5.3 Trigonometric Substitutions

Return more midterms?
Rough meaning of grades:
- 29–34 is A
- 23–28 is B
- 17–22 is C
- 11–16 is D

Regarding the quiz—if you do every homework problem that was assigned, you'll have a severe case of deja vu on the quiz! On the exam, we do not restrict ourselves like this, but you get to have a sheet of paper.

The first homework problem is to compute

$$\int_{\sqrt{2}}^{2} \frac{1}{x^3\sqrt{x^2-1}}dx. \tag{5.3.1}$$

Your first idea might be to do some sort of substitution, e.g., $u = x^2 - 1$, but $du = 2xdx$ is nowhere to be seen and this simply doesn't work. Likewise, integration by parts gets us nowhere. However, a technique called "inverse trig substitutions" and a trig identity easily dispenses with the above integral and several similar ones! Here's the crucial table:

Expression	Inverse Substitution	Relevant Trig Identity
$\sqrt{a^2-x^2}$	$x = a\sin(\theta), -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$	$1 - \sin^2(\theta) = \cos^2(\theta)$
$\sqrt{a^2+x^2}$	$x = a\tan(\theta), -\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$1 + \tan^2(\theta) = \sec^2(\theta)$
$\sqrt{x^2-a^2}$	$x = a\sec(\theta), 0 \le \theta < \frac{\pi}{2}$ or $\pi \le \theta < \frac{3\pi}{2}$	$\sec^2(\theta) - 1 = \tan^2(\theta)$

Inverse substitution works as follows. If we write $x = g(t)$, then

$$\int f(x)dx = \int f(g(t))g'(t)dt.$$

This is *not* the same as substitution. You can just apply inverse substitution to any integral directly—usually you get something even worse, but for the integrals in this section using a substitution can vastly improve the situation.

If g is a $1-1$ function, then you can even use inverse substitution for a definite integral. The limits of integration are obtained as follows.

$$\int_a^b f(x)dx = \int_{g^{-1}(a)}^{g^{-1}(b)} f(g(t))g'(t)dt. \tag{5.3.2}$$

To help you understand this, note that as t varies from $g^{-1}(a)$ to $g^{-1}(b)$, the function $g(t)$ varies from $a = g(g^{-1}(a)$ to $b = g(g^{-1}(b))$, so f is being integrated over exactly the same values. Note also that (5.3.2) once again illustrates Leibniz's brilliance in designing the notation for calculus.

Let's give it a shot with (5.3.1). From the table we use the inverse substition

$$x = \sec(\theta).$$

We get

$$\begin{aligned}
\int_{\sqrt{2}}^{2} \frac{1}{x^3\sqrt{x^2-1}}dx &= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{1}{\sec(\theta)}\sqrt{\sec^2(\theta)-1}\sec(\theta)\tan(\theta)d\theta \\
&= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{1}{\sec(\theta)}\tan(\theta)\sec(\theta)\tan(\theta)d\theta \\
&= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \cos^(\theta)d\theta \\
&= \frac{1}{2}\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} 1+\cos(2\theta)d\theta \\
&= \frac{1}{2}\left[\theta+\frac{1}{2}\sin(2\theta)\right]_{\frac{\pi}{4}}^{\frac{\pi}{3}} \\
&= \frac{\pi}{24}+\frac{\sqrt{3}}{8}-\frac{1}{4}
\end{aligned}$$

Wow! That was like magic. This is really an amazing technique. Let's use it again to find the area of an ellipse.

Example 5.3.1. Consider an ellipse with radii a and b, so it goes through $(0, \pm b)$ and $(\pm a, 0)$. An equation for the part of an ellipse in the first quadrant is

$$y = b\sqrt{1-\frac{x^2}{a^2}} = \frac{b}{a}\sqrt{a^2-x^2}.$$

Thus the area of the entire ellipse is

$$A = 4\int_0^a \frac{b}{a}\sqrt{a^2-x^2}\,dx.$$

The 4 is because the integral computes 1/4th of the area of the whole ellipse. So we need to compute

$$\int_0^a \sqrt{a^2-x^2}\,dx$$

Obvious substitution with $u = a^2 - x^2$...? nope. Integration by parts...? nope.

