now determine the extensions c and x.

Lemma 2.13. *The extensions c and x are unramified at 13, and the class y is nontrivial.*

Proof. The claim for x follows from the fact that $A_{\mathrm{old}}[\mathfrak{m}]$ is unramified at 13, by the criterion of Néron–Ogg–Shafarevich applied to $J_0(43)$. This determines the class of x uniquely. The class c corresponds to an extension of the form $\mathbf{Q}(\zeta_7, \sqrt[7]{D})$, where $D = 43 \cdot 13^k$ for some k or $D = 13$. To determine it explicitly we prove the following Sub-lemma, which is a variation on Lemma 2.10:

Sub-lemma 2.14. *Fix a D whose only prime divisors lie in the set $\{13, 43\}$. Let $K = \mathbf{Q}(\zeta_7, \sqrt[7]{D})$. Suppose that the class c becomes trivial over K. Let $\ell \equiv 1 \bmod 7$ and suppose that ℓ splits completely in K. Then η_ℓ is not a good prime. Equivalently,*

$$\eta_\ell = T_\ell - 1 - \ell \equiv 0 \bmod \mathfrak{I}.$$

Proof. Choosing an appropriate basis for $A[\mathfrak{I}]$ as a four dimensional $\mathbf{F}_7$ vector space we may write the action of $\sigma \in \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ as

$$\rho(\sigma) = \begin{pmatrix} 1 & x & y & a \\ 0 & \chi & 0 & 0 \\ 0 & 0 & \chi & c \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where χ is the cyclotomic character, and a, c, x, y are functions on $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$. Suppose that $\sigma = \mathrm{Frob}_\ell$, and that $\ell \equiv 1 \bmod 7$ so $\chi(\sigma) = 1$. We compute that

$$(\rho(\sigma) - 1)^2 = \begin{pmatrix} 0 & 0 & 0 & c(\sigma)y(\sigma) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

If ℓ splits completely in K then $c(\mathrm{Frob}_\ell) = 0$. Now letting $x = \rho(\sigma)$, and remembering that $\ell \equiv 1 \bmod 7$, we find that

$$0 = x^2 - T_\ell x + 1 = (x-1)^2 + \eta_\ell x = \eta_\ell x.$$

Since x is invertible, it follows that η_ℓ annihilates $A[\mathfrak{I}]$ and thus $\eta_\ell \in \mathfrak{I}$. □

The following table gives the smallest prime that splits completely in $\mathbf{Q}(\zeta_7, \sqrt[7]{D})$ for various D:[[Todo: Explain how computed.]]

D	43	$43 \cdot 13$	$43 \cdot 13^2$	$43 \cdot 13^3$	$43 \cdot 13^4$	$43 \cdot 13^5$	$43 \cdot 13^6$	13
ℓ	631	211	29	337	281	197	239	421
$T_\ell \in \mathfrak{I}$	Yes	No	No	No	No	No	No	No

Thus the only possibility for c is that it becomes trivial in $\mathbf{Q}(\zeta_7, \sqrt[7]{43})$. As a double check of our computations, we find the next few primes that split completely in $\mathbf{Q}(\zeta_7, \sqrt[7]{43})$. They are 659, 1009, 1289 and 1933. We check in all of these cases that $T_\ell \in \mathfrak{I}$. Thus $D = 43$ and c is unramified at 13. We also conclude from these calculations that $y \neq 0$, completing the proof of Lemma 2.13. □

2.3 The torsion subgroup of A and $A^\vee$

We may write the action of Galois on $A[\mathfrak{I}] = A[7, \mathfrak{m}^2]$ as follows:

$$\rho(\sigma) = \begin{pmatrix} 1 & x & y & a \\ 0 & \chi & 0 & b \\ 0 & 0 & \chi & c \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

Where x, y, a, b and c are all continuous functions of $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$. We may also assume that the first 2×2 matrix represents the action of Galois on $A_{\mathrm{old}}[\mathfrak{m}]$.

