Computations of elliptic units for real quadratic fields

Samit Dasgupta

September 3, 2004

Contents

Introduction

Elliptic units, which are obtained by evaluating modular units at quadratic imaginary arguments of the Poincaré upper half-plane, allow the analytic construction of abelian extensions of imaginary quadratic fields. The Kronecker limit formula relates the complex absolute values of these units to values of zeta functions, and allowed Stark to prove his rank one archimedean conjecture for abelian extensions of quadratic imaginary fields [9].

A conjectural construction of an analogous theory for real quadratic fields K was proposed in [2], by replacing the infinite prime of $\mathbf{Q}$ with a prime p that remains inert in K. The completion K_p is a quadratic unramified extension of $\mathbf{Q}_p$. The construction of [2] associates to a modular unit α and any $\tau \in K - \mathbf{Q}$ an element $u(\alpha, \tau) \in K_p^\times$ which is conjectured to be a p-unit in a specific narrow ring class field of K depending on τ and denoted H_τ (cf. [2, Conjecture 2.14], hereafter denoted Conjecture DD). In harmony with the fact that the role of ∞ is played by that of p, the construction of $u(\alpha, \tau)$ involves p-adic integration in a manner motivated by the definition of "Stark-Heegner points" given in [1] and generalized in [5].

The main theorems of [2] relate the p-adic valuation and p-adic logarithm of $u(\alpha, \tau)$ to values at 0 of partial zeta functions (classical and p-adic, respectively) attached to the extension H_τ/K, and thereby allow one to deduce Gross's p-adic analogue of Stark's conjecture (see [6]) for this extension from Conjecture DD .

In the present article we provide concrete computational evidence for Conjecture DD. The formulas of [2, §4.4] may be used to calculate the units $u(\alpha, \tau)$ to a high p-adic accuracy. Using 50 digits of p-adic accuracy for $p = 3, 5, 7, 11$ and all ground fields K of positive discriminant less than 500, we are able to recognize the units $u(\alpha, \tau)$ as algebraic numbers for the fixed modular unit

$$\alpha = \Delta(z)^2\Delta(4z)/\Delta(2z)^3$$

and all τ such that H_τ is the narrow Hilbert class field of K. In each case, the algebraic number approximated by $u(\alpha, \tau)$ is a p-unit in H_τ as predicted by Conjecture DD.

We begin by recalling the definition of $u(\alpha, \tau)$, and proving that a certain modular symbol of measures μ appearing in this definition is $\mathbf{Z}$-valued. In [2], it is only proven that μ is $\mathbf{Z}_p$-valued. In Sections 4 and 5 we describe the method and results of our computations that supply empirical evidence for Conjecture DD.

This work grew out of discussions with H. Darmon, and the computations were done in MAGMA on W. Stein's MECCAH machines at Harvard University. The results were included in my PhD. thesis at the University of California-Berkeley, under the guidance of K. Ribet. I am grateful to all of them for their assistance.

1 Definition of the units

Let N be a positive integer. A *modular unit* is a holomorphic nowhere vanishing function on $\mathcal{H}/\Gamma_0(N)$ that extends to a meromorphic function on the compact Riemann surface $X_0(N)(\mathbf{C})$. A typical example of such a unit is the modular function

$$\alpha(\tau) = \prod_{d|N} \Delta(d\tau)^{n_d} \tag{1}$$

for integers n_d such that $\sum_d n_d = 0$. We will assume that the modular unit α has no zero or pole at the cusp ∞ of the completed upper half plane $\mathcal{H}^* = \mathcal{H} \cup \mathbf{P}^1(\mathbf{Q})$. This assumption is equivalent to the equation

$$\sum_d n_d d = 0. \tag{2}$$

Let p be a prime number not dividing N. Given the modular unit α of level N, we may define a modular unit of level Np by the rule

$$\alpha^*(z) := \alpha(z)/\alpha(pz).$$

The logarithmic derivatives of α and α^* are given by

$$\mathrm{dlog}\,\alpha(z) = 2\pi i F_2(z)\,\mathrm{d}z, \qquad \mathrm{dlog}\,\alpha^*(z) = 2\pi i F_2^*(z)\,\mathrm{d}z, \tag{3}$$

where $F_2(z)$ and $F_2^*(z)$ are the weight two Eisenstein series on $\Gamma_0(N)$ and $\Gamma_0(Np)$, respectively, given by the formulae

$$F_2(z) = -24\sum_{d|N} dn_d E_2(dz), \qquad F_2^*(z) = F_2(z) - pF_2(pz). \tag{4}$$

Here $E_2(z)$ is the standard Eisenstein series of weight 2:

$$E_2(z) = \frac{1}{(2\pi i)^2}\left(\zeta(2) + \frac{1}{2}\sum_{\substack{m=-\infty\\ m\neq 0}}^{\infty}\sum_{n=-\infty}^{\infty}\frac{1}{(mz+n)^2}\right) = -\frac{1}{24} + \sum_{n=1}^{\infty}\sigma_1(n)q^n, \quad q = e^{2\pi i\tau}. \tag{5}$$

(We remark that the double series used to define E_2 is not absolutely convergent and the resulting expression is not invariant under $\mathbf{SL}_2(\mathbf{Z})$.)

Let $\mathcal{M} = \mathrm{Div}_0(\Gamma_0(N)\infty)$ denote the group of degree-zero divisors on set of cusps with denominator divisible by N. Note that $\mathcal{M}$ has a natural left action by $\Gamma_0(N)$. A *partial modular symbol* with values in a group A is simply a group homomorphism from $\mathcal{M}$ to A. If ψ is a partial modular symbol and $r, s \in \Gamma_0(N)\infty$, then one writes

$$\psi\{r \to s\} \text{ or } \psi_m \quad \text{for} \quad \psi([r]-[s]), \text{ where } m = [r]-[s] \in \mathcal{M}.$$

Assumption (2) implies that the differential $\mathrm{dlog}\,\alpha$ on $\mathcal{H}^*$ is regular on $\Gamma_0(N)\infty$, so we may define a partial modular symbol ψ by the rule

$$\psi\{r \to s\} := \frac{1}{2\pi i}\int_r^s \mathrm{dlog}\,\alpha = \int_r^s F_2(z)\,\mathrm{d}z,$$

where the complex line integral on the right side is taken along any smooth path P in $\mathcal{H}^*$ connecting the cusps r and s. The rational integer $\psi\{r \to s\}$ may be understood as the winding number of the closed loop $\alpha(P)$ around the origin in the complex plane. The function ψ (which was denoted m_α in [2]) is called the partial modular symbol attached to α, and its value will be expressed in terms of classical Dedekind sums in Section 2.

