# Chapter 6

# Sequences and Series

Exam 2: Wednesday at 7pm in PCYN 109
Today: Sequence and Series (§11.1-§11.2)
Next: §11.3 Integral Test, §11.4 Comparison Test

What is $\lim_{n\to\infty} \frac{1}{2^n}$?
What is $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \ldots$?
What is $\frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \frac{1}{243} + \ldots$?
What is $\frac{1}{1} + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \ldots$?

Our main goal in this chapter is to gain a working knowledge of power series and Taylor series of function with just enough discussion of the details of convergence to get by. The precise rigorous details are beyond the scope of this book.

## 6.1 Sequences

You may have encountered sequences long ago in earlier courses and they seemed very difficult. You know much more mathematics now, so they will probably seem easier. On the other hand, we're going to go very quickly.

*We will completely skip several topics from Chapter 11. I will try to make what we skip clear. Note that the homework has been modified to reflect the omitted topics.*

A sequence is an ordered list of numbers. These numbers may be real, complex, etc., etc., but in this book we will focus entirely on sequences of real numbers. For example,

$$\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \frac{1}{64}, \frac{1}{128}, \ldots, \frac{1}{2^n}, \ldots$$

Since the sequence is ordered, we can view it as a function with domain the natural numbers $= 1, 2, 3, \ldots$.

**Definition 6.1.1 (Sequence).** A *sequence* $\{a_n\}$ is a function $a : \mathbb{N} \to \mathbb{R}$ that takes a natural number $n$ to $a_n = a(n)$. The number $a_n$ is the *nth term*.

For example,

$$a(n) = a_n = \frac{1}{2^n},$$

which we write as $\{\frac{1}{2^n}\}$. Here's another example:

$$(b_n)_{n=1}^{\infty} = \left(\frac{n}{n+1}\right)_{n=1}^{\infty} = \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots$$

**Example 6.1.2.** The Fibonacci sequence $(F_n)_{n=1}^{\infty}$ is defined recursively as follows:

$$F_1 = 1,\ F_2 = 1,\ F_n = F_{n-2} + F_{n-1} \quad \text{for } n \geq 3.$$

Let's return to the sequence $\left(\frac{1}{2^n}\right)_{n=1}^{\infty}$. We write $\lim_{n\to\infty} \frac{1}{2^n} = 0$, since the terms get arbitrarily small.

**Definition 6.1.3.** If $(a_n)_{n=1}^{\infty}$ is a sequence then $\lim_{n\to\infty} a_n = L$ means that $a_n$ gets arbitrarily close to $L$ as $n$ get sufficiently large. Secret rigorous definition: For every $\varepsilon > 0$ there exists $B$ such that for $n \geq B$ we have $|a_n - L| < \varepsilon$.

This is exactly like what we did in the previous course when we considered limits of functions. If $f(x)$ is a function, the meaning of $\lim_{x\to\infty} f(x) = L$ is essentially the same. In fact, we have the following fact.

**Proposition 6.1.4.** *If $f$ is a function with $\lim_{x\to\infty} f(x) = L$ and $(a_n)_{n=1}^{\infty}$ is the sequence given by $a_n = f(n)$, then $\lim_{n\to\infty} a_n = L$.*

As a corollary, note that this implies that all the facts about limits that you know from functions also apply to sequences!

**Example 6.1.5.**

$$\lim_{n\to\infty} \frac{n}{n+1} = \lim_{x\to\infty} \frac{x}{x+1} = 1$$

**Example 6.1.6.** The converse of Proposition 6.1.4 is false *in general*, i.e., knowing the limit of the sequence converges doesn't imply that the limit of the function converges. We have $\lim_{n\to\infty} \cos(2\pi n) = 1$, but $\lim_{x\to\infty} \cos(2\pi x) = 1$ diverges. The converse is OK if the limit involving the function converges.

**Example 6.1.7.** Compute $\lim_{n\to\infty} \dfrac{n^3 + n + 5}{17n^3 - 2006n + 15}$. *Answer:* $\frac{1}{17}$.

## 6.2 Series

Conider the following sequence of partial sums:

$$a_N = \sum_{n=1}^{N} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^N}.$$

Can we compute

$$\sum_{n=1}^{\infty} \frac{1}{2^n}?$$

These partial sums look as follows:

$$a_1 = \frac{1}{2}, \qquad a_2 = \frac{3}{4}, \qquad a_{10} = \frac{1023}{1024}, \qquad a_{20} = \frac{1048575}{1048576}$$

It looks very likely that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n} = 1$, if it makes any sense. But does it?

In a moment we will *define*

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \lim_{N\to\infty} \sum_{n=1}^{N} \frac{1}{2^n} = \lim_{N\to\infty} a_N.$$

A little later we will show that $a_N = \frac{2^N-1}{2^N}$, hence indeed $\sum_{n=1}^{\infty} \frac{1}{2^n} = 1$.

**Definition 6.2.1.** If $(a_n)_{n=1}^{\infty}$ is a sequence, then

$$\sum_{n=1}^{\infty} a_n = \lim_{N\to\infty} \sum_{n=1}^{N} a_n = \lim_{N\to\infty} s_N$$

provided the limit exists. Otherwise we say that $\sum_{n=1}^{\infty} a_n$ *diverges.*

**Example 6.2.2 (Geometric series).** Consider $\sum_{n=1}^{\infty} ar^{n-1}$ for $a \neq 0$. Then

$$s_N = \sum_{n=1}^{N} ar^{n-1} = \frac{a(1-r^N)}{1-r}.$$

To see this, multiply both sides by $1-r$ and notice that all the terms in the middle cancel out. For what values of $r$ does $\lim_{N\to\infty} \frac{a(1-r^N)}{1-r}$ converge? If $|r| < 1$, then $\lim_{N\to\infty} r^N = 0$ and

$$\lim_{N\to\infty} \frac{a(1-r^N)}{1-r} = \frac{a}{1-r}.$$

If $|r| > 1$, then $\lim_{N\to\infty} r^N$ diverges, so $\sum_{n=1}^{\infty} ar^{n-1}$ diverges. If $r = \pm 1$, it's clear since $a \neq 0$ that the series also diverges (since the partial sums are $s_N = \pm Na$).

For example, if $a = 1$ and $r = \frac{1}{2}$, we get

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{1}{1-\frac{1}{2}},$$

as claimed earlier.