Theorem 2.15. *The rational 7-torsion subgroups of A and $A^\vee$ are exactly $\mathbf{Z}/7$.*

Proof. The cuspidal torsion subgroup of $A^\vee[7]$ is $\mathbf{Z}/7$, as can be computed.[[Todo: A calc.]] Suppose that there was an inclusion $\pi : \mathbf{Z}/7 \oplus \mathbf{Z}/7 \subseteq A^\vee[7]$. Then considering the Cartier dual there would be a section

$$\pi^\vee : A[\mathfrak{I}] \to \mu_7 \oplus \mu_7$$

which contradicts lemma 2.11. □

Note, that this just proves that the rational 7-torsion of $A^\vee$ has rank one. We prove now that there are no points of order 49.

Theorem 2.16. *The 7-Sylow subgroups of $A_{\mathrm{tors}}(\mathbf{Q})$ and $A^\vee_{\mathrm{tors}}(\mathbf{Q})$ have degree 7.*

Proof. Since all rational torsion of A is contained in $A[\mathfrak{m}]$ by Lemma 2.6, the first claim has already been proved. For the second, consider the variety $A' = A^\vee/\mu_7$. Then from our explicit description of the Galois action on 7-torsion we find that $\mathbf{Z}/7 \oplus \mathbf{Z}/7 \subset A'[7]$. Suppose that $\mathbf{Z}/49\mathbf{Z} \subset A^\vee_{\text{tors}}(\mathbf{Q})$. Then since the torsion of $A^\vee$ injects into the torsion of A' the torsion subgroup of A' must have order divisible by 343. This contradicts the multiple of torsion computed earlier.[[Todo: Ref to the calc.]] □

2.4 Conjectural Computations with the BSD conjecture

[[Todo: Here William writes down BSD and shows how in this case it implies that the odd part of Ш(A) is trivial.]]

2.5 Remarks on the Structure of the Tate Module

An important ingredient in our calculations was explicitly working with the action of the completed Hecke algebra $\mathbf{T}_\mathfrak{m}$ on the Tate module $A^{(\mathfrak{m})}$. What is the structure of this module? We know that as a $\mathbf{T}_\mathfrak{m}$ module, it is torsion free, and has "rank two" in the sense that $A^{(\mathfrak{m})} \otimes \mathbf{Q}$ is a free $\mathbf{T}_\mathfrak{m} \otimes \mathbf{Q}$ of rank two. In general the structure of finite rank torsion free modules over a local domain can be quite difficult (except in the case of a discrete valuation ring, of course!). However, certain special rings admit a nice description of their finite rank modules.

Definition 2.17. Let R be a local domain of dimension one with maximal ideal $\mathfrak{m}$. Let $\widetilde{R}$ denote the integral closure of R. Suppose that every ring R' such that $R \subseteq R' \subseteq \widetilde{R}$ is Gorenstein. Then R is a Bass ring.

For example, let $\widetilde{R}$ be the Witt vectors $W(\mathbf{F}_q)$ for $q = p^f$, let π be the reduction map $\pi : \widetilde{R} \to \mathbf{F}_q$, and let $R = \pi^{-1}(\mathbf{F}_p)$. Then R is a Bass ring. Taking the example $q = 49$ and $f = 2$ we find that $R = \mathbf{T}_\mathfrak{m}$ is a Bass ring.

The following theorem answers our questions about modules for Bass rings.

Theorem 2.18. *Let R be a Bass ring. Then any torsion free module of rank n is a direct sum of rank one modules M isomorphic to a subring R' satisfying*

$$R \subseteq R' \subseteq \widetilde{R}.$$

In our examples, this implies that the only rank one modules of $\mathcal{O}_\mathfrak{m}$ are either $\mathcal{O}_\mathfrak{m}$ or $\widetilde{\mathcal{O}}_\mathfrak{m}$. Since $\dim(\mathcal{O}_\mathfrak{m}/\mathfrak{m}) = 1$ and $\dim(\widetilde{\mathcal{O}}_\mathfrak{m}/\mathfrak{m}) = 2$ we conclude the following:

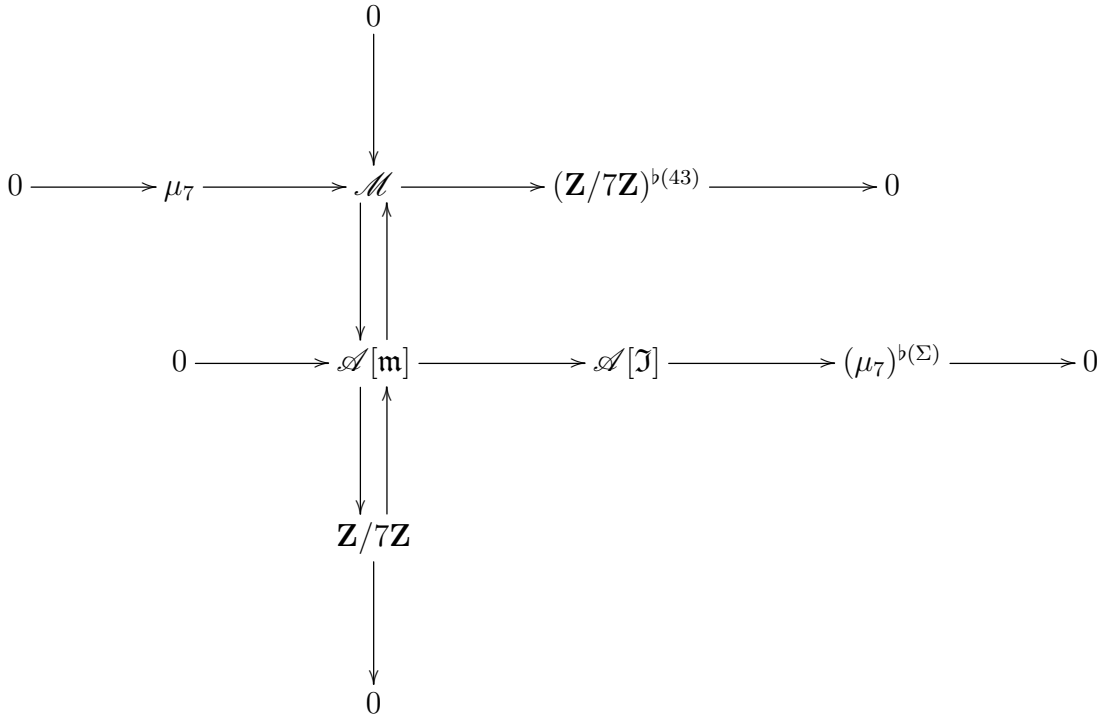
Lemma 2.19. *There is an isomorphism of Hecke modules*

$$A^{(\mathfrak{m})} \simeq \mathcal{O}_\mathfrak{m} \oplus \widetilde{\mathcal{O}}_\mathfrak{m}.$$

3 The Descent

The goal of this section is to prove that $Ш(A)[\mathfrak{m}]$ is trivial and that $A(\mathbf{Q})$ has rank 0.

It turns out to be easier to compute the descent on $A^\vee$ rather than A. Let $S = \mathrm{Spec}(\mathbf{Z})$. Let $\mathscr{A}$ be the Néron model of $A^\vee$ over S, and note that $\mathscr{A}[\mathfrak{J}]$ equals $\mathscr{A}[7] \otimes_{\mathbf{T}_7} \mathbf{T}_\mathfrak{m}$. For a group scheme G/S and a finite set of primes Σ let $G^{\flat(\Sigma)}$ denote the quasi-finite group scheme over S obtained from G by removing the special fibres at primes in Σ. We have the following exact diagram of quasi-finite group schemes over S:



Where the up and down arrows correspond to the fact that there exists sections, and Σ is a subset of $\{13, 43\}$. The inclusions $\mu_7, \mathbf{Z}/7\mathbf{Z} \to \mathscr{A}[\mathfrak{m}]$ exist because of the corresponding inclusions of Galois modules and the Néron mapping property applied to $\mathscr{A}$. The identity $\mathscr{M}/\mu_7 = (\mathbf{Z}/7\mathbf{Z})^{\flat(43)}$ similarly follows from the fact that the Galois module underlying $\mathscr{M}$ is ramified at 43 and unramified at 13 (the corresponding extension class is c, which we identified by Lemma 2.13) and the Néron mapping property. We note that $43 \in \Sigma$ because the extension class x is ramified at 43, by Lemma 2.13. Now we make the following observations on *fppf* cohomology for $S = \mathrm{Spec}(\mathbf{Z})$ (see for example [3], p.48):