The Eisenstein series of (4) and (5) are part of a natural *family* of Eisenstein series of varying weights. For even $k \geq 2$, consider the standard Eisenstein series of weight k:

$$E_k(z) = \frac{2(k-1)!}{(2\pi i)^k}\sum_{m,n=-\infty}^{\infty}{}' \frac{1}{(mz+n)^k} = -\frac{B_k}{2k} + \sum_{n=1}^{\infty}\sigma_{k-1}(n)q^n. \tag{6}$$

Define likewise the higher weight Eisenstein series

$$\begin{aligned} F_k(z) &= -24 \sum_{d|N} n_d \cdot d \cdot E_k(dz) \\ &= -\frac{48(k-1)!}{(2\pi i)^k} \sideset{}{'}\sum_{m,n=-\infty}^{\infty} \left(\frac{1}{(mz+n)^k} \sum_{d|(N,m)} n_d d \right) \\ &= -24 \sum_{n=1}^{\infty} \sigma_{k-1}(n) \sum_{d|N} n_d d q^{nd}. \end{aligned} \tag{7}$$

The F_k are modular forms of weight k on $\Gamma_0(N)$ that are holomorphic on the upper half plane. Note that these Eisenstein series have no constant term and hence are holomorphic at the cusp ∞. We also define, for the purpose of p-adic interpolation, the function

$$F_k^*(z) = F_k(z) - p^{k-1} F_k(pz).$$

We extend the definition of $E_k(z)$ and $F_k(z)$ to all $k \geq 2$ by letting $E_k = F_k = 0$ for k odd.

Let $\mathbf{X}$ denote the subspace of $\mathbf{Z}_p \times \mathbf{Z}_p$ consisting of all pairs (a, b) such that a and b are not both divisible by p in $\mathbf{Z}_p$. This space of "primitive vectors" makes an appearance in the earlier work of Greenberg and Stevens [7]. Denote by $\mathrm{Meas}(\mathbf{X}, \mathbf{Z})$ the group of additive measures on $\mathbf{X}$ with values in $\mathbf{Z}$; this is the set of functions ν which assign to each compact open set $U \subset \mathbf{X}$ an element $\nu(U) \in \mathbf{Z}$ such that

- $\nu(U \cup V) = \nu(U) + \nu(V)$ for disjoint compact open U and V, and
- $\nu(\mathbf{X}) = 0$.

The crucial technical ingredient in the definition of $u(\alpha, \tau)$ is the following result (see [2, Theorem 4.2]).

Theorem 1.1. *Let α be fixed as above. There is a unique* $\mathrm{Meas}(\mathbf{X}, \mathbf{Z})$*-valued partial modular symbol μ such that for every homogeneous polynomial $h(x, y) \in \mathbf{Z}[x, y]$ of degree $k-2$,*

$$\int_{\mathbf{X}} h(x, y)\, \mathrm{d}\mu\{r \to s\}(x, y) = \mathrm{Re}\left((1 - p^{k-2}) \int_r^s h(z, 1) F_k(z)\, \mathrm{d}z \right). \tag{8}$$

The p-adic integral on the left side of (8) is defined to be

$$\lim_{||\mathcal{U}|| \to 0} \sum_{U \in \mathcal{U}} h(x_U, y_U) \cdot \mu\{r \to s\}(U) \in \mathbf{Z}_p,$$

where $\mathcal{U}$ is a cover of $\mathbf{X}$ by disjoint compact opens, (x_U, y_U) is an arbitrary point of $U \in \mathcal{U}$, and the p-adic limit is taken over uniformly finer covers $\mathcal{U}$. Implicit in the statement of

Theorem 1.1 is the fact that the real numbers on the right side of (8) are in fact rational; we will give explicit formulas for these numbers in terms of generalized Dedekind sums in Section 2. In Section 3, we will prove that the partial modular symbol of measures μ is in fact $\mathbf{Z}$-valued (rather than just $\mathbf{Z}_p$-valued, as proven in [2]).

The space $\mathbf{X}$, viewed as a subspace of the larger space $\mathbf{Y} := \mathbf{Q}_p^2 - \{0\}$, forms a fundamental domain for the action of multiplication by p on $\mathbf{Y}$. Hence the measures $\mu\{r \to s\}$ can be extended uniquely to measures on $\mathbf{Y}$ which are invariant under multiplication by p:

$$\mu\{r \to s\}(pU) = \mu\{r \to s\}(U)$$

for all compact open $U \subset \mathbf{Y}$. The group $\mathbf{GL}_2(\mathbf{Q}_p)$ acts on $\mathbf{Y}$ by left multiplication by viewing the elements of $\mathbf{Y}$ as column vectors. The partial modular symbol of measures μ satisfies the following additional properties:

- For all
$$\gamma \in \tilde{\Gamma} := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{GL}_2^+(\mathbf{Z}[1/p]) : N|c \right\}$$
and all compact open $U \subset \mathbf{Y}$,
$$\mu\{\gamma r \to \gamma s\}(\gamma U) = \mu\{r \to s\}(U).$$

- For every homogeneous polynomial $h(x, y) \in \mathbf{Z}[x, y]$ of degree $k - 2$,
$$\int_{\mathbf{Z}_p \times \mathbf{Z}_p^\times} h(x, y)\, \mathrm{d}\mu\{r \to s\}(x, y) = \mathrm{Re}\left(\int_r^s h(z, 1) F_k^*(z)\, \mathrm{d}z \right). \tag{9}$$

We are now ready to define $u(\alpha, \tau)$. Let K be a real quadratic field such that p is inert in K, a let $\tau \in K - \mathbf{Q}$. Assume that the reduction of τ modulo p, which is an element of $\mathbf{P}^1(\mathbf{F}_{p^2})$, does not lie in $\mathbf{P}^1(\mathbf{F}_p)$. Choose the real embedding of K in which τ is greater than its Galois conjugate. Denote by Γ_τ the stabilizer of τ in

$$\Gamma := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{SL}_2(\mathbf{Z}[1/p]) : N|c \right\} \subset \tilde{\Gamma},$$

acting via linear fractional transformations. Let γ_τ be the unique element of Γ_τ whose image generates the quotient $\Gamma_\tau/\langle \pm 1 \rangle \cong \mathbf{Z}$ such that

$$\gamma_\tau \begin{pmatrix} \tau \\ 1 \end{pmatrix} = \varepsilon \begin{pmatrix} \tau \\ 1 \end{pmatrix}$$

with $\varepsilon > 1$. Write $\gamma_\tau = \begin{pmatrix} a & b \\ Nc & d \end{pmatrix}$.

We define the element $u(\alpha,\tau) \in K_p^\times$ by the formula

$$u(\alpha,\tau) = p^{\psi\{\infty\to a/Nc\}} \cdot \times\!\!\!\!\!\!\int_{\mathbf{X}} (x-y\tau)d\mu\{\infty\to a/Nc\}(x,y). \tag{10}$$

The p-adic multiplicative integral on the right side of (10) is defined to be

$$\lim_{||\mathcal{U}||\to 0} \prod_{U\in\mathcal{U}} (x_U - y_U\tau)^{\mu\{\infty\to a/Nc\}(U)} \in \mathcal{O}_p^\times,$$

with the notation as in (1). Here $\mathcal{O}_p$ denotes the ring of integers of K_p. Note that the definition of the multplicative integral is contingent on Proposition 3.2 below, which states that μ is $\mathbf{Z}$-valued; this constitutes an improvement over the definition of [2], where $u(\alpha,\tau)$ was defined only up to a root of unity in $K_p^\times$.