Let's try inverse substitution. The table above suggests using $x = a\sin(\theta)$, so $dx = a\cos(\theta)d\theta$. We get

$$\int_0^{\frac{\pi}{2}} \sqrt{a^2-a^2\sin^2(\theta)}d\theta = a^2\int_0^{\frac{\pi}{2}} \cos^2(\theta)d\theta \tag{5.3.3}$$

$$= \frac{a^2}{2}\int_0^{\frac{\pi}{2}} 1+\cos(2\theta)d\theta \tag{5.3.4}$$

$$= \frac{a^2}{2}\left[\theta+\frac{1}{2}\sin(2\theta)\right]_0^{\frac{\pi}{2}} \tag{5.3.5}$$

$$= \frac{a^2}{2}\cdot\frac{\pi}{2} = \frac{\pi a^2}{4}. \tag{5.3.6}$$

Thus the area is

$$4\frac{b}{a}\frac{\pi a^2}{4} = \pi ab.$$

Consistency Check: If the ellipse is a circle, i.e., $a = b = r$, this is πr^2, which is a well-known formula for the area of a circle.

Remark 5.3.2. Trigonometric substitution is useful for functions that involve $\sqrt{a^2 - x^2}$, $\sqrt{x^2 + a^2}$, $\sqrt{x^2 - a}$, but *not all at once!*. See the above table for how to do each.

One other important technique is to use completing the square.

Example 5.3.3. Compute $\int \sqrt{5 + 4x - x^2}\, dx$. We *complete the square*:

$$5 + 4x - x^2 = 5 - (x - 2)^2 + 4 = 9 - (x - 2)^2.$$

Thus

$$\int \sqrt{5 + 4x - x^2}\, dx = \int \sqrt{9 - (x - 2)^2}\, dx.$$

We do a usual substitution to get rid of the $x - 2$. Let $u = x - 2$, so $du = dx$. Then

$$\int \sqrt{9 - (x - 2)^2}\, dx = \int \sqrt{9 - y^2}\, dy.$$

Now we have an integral that we can do; it's almost identical to the previous example, but with $a = 9$ (and this is an indefinite integral). Let $y = 3\sin(\theta)$, so $dy = 3\cos(\theta)d\theta$. Then

$$\begin{aligned}
\int \sqrt{9 - (x - 2)^2}\, dx &= \int \sqrt{9 - y^2}\, dy \\
&= \int \sqrt{3^2 - 3^2 \sin^2(\theta)} 3\cos(\theta) d\theta \\
&= 9 \int \cos^2(\theta)\, d\theta \\
&= \frac{9}{2} \int 1 + \cos(2\theta) d\theta \\
&= \frac{9}{2}\left(\theta + \frac{1}{2}\sin(2\theta)\right) + c
\end{aligned}$$

Of course, we *must transform* back into a function in x, and that's a little tricky. Use that

$$x - 2 = y = 3\sin(\theta),$$

so that

$$\theta = \sin^{-1}\left(\frac{x - 2}{3}\right).$$

$$\begin{aligned}
\int \sqrt{9-(x-2)^2}\,dx &= \cdots \\
&= \frac{9}{2}\left(\theta + \frac{1}{2}\sin(2\theta)\right) + c \\
&= \frac{9}{2}\left[\sin^{-1}\left(\frac{x-2}{3}\right) + \sin(\theta)\cos(\theta)\right] + c \\
&= \frac{9}{2}\left[\sin^{-1}\left(\frac{x-2}{3}\right) + \left(\frac{x-2}{3}\right)\cdot\left(\frac{\sqrt{9-(x-2)^2}}{3}\right)\right] + c.
\end{aligned}$$

Here we use that $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$. Also, to compute $\cos(\sin^{-1}\left(\frac{x-2}{3}\right))$, we draw a right triangle with side lengths $x-2$ and $\sqrt{9-(x-2)^2}$, and hypotenuse 3.

Example 5.3.4. Compute

$$\int \frac{1}{\sqrt{t^2-6t+13}}dt$$

To compute this, we complete the square, etc.

$$\int \frac{1}{\sqrt{t^2-6t+13}}dt = \int \frac{1}{\sqrt{(t-3)^2+4}}dt$$

[[Draw triangle with sides 2 and $t-3$ and hypotenuse $\sqrt{(t-3)^2+4}$. Then

$$\begin{aligned}
t-3 &= 2\tan(\theta) \\
\sqrt{(t-3)^2+4} &= 2\sec(\theta) = \frac{2}{\cos(\theta)} \\
dt &= 2\sec^2(\theta)d\theta
\end{aligned}$$

Back to the integral, we have

$$\begin{aligned}
\int \frac{1}{\sqrt{(t-3)^2+4}}dt &= \int \frac{2\sec^2(\theta)}{2\sec(\theta)}d\theta \\
&= \int \sec(\theta)d\theta \\
&= \ln|\sec(\theta)+\tan(\theta)| + c \\
&= \ln\left|\sqrt{(t-3)^2+4}2 + \frac{t-3}{2}\right| + c.
\end{aligned}$$