Lemma 3.1. *The following hold.*

1. $\mathrm{H}^i(S, \mathbf{Z}/7\mathbf{Z}) = \mathrm{H}^i(S, \mu_7) = 0$ *for* $i = 1, 2$.
2. $\mathrm{H}^1(S, (\mathbf{Z}/7\mathbf{Z})^{\flat(43)}) = 0$, *and* $\mathrm{H}^2(S, (\mathbf{Z}/7\mathbf{Z})^{\flat(43)})$ *is one dimensional.*
3. $\mathrm{H}^1(S, \mathscr{M}) = 0$, *and* $\mathrm{H}^2(S, \mathscr{M}) = \mathrm{H}^2(S, (\mathbf{Z}/7\mathbf{Z})^{\flat(43)})$.

4. $\mathrm{H}^1(S, \mathscr{A}[\mathfrak{m}]) = 0$, *and* $\mathrm{H}^2(S, \mathscr{A}[\mathfrak{m}]) = \mathrm{H}^2(S, (\mathbf{Z}/7\mathbf{Z})^{\flat(43)})$.

5. $\mathrm{H}^1(S, \mathscr{A}[\mathfrak{I}]) = 0$.

6. *If* Φ *is the component group of* $\mathscr{A}$ *then* $\mathrm{H}^0(S, \Phi) \otimes \mathbf{Z}_7 = \mathbf{Z}/7\mathbf{Z}$.

7. *The map* $\mathrm{H}^0(\mathscr{A}) \to \mathrm{H}^0(\Phi)$ *is surjective.*

Proof. We prove each item in turn.

1. This follows from the discussion in [3], p.48, p.49.

2. Again we follow [3], p.48, p.49. There is an exact sequence
$$0 \to (\mathbf{Z}/7\mathbf{Z})^{\flat(43)} \to \mathbf{Z}/7\mathbf{Z} \to \phi \to 0$$
for some skyscraper sheaf ϕ supported in characteristic 43. Taking global sections we see that $\mathrm{H}^0(S, \mathbf{Z}/7\mathbf{Z})$ surjects onto $\mathrm{H}^0(S, \phi) = \mathbf{Z}/7\mathbf{Z}$ and thus $\mathrm{H}^1(S, (\mathbf{Z}/7\mathbf{Z})^{\flat(43)})$ injects into $\mathrm{H}^1(S, \mathbf{Z}/7\mathbf{Z}) = 0$. On the other hand, from part one we also conclude that $\mathrm{H}^2(S, (\mathbf{Z}/7\mathbf{Z})^{\flat(43)}) = \mathrm{H}^1(S, \phi)$ is one dimensional.

3. This follows from the long exact sequence of cohomology, and parts one and two.

4. Since $\mathscr{A}[\mathfrak{m}] = \mathscr{M} \oplus \mathbf{Z}/7\mathbf{Z}$, we have $\mathrm{H}^i(S, \mathscr{A}[\mathfrak{m}]) = \mathrm{H}^i(S, \mathscr{M}) \oplus \mathrm{H}^i(S, \mathbf{Z}/7\mathbf{Z})$.