The element $u(\alpha,\tau) \in K_p^\times$ is conjectured to lie in a ring class field extension of K. To be precise, assume that the minimal quadratic polynomial with integer coefficients satisfied by τ has the form

$$A\tau^2 + B\tau + C = 0, \quad (A,B,C) = 1, \quad A > 0$$

where

$$N|A \text{ and } D = B^2 - 4AC \text{ is relatively prime to } Np. \tag{11}$$

Conjecture DD. *The element $u(\alpha,\tau) \in K_p^\times$ is a p-unit in the narrow ring class field H_D attached to the discriminant $D > 0$.*

2 Dedekind sums

In this section we relate the integrals appearing in the right of (8) and (9) to generalized Dedekind sums. The explicitly calculable formulas of this section will be used in the computations of $u(\alpha,\tau)$.

The classical Bernoulli polynomials B_n are defined by the power series

$$\frac{e^{tx}}{e^t - 1} = \sum_{n=0}^{\infty} \frac{B_n(x)}{n!} t^{n-1}.$$

We modify the standard Bernoulli polynomials and make them periodic by defining

$$\tilde{B}_s(x) := \begin{cases} 0 & \text{if } s = 1 \text{ and } x \in \mathbf{Z} \\ B_s(\{x\}) = B_s(x - [x]) & \text{otherwise.} \end{cases}$$

Let s and t be positive integers. For a and c relatively prime and $c \geq 1$, the generalized Dedekind sum $D_{s,t}(a/c)$ is defined by

$$D_{s,t}(a/c) := \frac{c^{s-1}}{st} \sum_{h=1}^{c} \tilde{B}_s(h/c)\tilde{B}_t(ha/c).$$

In terms of these generalized Dedekind sums, we have from [2, §4.4 and §4.7]

$$\mathrm{Re}\left[\int_{\infty}^{\frac{a}{Nc}} z^n F_k(z)\,\mathrm{d}z\right] = -12\sum_{\ell=0}^{n}\binom{n}{\ell}\left(\frac{a}{Nc}\right)^{n-\ell}(-1)^{\ell}\sum_{d|N}\frac{n_d}{d^{\ell}}D_{k-\ell-1,\ell+1}\left(\frac{a}{Nc/d}\right), \tag{12}$$

$$\begin{aligned}\mathrm{Re}\left[\int_{\infty}^{\frac{a}{Nc}} z^n F_k^*(z)\,\mathrm{d}z\right] &= -12\sum_{\ell=0}^{n}\binom{n}{\ell}\left(\frac{a}{Nc}\right)^{n-\ell}(-1)^{\ell}\times \\ &\sum_{d|N}\frac{n_d}{d^{\ell}}\left[D_{k-\ell-1,\ell+1}\left(\frac{a}{Nc/d}\right)-p^{k-\ell-2}D_{k-\ell-1,\ell+1}\left(\frac{pa}{Nc/d}\right)\right].\end{aligned} \tag{13}$$

Equation (12) for $k=2$ and $n=0$ yields the formula

$$\psi\{\infty\to a/Nc\} = -12\sum_{d|N} n_d D\left(\frac{a}{Nc/d}\right),$$

where $D = D_{1,1}$ is the classical Dedekind sum.

3 Integrality of the measures

In this section we prove that the measures μ_m, which are proven in [2] to be only $\mathbf{Z}_p$-valued, actually take on integer values. We begin by reviewing the single-variable measures arising from Bernoulli polynomials; our presentation is motivated by [8, §10.2]. Let $e \geq 1$ be a positive integer divisible by N but not by p, and let

$$Z = \lim_{\leftarrow} \mathbf{Z}/ep^n\mathbf{Z} \cong \mathbf{Z}/e\mathbf{Z}\times\mathbf{Z}_p.$$

For each integer $k \geq 1$, define a $\mathbf{Z}_p$-valued measure $\mathcal{F}_k$ on Z corresponding to the Eisenstein series F_{2k} by the rule

$$\mathcal{F}_k(a+ep^n\cdot Z) := \sum_{d|N} n_d\left(\frac{ep^n}{d}\right)^{k-1}\cdot\frac{1}{k}\cdot\tilde{B}_k\left(\frac{a}{ep^n/d}\right)$$

for each integer a. The distribution relation

$$\sum_{a=1}^{f}\tilde{B}_k\left(x+\frac{a}{f}\right) = f^{k-1}\tilde{B}_k(x) \tag{14}$$

for Bernoulli polynomials demonstrates that $\mathcal{F}_k$ is indeed a distribution for each $k \geq 1$. Furthermore, the proof of the following proposition shows that these measures are actually $\mathbf{Z}_p$-valued. For $x \in Z$, let x_p denote the projection of x onto $\mathbf{Z}_p$.

Proposition 3.1. *For every compact open set $U \subset Z$ we have*

$$\mathcal{F}_k(U) = \int_U x_p^{k-1}\,\mathrm{d}\mathcal{F}_1(x).$$

Proof. It suffices to consider U of the form $U = a + ep^n Z$ for integers a. We will prove that

$$\mathcal{F}_k(U) \equiv a^{k-1}\mathcal{F}_1(U) \pmod{p^{n-\epsilon}\mathbf{Z}_p} \tag{15}$$

where ϵ depends only on k. The key fact is that the Bernoulli polynomial $B_k(x)$ begins $x^k - \frac{1}{2}kx^{k-1} + \cdots$. Therefore

$$\mathcal{F}_k(U) \equiv \sum_{d|N} n_d \left(\frac{ep^n}{d}\right)^{k-1} \frac{1}{k}\left(\left(\frac{da}{ep^n} - \left[\frac{da}{ep^n}\right]\right)^k - \frac{k}{2}\left(\frac{da}{ep^n} - \left[\frac{da}{ep^n}\right]\right)^{k-1}\right) \tag{16}$$

modulo $p^{n-\epsilon}\mathbf{Z}_p$, where ϵ is the largest power of p appearing in the denominators of the coefficients of $B_k(x)/k$. The congruence (16) yields:

$$\begin{aligned}\mathcal{F}_k(U) &\equiv \sum_{d|N} \frac{n_d}{k}\left(\left(\frac{d}{ep^n}\right)a^k - ka^{k-1}\left[\frac{da}{ep^n}\right] - \frac{k}{2}a^{k-1}\right) \pmod{p^{n-\epsilon}\mathbf{Z}_p} \\ &\equiv -\sum_{d|N} n_d a^{k-1}\left[\frac{da}{ep^n}\right], \qquad (17)\end{aligned}$$

where (17) uses $\sum n_d = \sum n_d d = 0$. Meanwhile we find

$$a^{k-1}\mathcal{F}_1(U) = a^{k-1}\sum_{d|n} n_d\left(\frac{d}{ep^n} - \left[\frac{da}{ep^n}\right]\right) = -\sum_{d|N} n_d a^{k-1}\left[\frac{da}{ep^n}\right]. \tag{18}$$

Equations (17) and (18) yield (15), proving the proposition. □

The measures $\mathcal{F}_k$ may be used to calculate the modular symbol of measures μ. Let the fraction a/Nc be fixed; we will write ν for $\mu\{\infty \to a/Nc\}$.

Let V be a compact open subset of $\mathbf{Z}_p^\times$, and let $f_i = \sum_{n=0}^{d_i} c_n(i)y^n$ be a sequence of polynomials such that $\lim_{i\to\infty} f_i(x)$ is the characteristic function of V. Then equations (8) and (12) for the moments of ν yield

$$\nu(\mathbf{Z}_p \times V) = \lim_{i\to\infty} -12\sum_{d|N} n_d \sum_{n=0}^{d_i}(1-p^n)c_n(i)\cdot D_{n+1,1}\left(\frac{a}{Nc/d}\right). \tag{19}$$

From the distribution relation (14) with $k=1$, we have

$$\begin{aligned}D_{n+1,1}\left(\frac{a}{Nc/d}\right) &= \left(\frac{Nc}{d}\right)^n \sum_{h=1}^{Nc/d} \frac{\tilde{B}_{n+1}\left(\frac{h}{Nc/d}\right)}{n+1}\cdot\tilde{B}_1\left(\frac{ha}{Nc/d}\right) \\ &= \left(\frac{Nc}{d}\right)^n \sum_{h=1}^{Nc} \frac{\tilde{B}_{n+1}\left(\frac{h}{Nc/d}\right)}{n+1}\cdot\tilde{B}_1\left(\frac{ha}{Nc}\right).\end{aligned}$$

Hence (19) becomes

$$\nu(\mathbf{Z}_p \times V) = \lim_{i\to\infty} -12 \sum_{h=1}^{Nc} \tilde{B}_1\left(\frac{ha}{Nc}\right) \sum_{n=0}^{d_i} (1-p^n) \sum_{d|N} n_d \left(\frac{Nc}{d}\right)^n \frac{\tilde{B}_{n+1}\left(\frac{h}{Nc/d}\right)}{n+1} c_n(i). \quad (20)$$

Write $Nc = ep^r$ with p not dividing e. Then N divides e, and in terms of the measure $\mathcal{F}_1$ above we have

$$\sum_{n=0}^{d_i} \sum_{d|N} n_d \left(\frac{Nc}{d}\right)^n \frac{\tilde{B}_{n+1}\left(\frac{h}{Nc/d}\right)}{n+1} c_n(i) = \int_{h+ep^rZ} (f_i(x_p) - f_i(px_p))\, \mathrm{d}\mathcal{F}_1(x)$$

by Proposition 3.1. Let us now specify V of the form $V = b + p^s\mathbf{Z}_p$, with $s \geq r$ and $b \in \mathbf{Z}_p^\times$. In the limit as $i \to \infty$, the value $f_i(x_p)$ approaches 1 or 0 according to whether $x_p \in V$, and $f_i(px_p)$ approaches 0. Therefore (20) becomes

$$\begin{aligned} \nu(\mathbf{Z}_p \times V) &= -12 \sum_{h=1}^{Nc} \tilde{B}_1\left(\frac{ha}{Nc}\right) \mathcal{F}_1(\{x \in h + ep^rZ : x_p \in V\}) \\ &= -12 \sum_{\substack{h=1 \\ h\in b+p^r\mathbf{Z}_p}}^{Nc} \tilde{B}_1\left(\frac{ha}{ep^r}\right) \sum_{d|N} n_d \tilde{B}_1\left(\frac{y}{ep^s/d}\right) \quad (21) \end{aligned}$$

where y is an integer such that $y \equiv h \pmod{e}$ and $y \equiv b \pmod{p^s}$. Fixing one such y for each h we obtain

$$\begin{aligned} \nu(\mathbf{Z}_p \times V) &= 12 \sum_{\substack{h=1 \\ h\in b+p^r\mathbf{Z}_p}}^{Nc} \tilde{B}_1\left(\frac{ha}{ep^r}\right) \sum_{d|N} n_d \left[\frac{y}{ep^s/d}\right] \quad (22) \\ &\equiv 12 \frac{a}{Nc} \sum_{\substack{h=1 \\ h\in b+p^r\mathbf{Z}_p}}^{Nc} h \sum_{d|N} n_d \left[\frac{y}{ep^s/d}\right] \pmod{\mathbf{Z}}, \end{aligned}$$

where (22) uses $\sum n_d = \sum n_d d = 0$. Hence to prove integrality, it suffices to consider the case $a = 1$. For this purpose, we return to (21) with $a = 1$ and rewrite the expression in terms of a generalized Dedekind sum:

$$\nu(\mathbf{Z}_p \times V) = -12 \sum_{d|N} n_d \sum_{\substack{h=1 \\ h\in b+p^r\mathbf{Z}_p}}^{ep^r/d} \tilde{B}_1\left(\frac{h}{ep^r/d}\right) \tilde{B}_1\left(\frac{y}{ep^s/d}\right). \quad (23)$$

The inner sum is the generalized Dedekind sum denoted $C(1, 1, p^{s-r}, \frac{e}{d}, \frac{ke/d}{p^s}, 0)$ in [10], where k is an integer chosen so that $ke/d \equiv b \pmod{p^s}$. The reciprocity law governing these

Dedekind sums ([10, Theorem 2]) shows that this value equals

$$\begin{aligned}
\sum_{\substack{h=1 \\ h\in b+p^r\mathbf{Z}_p}}^{ep^r/d} \tilde{B}_1\left(\frac{h}{ep^r/d}\right)\tilde{B}_1\left(\frac{y}{ep^s/d}\right) &= \frac{(e/d)^2}{2p^{s-r}} - \sum_{i=1}^{p^{s-r}} \tilde{B}_1\left(\frac{i}{p^{s-r}}\right)\tilde{B}_1\left(\frac{(ip^r-k)e/d}{p^{s-r}}\right) \\
&\quad + \frac{p^{s-r}}{2e/d}\tilde{B}_2\left(-\frac{b}{p^r}\right) + \frac{p^{s-r}}{2e/d}\tilde{B}_2\left(\frac{b}{p^s}\right) \qquad (24) \\
&\quad - \tilde{B}_1\left(\frac{b}{p^s}\right). \qquad (25)
\end{aligned}$$