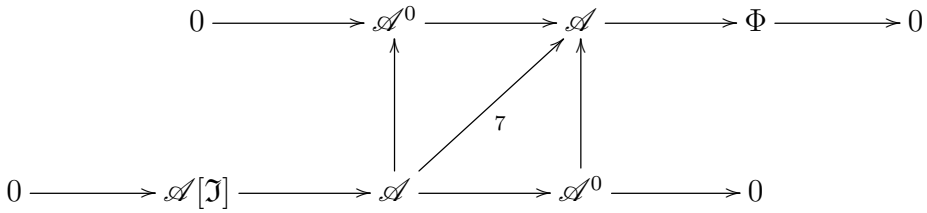
5. The long exact sequence of cohomology gives an exact sequence
$$0 \to \mathrm{H}^1(S, \mathscr{A}[\mathfrak{I}]) \to \mathrm{H}^1(S, (\mu_7)^{\flat(\Sigma)}) \to \mathrm{H}^2(S, \mathscr{A}[\mathfrak{m}])$$
Now the exact sequence
$$0 \to (\mu_7)^{\flat(\Sigma)} \to \mu_7 \to \psi \to 0$$
shows that $\mathrm{H}^0(S, \psi) = \mathrm{H}^1(S, (\mu_7)^{\flat(\Sigma)})$. Since
$$\mathrm{H}^0(S, \psi) = \bigoplus_{p \in \Sigma} \mu_7(\mathbf{F}_p)$$
and since $\mathbf{F}_{13}$ has no 7th roots of unity we see that the dimension of $\mathrm{H}^1(S, (\mu_7)^{\flat(\Sigma)})$ is the dimension of $\mu_7(\mathbf{F}_{43})$, which is one. On the other hand, the target of the coboundary map $\mathrm{H}^2(S, \mathscr{A}[\mathfrak{m}]) \simeq \mathrm{H}^1(S, (\mathbf{Z}/7\mathbf{Z})^{\flat(43)})$ is one dimensional also. Thus it suffices to show that the coboundary map
$$\mu_7(\mathbf{F}_{43}) \simeq H^1(S, (\mu_7)^{\flat(\Sigma)}) \longrightarrow \mathrm{H}^1(S, (\mathbf{Z}/7\mathbf{Z})^{\flat(43)}) \simeq H^1(S, \phi)$$
is nontrivial. Perhaps more helpfully, this boundary map is the same as the one coming from the associated long exact sequence of the following long exact sequence:
$$0 \to (\mathbf{Z}/7\mathbf{Z})^{\flat(43)} \to \mathscr{N} \to (\mu_7)^{\flat(\Sigma)} \to 0,$$
Where $\mathscr{N} = \mathscr{A}[\mathfrak{I}]/(\mathbf{Z}/7\mathbf{Z} \oplus \mu_7)$. The Galois extension corresponding to the generic fibre of $\mathscr{N}$ is given by extension class y.[[Todo: To myself, this is what needs to be completed.]]

6. This follows from the [3], Appendix, Thm. A.1.

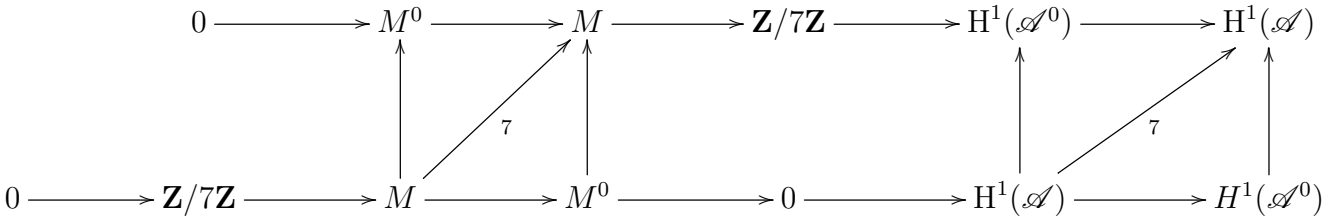
7. This follows from the identification of Φ above and a computation...[[Todo: Is this right? Or is it obvious]]

□

Let us consider the usual descent sequence:



Let M be the Mordell–Weil group of A. By the Néron mapping property $M = \mathscr{A}(S)$. Let $M^0 = \mathscr{A}^0(S)$. Taking cohomology and using parts 5 & 7 of Lemma 3.1, we get the following exact diagram[[Todo: William, make the map $M \to \mathbf{Z}/7\mathbf{Z}$ a surjective arrow]], where by abuse of notation we consider everything (as in [3] p.150) in the category of $\mathbf{T}$-modules modulo the category of $\mathbf{T}$-modules whose support lies in some finite set Δ containing all primes of characteristic seven besides $\mathfrak{m}$.