Using this expression in equation (23), the terms from lines (24) and (25) vanish since $\sum n_d d = 0$. The remaining line yields only terms in $\mathbf{Z}[1/p]$. Since we know that ν is $\mathbf{Z}_p$-valued, we thus conclude that $\nu(\mathbf{Z}_p \times V) \in \mathbf{Z}$. Since the $\tilde{\Gamma}$ translates of the sets $\mathbf{Z}_p \times V$ form a basis of compact opens for $\mathbf{Q}_p^2 - \{0\}/p^{\mathbf{Z}} \cong \mathbf{X}$, the $\tilde{\Gamma}$-invariance of μ therefore implies:

Proposition 3.2. *The modular symbol of measures μ is $\mathbf{Z}$-valued.*

4 Method

As before, let K denote a real quadratic field in which p is inert. The completion K_p is a quadratic unramified extension of $\mathbf{Q}_p$. Let β be a primitive (p^2-1)st root of unity in $K_p^\times$, and let $\log_\beta$ denote the discrete logarithm with base β:

$$\log_\beta : K_p^\times \to \mathbf{Z}/(p^2-1)\mathbf{Z}, \text{ where } \frac{x}{p^{\mathrm{ord}_p(x)}\beta^{\log_\beta(x)}} \in 1 + p\mathcal{O}_p.$$

We then have the decomposition

$$K_p^\times \cong \mathbf{Z} \times \mathbf{Z}/(p^2-1)\mathbf{Z} \times \mathcal{O}_p \text{ given by } x \mapsto (\mathrm{ord}_p(x), \log_\beta(x), \log_p(x)).$$

For $x = u(\alpha, \tau)$ and $\gamma_\tau = \begin{pmatrix} a & * \\ Nc & * \end{pmatrix}$, these three components are given by the formulas

$$\mathrm{ord}_p(u(\alpha,\tau)) = -12\sum_{d|N} n_d \cdot D\left(\frac{a}{Nc/d}\right), \qquad (26)$$

$$\log_\beta(u(\alpha,\tau)) = \int_{\mathbf{X}} \log_\beta(x - y\tau)\, \mathrm{d}\mu\{\infty \to a/Nc\}(x,y), \qquad (27)$$

$$\log_p(u(\alpha,\tau)) = \int_{\mathbf{X}} \log_p(x - y\tau)\, \mathrm{d}\mu\{\infty \to a/Nc\}(x,y). \qquad (28)$$

The computations of (26) and (27) are easy to execute in practice (note that for (27) it suffices to take a cover of $\mathbf{X}$ in which x and y are determined modulo p), so we only elaborate upon the computation of (28).

Suppose we are content to calculate (28) to an accuracy of M p-adic digits. Let $m = [\infty] - [a/Nc] \in \mathcal{M}$. Then $\log_p(u(\alpha,\tau))$ is equal to

$$\int_{\mathbf{Z}_p\times\mathbf{Z}_p^\times} \log_p(y)\,\mathrm{d}\mu_m(x,y) + \int_{\mathbf{Z}_p\times\mathbf{Z}_p^\times} \log_p\left(\frac{x}{y}-\tau\right)\mathrm{d}\mu_m(x,y) \tag{29}$$

$$+\int_{\mathbf{Z}_p^\times\times p\mathbf{Z}_p} \log_p(x)\,\mathrm{d}\mu_m(x,y) + \int_{\mathbf{Z}_p^\times\times p\mathbf{Z}_p} \log_p\left(1-\frac{y\tau}{x}\right)\mathrm{d}\mu_m(x,y). \tag{30}$$

The first term of (29) is independent of τ. To evaluate this term to an accuracy of p^M, one finds a polynomial $f(y)$ that is congruent to $\log_p(y)$ modulo p^M for all $y \in \mathbf{Z}_p^\times$. This can be done as follows. For each $i = 1,\dots,p-1$, let

$$g_i(y) = \prod_{\substack{j=1\\ j\neq i}}^{p-1} (y-j)^M,$$

and let $h_i(y)$ denote the power series of $\log_p(y)/g_i(y)$ on the residue disc $i + p\mathbf{Z}_p$, truncated after $M + \log M$ terms (the extra $\log M$ terms account for the denominators divisible by powers of p in the power series of $\log_p$). Then letting $f_i(y) = g_i(y)h_i(y)$ and $f(y) = \sum_{i=1}^{p-1} f_i(y)$ produces the desired polynomial; it has degree $p(M + \log M)$. The first term of (29) may then be evaluated be replacing $\log_p(y)$ by $f(y)$ and using (9) and (13) to evaluate the integral of y^n on $\mathbf{Z}_p \times \mathbf{Z}_p^\times$ against the measure μ_m.

There is a $\mathbf{Z}_p^\times$-bundle map

$$\pi : \mathbf{X} \to \mathbf{P}^1(\mathbf{Q}_p) \quad \text{given by} \quad (x,y) \mapsto x/y.$$

Given the measure μ_m on $\mathbf{X}$, its push forward to $\mathbf{P}^1(\mathbf{Q}_p)$ is defined by the rule

$$\overline{\mu}_m(U) = \pi_*\mu_m(U) = \mu_m(\pi^{-1}(U)) \in \mathbf{Z}$$

for compact open $U \subset \mathbf{P}^1(\mathbf{Q}_p)$ (the measure-valued partial modular symbol $\overline{\mu}$ was denoted μ_α in [2]). The second term of (29) may be recognized as a push forward from $\mathbf{X}$ to $\mathbf{P}^1(\mathbf{Q}_p)$, and equals

$$\begin{aligned}\int_{\mathbf{Z}_p} \log_p(t-\tau)\,\mathrm{d}\overline{\mu}_m(t) &= \sum_{i=0}^{p-1}\int_{i+p\mathbf{Z}_p} \log_p(t-i+(i-\tau))\,\mathrm{d}\overline{\mu}_m(t)\\ &= \sum_{i=0}^{p-1}\left[\log_p(\tau-i)\overline{\mu}_m(i+p\mathbf{Z}_p) + \int_{i+p\mathbf{Z}_p}\log_p\left(1-\frac{t-i}{\tau-i}\right)\mathrm{d}\overline{\mu}_m(t)\right].\end{aligned}$$