From this diagram we easily conclude that $M \simeq \mathbf{Z}/7\mathbf{Z}$. It also follows that the multiplication by 7 map on $H^1(\mathscr{A})$ is injective (modulo $\mathbf{T}$-modules with support outside $\mathfrak{m}$), and thus that that $H^1(\mathscr{A})[\mathfrak{m}] = 0$. By [4], Prop, p.263, the Tate–Shafarevich group Ш(A) injects into $H^1(\mathscr{A})$, and thus it follows that Ш$(A)[\mathfrak{m}] = 0$.

4 Algorithmic Computations

In this section we explain how to use explicit computations to verify several key assertions that appear elsewhere in this paper. All computations below were done using Sage (see [S+11]).

4.1 The quotient by the Eisenstein ideal

This section is about the quotient $\mathbf{T}/\mathscr{I}$ that arises in Section 2. Our strategy is useful more generally for computing $\mathbf{T}/I$ for any Hecke algebra $\mathbf{T}$ and ideal I. The main ideas are (1) the Sturm bound, (2) embedding $\mathbf{T}$ in a $\mathbf{Q}$ vector space V of dimension $\dim S_2(\Gamma_0(N))$, which replaces linear algebra in a space of dimension n^2 by linear algebra in a space of dimension n, and (3) a method for computing the $\mathbf{T}$-ideal generated by given elements of $\mathbf{T}$ in terms of the representation (2). The entire computation involves only linear algebra, and makes no use of Groebner basis, so the complexity is easy to understand, and there is a natural mod p analogue of the algorithm.

Using modular symbols, we compute a $\mathbf{T}$-module S that is isomorphic to $S_2(\Gamma_0(N); \mathbf{Q})$, and for the purposes of our computations we work instead with S:

```
sage: S = ModularSymbols(559, sign=1).cuspidal_subspace(); S
Modular Symbols subspace of dimension 49 of Modular Symbols space of dimension 52 for
Gamma_0(559) of weight 2 with sign 1 over Rational Field
```

We view the Hecke operators as matrices acting from the right on row vectors in $V = \mathbf{Q}^{49}$. We choose a vector v such that the map $\mathbf{T} \to V$ given by $t \mapsto v.t$ is injective. With respect to our chosen basis, the vector $v = (1, 0, 0, \ldots, 0, 0)$ works; moreover, $vT_1, vT_2, \ldots, vT_{49}$ are linearly independent elements of V:

```
sage: def hecke_image(v, B): return [ v * S.hecke_matrix(i) for i in [1..B] ]
sage: v = vector(QQ, 49); v[0] = 1
sage: span(hecke_image(v, 49)).dimension()
49
```

To obtain an isomorphic copy of the exact Hecke algebra $\mathbf{T}$ as a $\mathbf{Z}$-submodule $L \subset V$, we use the Sturm bound:

```
sage: S.sturm_bound()
103
sage: L = span(ZZ, hecke_image(v,103))
```

Now that we know L we can try a smaller bound in place of 103; we find that 52 works, which means that $T_1, \ldots, T_{52}$ generate $\mathbf{T}$ as a $\mathbf{Z}$-module (no smaller number works).

```
sage: L2 = span(ZZ, hecke_image(v, 52))
sage: L2.index_in(L)
1
sage: span(ZZ,hecke_image(v,51)).index_in(L)
2
```

Thus the T_n, for $n \leq 52$, are generators for $\mathbf{T}$ as a $\mathbf{Z}$-module, hence generate $\mathbf{T}$ as a ring. Each T_n can be written as a polynomial over $\mathbf{Z}$ in the T_p for primes $p < 52$. Thus $\mathscr{I}$ is the ideal generated by $T_\ell - (1 + \ell)$ for primes $\ell \neq 13, 43$ with $\ell < 52$ and $T_{13} + 1$ and $T_{43} - 1$. We let M be the $\mathbf{Z}$-module (not ideal) generated by these elements:

```
sage: gens = [(S.hecke_matrix(ell) - (ell+1)) for ell in primes(52) if 559%ell != 0]
sage: gens.extend([S.hecke_matrix(13) + 1, S.hecke_matrix(43) - 1])
```

```
sage: M = span(ZZ, [v*g for g in gens])
```