The last integrand may be expanded as a power series in the residue disc $i + p\mathbf{Z}_p$, and hence to calculate the integral to an accuracy of p^M it suffices to calculate the moments

$$\int_{i+p\mathbf{Z}_p} (t-i)^n\,\mathrm{d}\overline{\mu}_m(t) = p^n \int_{\mathbf{Z}_p} u^n\,\mathrm{d}\overline{\mu}_{P_i^{-1}m}(u) \tag{31}$$

for $n = 0, \dots, M-1$, where $P_i = \begin{pmatrix} p & i \\ 0 & 1 \end{pmatrix}$, and (31) uses the invariance of $\overline{\mu}$ under $P_i \in \tilde{\Gamma}$. Writing $P_i^{-1}m = \tilde{w} = [\infty] - [w]$, we calculate (31) by pulling back to $\mathbf{X}$:

$$\begin{aligned} \int_{\mathbf{Z}_p} u^n \, \mathrm{d}\overline{\mu}_{\tilde{w}}(u) &= \int_{\mathbf{Z}_p \times \mathbf{Z}_p^\times} x^n y^{-n} \, \mathrm{d}\mu_{\tilde{w}}(x,y) \\ &= \lim_{\substack{j\to\infty \\ g=(p-1)p^j}} \int_{\mathbf{Z}_p \times \mathbf{Z}_p^\times} x^n y^{g-n} \, \mathrm{d}\mu_{\tilde{w}}(x,y) \\ &= -\lim_{\substack{j\to\infty \\ g=(p-1)p^j}} 12 \sum_{\ell=0}^{n} \binom{n}{l} w^{n-\ell} (-1)^\ell \sum_{d|N} n_d d^{-\ell} D_{g-\ell+1,\ell+1}(dw), \end{aligned} \tag{32}$$

Writing $w = b/ep^r$ (with $p \nmid e$ and $N|e$) and employing the distribution relation (14), expression (32) maybe expressed in terms of the single-variable measures of Section 3:

$$\begin{aligned} \lim_{\substack{j\to\infty \\ g=(p-1)p^j}} \sum_{d|N} n_d d^{-\ell} D_{g-\ell+1,\ell+1}\left(\frac{bd}{ep^r}\right) &= \sum_{h=1}^{ep^r} \frac{\tilde{B}_{\ell+1}(\frac{hb}{ep^r})}{\ell+1} \lim_{\substack{j\to\infty \\ g=(p-1)p^j}} \mathcal{F}_{g-\ell+1}(h + ep^r \cdot Z) \\ &= \sum_{h=1}^{ep^r} \frac{\tilde{B}_{\ell+1}(\frac{hb}{ep^r})}{\ell+1} \int_{h+ep^r Z} x^{-\ell} \, \mathrm{d}\mathcal{F}_1(x), \end{aligned} \tag{33}$$

by Proposition 3.1. The integrals of (33) may be computed modulo p^M by expanding $x^{-\ell}$ as a power series, and using Proposition 3.1 to calculate the moments of $\mathcal{F}_1$.

The terms of (30) may be calculated similarly using the methods described above for (29). Our method has broken down the computation of (28) into two parts. The first step is the calculation of the following integrals, independent of τ:

1. $\int_{\mathbf{Z}_p \times \mathbf{Z}_p^\times} \log_p(y) \, \mathrm{d}\mu_m(x,y)$,
2. $\int_{i+p\mathbf{Z}_p} (t-i)^n \, \mathrm{d}\overline{\mu}_m(t), i = 0, \dots, p=1, n = 0, \dots, M-1$,
3. $\int_{\mathbf{Z}_p^\times \times p\mathbf{Z}_p} \log_p(x) \, \mathrm{d}\mu_m(x,y)$,
4. $\int_{\mathbf{P}^1(\mathbf{Q}_p) - \mathbf{Z}_p} t^{-n} \, \mathrm{d}\overline{\mu}_m(t), n = 0, \dots M-1$.

(The last moment arises in the computation of (30).) The second step is to calculate $\log_p u(\alpha, \tau)$ from these integrals, using the decomposition of (29)–(30).

Hence our algorithm is to execute one program, which calculates for a given α, p, and M, the integrals (1)–(4) once and for all as m ranges over a $\Gamma_0(N)$-module basis for $\mathcal{M}$, to an accuracy of p^M. (Using the $\Gamma_0(N)$-invariance of the indefinite integral, it suffices to calculate (1)–(4) for a $\Gamma_0(N)$-module basis of $\mathcal{M}$ in order to evaluate the indefinite integral for all $m \in \mathcal{M}$.) This program executes $O(pM^2)$ operations. The output is stored in a file.

A second program is then run, inputting the integrals (1)–(4) from the output file of the first program, and calculating $u(\alpha, \tau)$ to an accuracy of p^M as described above. This calculation executes $O(pM)$ operations, and hence is rather quick even when M is large. Thus to compute the p-units $u(\alpha, \tau)$ to a high accuracy for various real quadratic fields K, it suffices to execute the (much slower) first program only once.

5 Results

The methods of Section 4 were used with the modular unit

$$\alpha(z) = \Delta(z)^2 \Delta(2z)^{-3} \Delta(4z)$$

of level $N = 4$ and various p. A p-adic accuracy of $M = 50$ digits was used. In our calculations, we restricted to fields K and $\tau \in K$ so that the resulting elements $u(\alpha, \tau)$ would be (conjecturally) defined over the narrow Hilbert class field H^+ of K. Assumption (11) leads us to restrict to K of discriminant D congruent to 1 modulo 8. For each p we considered all $D < 500$ for which p is inert in K and $\mathcal{O}_K$ contains no unit of norm -1.

The programming language MAGMA was used for the computations. In each case, representatives for each of the h classes of quadratic forms were used to produce a τ_i (see [1, §5.2]) and the corresponding $u(\alpha, \tau_i) \in K_p^\times$. Conjecture 2.14 of [2] predicts that the conjugate of $u(\alpha, \tau_i) \in H^+$ over H is $u(\alpha, \tau_i)^{-1}$. Thus the characteristic polynomial of the $u(\alpha, \tau_i) \in H^+$ over K should be

$$P(x) = \prod_{i=1}^{h} (x - u(\alpha, \tau_i))(x - u(\alpha, \tau_i)^{-1}).$$

The polynomial $P(x)$ is computed in $K_p[x]$ to an accuracy of 50 p-adic digits, and a simple algorithm involving shortest lattice vectors (see [3, §1.6]) is used to recognize the resulting p-adic numbers as elements of K.

Remark 5.1. The modular symbol ψ attached to α actually takes values in $3\mathbf{Z}$, since α is the cube of the modular function

$$\eta(z)^8 \eta(2z)^{-12} \eta(4z)^4$$

of level 4. In order to minimize the heights of the points $u(\alpha, \tau)$, it is preferable to replace ψ with $\psi/3$.

Furthermore, after executing our algorithm, it was clear that in most cases our p-units were still powers of smaller units. If the integers $\mathrm{ord}_p(u(\alpha, \tau_i))$ and

$$\mu\{\infty \to a/Nc\}((u + p\mathbf{Z}_p) \times (v + p\mathbf{Z}_p))$$

for $(u, v) \in \mathbf{X}$ are divisible by a common integer r relatively prime to p, then formulas (26)–(28) yield a canonical rth root of $u(\alpha, \tau)$ in $K_p^\times$, by replacing ψ by ψ/r. In each case where $\mathrm{ord}_p\, u(\alpha, \tau) \neq 0$, we calculated the largest r for which this was the case.

The tables below present our results; we list for each discriminant the class number h of $\mathcal{O}_K$ (so $[H^+ : K] = 2h$), the maximal value of r as described in Remark 5.1, the values $\frac{1}{r}\,\mathrm{ord}_p\, u(\alpha, \tau)$, and the polynomial $P(x)$ of the $u(\alpha, \tau)^{1/r}$ scaled to clear powers of p from the denominator. In each case the polynomials produced are indeed characteristic polynomials of p-units in H^+. In many cases, the units listed are powers of smaller p-units in H^+; in these cases, the polynomial $P(x)$ of the largest root lying in H^+ is listed in the table on the following line (with the root taken implied by the value of r). This root is not necessarily uniquely defined, depending on the presence of roots of unity in H^+.

Remark 5.2. Since the units we produce conjecturally have trivial valuation at each place not lying above p, they are determined uniquely by their valuations at the places above p. In particular, when the class number of K is 1 and $\mathrm{ord}_p\, u(\alpha, \tau) = 0$, we expect $u(\alpha, \tau)$ to be a root of unity. To produce non-trivial units in this case, one must work with a different modular unit α to avoid the "accidental zero" caused by the particular linear combination of Δ-functions weighted by n_d used to define α.

Similarly, if K has class number 2 and the values $\mathrm{ord}_p\, u(\alpha, \tau_i)$ for the distinct Γ-orbits of τ_i are equal, then we expect the corresponding units to be equal, and our polynomial $P(x)$ to factor as a square. A different modular unit must be used to generate the full narrow Hilbert class field. These features of the construction are evident in the tables.

References

[1] H. Darmon. *Integration on $\mathcal{H}_p \times \mathcal{H}$ and arithmetic applications.* Ann. of Math. (2) **154** (2001), no. 3, 589–639.

[2] H. Darmon, S. Dasgupta. Elliptic units for real quadratic fields, to appear in Ann. of Math.

[3] H. Darmon, R. Pollack. *The efficient calculation of Stark–Heegner points via overconvergent modular symbols,* submitted.

[4] S. Dasgupta. PhD thesis, University of California-Berkeley, May 2004.

[5] S. Dasgupta. *Stark–Heegner points on modular jacobians*, in progress.

Table 1: Characteristic Polynomial of $u(\alpha, \tau)$ for $p = 3$

D	h	r	$\mathrm{ord}_p\, u(\alpha,\tau)^{1/r}$	$P(x)$
161	1	6	± 0	$x^2 - 2x + 1$
209	1	6	± 6	$729x^2 + 1358x + 729$
		36	± 1	$3x^2 + 5x + 3$
305	2	6	$\pm 2,\ \pm 4$	$6561x^4 - \frac{675\sqrt{D}+3987}{2}x^3 + \frac{75\sqrt{D}+4607}{2}x^2 - \frac{675\sqrt{D}+3987}{2}x + 6561$
		12	$\pm 1,\ \pm 3$	$81x^4 - \frac{9\sqrt{D}+345}{2}x^3 + \frac{15\sqrt{D}+419}{2}x^2 - \frac{9\sqrt{D}+345}{2}x + 81$
329	1	6	± 0	$x^2 - 2x + 1$
377	2	12	$\pm 1, \pm 3$	$81x^4 - \frac{21\sqrt{D}+207}{2}x^3 + \frac{21\sqrt{D}+499}{2}x^2 - \frac{21\sqrt{D}+207}{2}x + 81$
473	3	6	$\pm 2, \pm 2, \pm 6$	$3^{10}x^6 + \frac{15795\sqrt{D}+101493}{2}x^5 + \frac{12285\sqrt{D}+620541}{2}x^4 + \frac{34905\sqrt{D}+336763}{2}x^3 + \frac{12285\sqrt{D}+620541}{2}x^2 + \frac{15795\sqrt{D}+101493}{2}x + 3^{10}$
		12	$\pm 1, \pm 1, \pm 3$	$243x^6 + \frac{-9\sqrt{D}+945}{2}x^5 + \frac{15\sqrt{D}+1167}{2}x^4 + \frac{21\sqrt{D}+815}{2}x^3 + \frac{15\sqrt{D}+1167}{2}x^2 + \frac{-9\sqrt{D}+945}{2}x + 243$
497	1	6	± 0	$x^2 - 2x + 1$

[6] B.H. Gross. *p-adic L-series at $s = 0$.* J. Fac. Sci. Univ. Tokyo Sect. IA Math. **28** (1981), no. 3, 979–994 (1982).

[7] R. Greenberg, G. Stevens, *p-adic L-functions and p-adic periods of modular forms.* Invent. Math. **111** (1993), no. 2, 407–447.

[8] S. Lang. *Cyclotomic fields I and II.* Combined second edition. Graduate Texts in Mathematics, 121. Springer-Verlag, New York, 1990.

[9] H. M. Stark. *L-functions at $s = 1$. IV. First derivatives at $s = 0$.* Adv. in Math. **35** (1980), no. 3, 197–235.

[10] U. Halbritter. *Some new reciprocity formulas for generalized Dedekind sums.* Results Math. **8** (1985), no. 1, 21–46.

Table 2: Characteristic Polynomial of $u(\alpha, \tau)$ for $p = 5$

D	h	r	$\mathrm{ord}_p\, u(\alpha,\tau)^{1/r}$	$P(x)$
33	1	6	± 2	$25x^2 + \frac{3\sqrt{D}-49}{2}x + 25$
		12	± 1	$5x^2 + \frac{3\sqrt{D}-1}{2}x + 5$
57	1	12	± 1	$5x^2 + \frac{-\sqrt{D}+9}{2}x + 5$
177	1	6	± 6	$5^6x^2 + \frac{4011\sqrt{D}+5231}{2}x + 5^6$
		12	± 3	$125x^2 + \frac{21\sqrt{D}-191}{2}x + 125$
217	1	6	± 0	$x^2 - 2x + 1$
273	2	6	$\pm 2, \pm 2$	$(25x^2 + \frac{3\sqrt{D}-41}{2}x + 25)^2$
		12	$\pm 1, \pm 1$	$(5x^2 + \frac{-\sqrt{D}+3}{2}x + 5)^2$
297	1	12	± 3	$125x^2 - 74x + 125$
		36	± 1	$5x^2 + x + 5$
377	2	6	$\pm 2, \pm 6$	$5^8x^4 + \frac{30375\sqrt{D}+533925}{2}x^3 + \frac{76545\sqrt{D}+102167}{2}x^2 + \frac{30375\sqrt{D}+533925}{2}x + 5^8$
		12	$\pm 1, \pm 3$	$625x^4 + \frac{75\sqrt{D}-655}{2}x^3 + \frac{-15\sqrt{D}+1447}{2}x^2 + \frac{75\sqrt{D}-655}{2}x + 625$
393	1	6	± 10	$5^{10}x^2 + 2275534x + 5^{10}$
		12	± 5	$3125x^2 + 4154x + 3125$
417	1	6	± 6	$5^6x^2 - \frac{2109\sqrt{D}+18929}{2}x + 5^6$
		12	± 3	$125x^2 + \frac{19\sqrt{D}+111}{2}x + 125$
473	3	6	$\pm 2, \pm 2, \pm 6$	$5^{10}x^6 + \frac{-253125\sqrt{D}-4501875}{2}x^5 + \frac{496125\sqrt{D}+5836125}{2}x^4 + \frac{-59535\sqrt{D}-13546883}{2}x^3 + \frac{496125\sqrt{D}+5836125}{2}x^2 + \frac{-253125\sqrt{D}-4501875}{2}x + 5^{10}$
		12	$\pm 1, \pm 1, \pm 3$	$3125x^6 + \frac{-1125\sqrt{D}-1475}{2}x^5 + \frac{225\sqrt{D}+47345}{2}x^4 + \frac{-2655\sqrt{D}-6797}{2}x^3 + \frac{225\sqrt{D}+47345}{2}x^2 + \frac{-1125\sqrt{D}-1475}{2}x + 3125$
497	1	6	± 0	$x^2 - 2x + 1$