We now enlarge M by multiplying by all Hecke operators T_ℓ for $\ell < 52$. Note that we compute $(vT_\ell)g$ instead of $v(T_\ell g)$, since the former involves only vector-matrix multiplcation, whereas the later involves matrix-matrix multiplication.

```
sage: for p in primes(53): M += span(ZZ, [(v*S.hecke_matrix(p))*g for g in gens])
sage: M.index_in(L)
14
```

We enlarge again and find that M does not get any bigger, which proves that M is now $\mathbf{T}$-invariant, hence for this enlarged M we have $M = \mathscr{I}$, so $T/\mathscr{I} \cong \mathbf{Z}/14\mathbf{Z}$, as claimed:

```
sage: for p in primes(53): M += span(ZZ, [(v*S.hecke_matrix(p))*g for g in gens])
sage: M.index_in(L)
14
```

4.2 The 15-dimensional abelian variety A

Let F be the characteristic polynomial of T_2 acting on $S_2(\Gamma_0(559))_{\text{new}}$, which we compute as follows using modular symbols:

```
sage: N = ModularSymbols(559, sign=1).cuspidal_subspace().new_subspace()
sage: F = N.T(2).charpoly()
```

We find that exactly one of the irreducible factor h of F has 3 as a root modulo 7, and that factor occurs with multiplicity one:

```
sage: v = [Z for Z in factor(F) if Z[0].change_ring(GF(7))(3) == 0]; v
[(x^15 - 2*x^14 - 22*x^13 + 43*x^12 + 187*x^11 - 354*x^10 - 769*x^9 + 1395*x^8 + 1553*x^7
        - 2684*x^6 - 1328*x^5 + 2265*x^4 + 241*x^3 - 606*x^2 + 33*x + 13, 1)]
sage: h = v[0][0]
```

Let $A = \ker(h(T_2))^0 \subset J_0(559)$ be the identity component of the kernel of $h(T_2)$, which is a 15-dimensional abelian variety. The above calculation shows that this is the only simple abelian subvariety of $J_0(559)_{\text{new}}$ with $A[(7, T_2 - 3)] \neq 0$. Since there is some simple B with $B[\mathfrak{m}] \neq 0$ and $(7, T_2 - 3) \subset \mathfrak{m}$, this A must be it.

4.3 The Index of $\mathcal{O}_\mathfrak{m}$ in its integral closure

Let $\mathcal{O}$ be the ring generated by the Hecke algebra acting on A. This ring contains with finite index the ring $R = \mathbf{Z}[a_2]$, where a_2 is a root of the polynomial h from Section 4.2 above. The integral closure of $\mathcal{O}$ is the integral closure $\mathcal{O}_K$ of R, and R has index $7^2 \cdot 499$ in $\mathcal{O}_K$:

```
sage: K.<a> = NumberField(h)
sage: OK = K.maximal_order()
sage: factor(K.order(a).index_in(OK))
7^2 * 499
```

To compute the index of $\mathcal{O}$ in $\mathcal{O}_K$, we compute the index $[\mathcal{O} : R]$, which we do using a similar trick to the one in Section 4.1 above. First we compute the simple factor of the modular symbols space corresponding to our 15-dimensional A:

```
sage: M = ModularSymbols(559, sign=1)
sage: D = M.cuspidal_subspace().new_subspace().decomposition()[-1]; D
Modular Symbols subspace of dimension 15 of Modular Symbols space of dimension 52 for Gamma_0(559) of weigh
```

Next, we find the "rational period mapping", which is by definition some homomorphism $M \to V = \mathbf{Q}^{15}$ with kernel the largest $\mathbf{T}$-stable complement of the $\mathbf{T}$-module D. This map is called a "period mapping", because the kernel is the same as the kernel of the period mapping got by integrating cuspidal modular symbols against the cusp form f and its Galois conjugates. It is computed by using linear algebra over $\mathbf{Q}$ to find the subspace of $\mathrm{Hom}(M, \mathbf{Q})$ on which $h(T_2) = 0$, where h is as in Section 4.2, which amounts to finding the kernel of $h(T_2^t)$.

```
sage: phi = D.rational_period_mapping()
```