Table 3: Characteristic Polynomial of $u(\alpha, \tau)$ for $p = 7$

D	h	r	$\mathrm{ord}_p\, u(\alpha,\tau)^{1/r}$	$P(x)$
33	1	6	± 2	$49x^2 + 94x + 49$
		12	± 1	$7x^2 + 2x + 7$
129	1	12	± 1	$7x^2 - 2x + 7$
201	1	6	± 2	$49x^2 + 94x + 7$
		12	± 1	$7x^2 + 2x + 7$
209	1	12	± 3	$343x^2 - 610x + 343$
		36	± 1	$7x^2 + 5x + 7$
297	1	6	± 6	$7^6x^2 + 153502x + 7^6$
		36	± 1	$7x^2 + 2x + 7$
321	3	6	$\pm 2, \pm 2, \pm 6$	$7^{10}x^6 - \frac{1188495\sqrt{D}+567084987}{2}x^5 + \frac{-557865\sqrt{D}+433702773}{2}x^4 + \frac{5083155\sqrt{D}-475485877}{2}x^3 + \frac{-557865\sqrt{D}+433702773}{2}x^2 - \frac{1188495\sqrt{D}+567084987}{2}x + 7^{10}$
		12	$\pm 1, \pm 1, \pm 3$	$7^5x^6 - \frac{2205\sqrt{D}+53361}{2}x^5 + \frac{3465\sqrt{D}+48699}{2}x^4 - \frac{4455\sqrt{D}+21791}{2}x^3 + \frac{3465\sqrt{D}+48699}{2}x^2 - \frac{2205\sqrt{D}+53361}{2}x + 7^5$
377	2	6	$\pm 2, \pm 6$	$7^8x^4 + \frac{-1210545\sqrt{D}+3900253}{2}x^3 + \frac{-172935\sqrt{D}+31066815}{2}x^2 + \frac{-1210545\sqrt{D}+3900253}{2}x + 7^8$
		12	$\pm 1, \pm 3$	$2401x^4 + \frac{315\sqrt{D}+10017}{2}x^3 + \frac{405\sqrt{D}+15155}{2}x^2 + \frac{315\sqrt{D}+10017}{2}x + 2401$
465	2	6	± 4 , ± 4	$(2401x^2 - 4034x + 2401)^2$
		24	± 1 , ± 1	$(7x^2 + 2x + 7)^2$
489	1	6	± 2	$49x^2 + 94x + 49$
		12	± 1	$7x^2 + 2x + 7$

Table 4: Characteristic Polynomial of $u(\alpha, \tau)$ for $p = 11$

D	h	r	$\operatorname{ord}_p u(\alpha,\tau)^{1/r}$	$P(x)$
57	1	6	± 2	$121x^2 - \frac{15\sqrt{D}+233}{2}x + 121$
		12	± 1	$11x^2 - \frac{5\sqrt{D}+3}{2}x + 11$
105	2	6	$\pm 2, \pm 2$	$(121x^2 - \frac{39\sqrt{D}+73}{2}x + 121)^2$
		12	$\pm 1, \pm 1$	$(11x^2 - \frac{3\sqrt{D}+13}{2}x + 11)^2$
129	1	6	± 2	$121x^2 + \frac{-21\sqrt{D}+199}{2}x + 121$
		12	± 1	$11x^2 + \frac{\sqrt{D}+21}{2}x + 11$
161	1	6	± 0	$x^2 - 2x + 1$
217	1	6	± 0	$x^2 - 2x + 1$
249	1	6	± 6	$11^6 - \frac{167295\sqrt{D}+3198553}{2}x + 11^6$
		12	± 3	$11^3 - \frac{285\sqrt{D}+587}{2}x + 11^3$
305	2	6	$\pm 2, \pm 6$	$11^8x^4 + \frac{10372725\sqrt{D}+344443077}{2}x^3 + \frac{23917275\sqrt{D}-61466353}{2}x^2 + \frac{10372725\sqrt{D}+344443077}{2}x + 11^8$
		12	$\pm 1, \pm 3$	$11^4x^4 + \frac{-2475\sqrt{D}+6853}{2}x^3 + \frac{-225\sqrt{D}+44467}{2}x^2 + \frac{-2475\sqrt{D}+6853}{2}x + 11^4$
321	3	6	$\pm 2, \pm 2, \pm 6$	$11^{10}x^6 - \frac{1967882169\sqrt{D}+60603418095}{2}x^5 + \frac{10953497049\sqrt{D}+178199983335}{2}x^4 - \frac{13842699651\sqrt{D}+210615242059}{2}x^3 + \frac{10953497049\sqrt{D}+178199983335}{2}x^2 - \frac{1967882169\sqrt{D}+60603418095}{2}x + 11^{10}$
		12	$\pm 1, \pm 1, \pm 3$	$11^5x^6 + \frac{-4719\sqrt{D}+427251}{2}x^5 + \frac{-37257\sqrt{D}+801537}{2}x^4 + \frac{-55935\sqrt{D}+531929}{2}x^3 + \frac{-37257\sqrt{D}+801537}{2}x^2 + \frac{-4719\sqrt{D}+427251}{2}x + 11^5$
329	1	6	± 0	$x^2 - 2x + 1$
393	1	6	± 10	$11^{10}x^2 - 50395911602x + 11^{10}$
		12	± 5	$11^5x^2 - 319798x + 11^5$
417	1	6	± 6	$11^6x^2 + \frac{174795\sqrt{D}-2882153}{2}x + 11^6$
		12	± 3	$11^3x^2 + \frac{215\sqrt{D}-813}{2}x + 11^3$
497	1	6	± 0	$x^2 - 2x + 1$