We use φ to define a homomorphism $\mathbf{T} \to \mathbf{Q}^{15}$ by $t \mapsto \varphi(vt)$, where v is an element of M so that this map is nonzero. We just choose the first basis vector v of M. Next we compute the image of $\mathcal{O}$ in V via this map, using a function that efficiently computes the action of Hecke operators on a specific basis vector of M.

```
sage: O = span(ZZ, [phi(M._hecke_image_of_ith_basis_vector(n, 0)) for n in [1..52]])
```

Now that we have $\mathcal{O}$, we next compute the image of $\mathbf{Z}[a_2]$, which is the $\mathbf{Z}$-module spanned by $1, a_2, \ldots, a_2^{14}$, which is the same as the images of $1, T_2, \ldots, T_2^{14}$ under our above map. To avoid matrix multiplication, we instead compute the iterates of v under the action of T_2, and take their image under φ.

```
sage: T2 = M.hecke_matrix(2)
sage: R = span(ZZ, [phi(v) for v in T2.iterates(M.0.element(), 15)])
```

Finally, we observe that the index of R in $\mathcal{O}$ is $7 \cdot 499$, hence the image of $\mathcal{O}$ in $\mathcal{O}_K$ is 7.

```
sage: factor(R.index_in(O))
7 * 499
```

The ideal 7 factors in $\mathcal{O}_K$ as a product $\mathfrak{p}_2\mathfrak{p}_6\mathfrak{p}_7$ with $\mathfrak{p}_i$ of degree i:

```
sage: [P.residue_class_degree() for P, e in K.factor(7)]
[7, 6, 2]
```

Note that R is already maximal at the two primes over 7 of degrees 6 and 7, since the degree 6 and degree 7 factors already appear with multiplicity 1 modulo 7:

```
sage: h.factor_mod(7)
(x + 4)^2 * (x^6 + 4*x^5 + 4*x^4 + 2*x^2 + x + 6) *
               (x^7 + 3*x^5 + 2*x^4 + 4*x^3 + x^2 + 3*x + 4)
```

Thus the prime over 7 at which we maximize to go from $\mathcal{O}$ to $\mathcal{O}_K$ is the one that contains $T_2 - 3$, i.e., the ideal $\mathfrak{m}$. We finally conclude that $[(\mathcal{O}_K)_{\mathfrak{p}_2} : \mathcal{O}_{\mathfrak{m}}] = 7$.

Remark 4.1. In Sage, we could compute the q-expansion of a newform associated to A by typing `D.q_eigenform(53,'a2')`; however, the resulting coefficients would then be expressed in terms of a power basis for $\mathbf{Q}(a_2)$, which involves fairly large numbers, making the rest of the calculation of the index much less efficient. This would not scale well to bigger computations.

4.4 The modular degree of A

The modular degree of A is by definition the square root of the degree of the map $A \to A^\vee$ induced by $A \to J_0(N) \cong J_0(N)^\vee \to A^\vee$. To compute the modular degree of A we use the modular degree function in Sage, which is an implementation of the algorithm described in [KS00, §3.1].

```
sage: M = ModularSymbols(559, sign=1)
sage: D = M.cuspidal_subspace().new_subspace().decomposition()[-1]; D
Modular Symbols subspace of dimension 15 of Modular Symbols space of dimension 52
for Gamma_0(559) of weight 2 with sign 1 over Rational Field
```

```
sage: A = D.abelian_variety()
sage: factor(A.modular_degree())      # long time -- 3 minutes!
2^21 * 7 * 31
```

If we are just interested in the odd part of the modular degree, we can compute a quantity that is equal to the modular degree up to a power of 2 much more quickly as follows:

```
sage: phi = D.integral_period_mapping()
sage: factor(matrix([phi(a) for a in D.integral_basis()]).det())
-1 * 2^20 * 7 * 31
```

In any case, we conclude that 7 exactly divides $\deg(A \to A^\vee)$, as claimed.

4.5 The kernels of ideals acting on A

4.6 Simultaneous pathologies

Compute the Eisenstein maximal ideals of Hecke that are primes of fusion and self-fusion for all levels up to 559? If this is even possible...

4.7 Computing the minimal degree of an isogeny to the dual

4.8 Table of minimal isogeny degrees

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