1.4 Using Integration to Determine Areas Between Curves

Today is 2006-01-18.
Quiz reminder: Friday, Jan 20 (describe format)
How was your weekend?
Mine was great—I wrote open source math software nonstop for days on end!

This section is about how to compute the area of fairly general regions in the plane. Regions are often described as the area enclosed by the graphs of several curves. ("My land is the plot enclosed by that river, that fence, and the highway.")

Recall that the integral $\int_a^b f(x)dx$ has a geometric interpretation as the signed area between the graph of $f(x)$ and the x-axis. We defined area by subdividing, adding up approximate areas (use points in the intervals) as Riemann sum, and taking the limit. Thus we defined area as a limit of Riemann sums. The fundamental theorem of calculus asserts that we can compute areas exactly when we can finding antiderivatives.

Instead of considering the area between the graph of $f(x)$ and the x-axis, we consider more generally two graphs, $y = f(x)$, $y = g(x)$, and assume for simplicity that $f(x) \geq g(x)$ on an interval $[a, b]$. Again, we approximate the area *between* these two curves as before using Riemann sums. Each approximating rectangle has width $(b - a)/n$ and height $f(x) - g(x)$, so

$$\text{Area bounded by graphs} \sim \sum [f(x_i) - g(x_i)]\Delta x.$$

Note that $f(x) - g(x) \geq 0$, so the area is nonnegative. From the definition of integral we see that the exact area is

$$\text{Area bounded by graphs } = \int_a^b (f(x) - g(x))dx. \tag{1.4.1}$$

Why did we make a big deal about approximations instead of just writing down (1.4.1)? Because having a sense of how this area comes directly from a Riemann sum is very important. But, what is the point of the Riemann sum if all we're going to do is write down the integral? The sum embodies the geometric manifestation of the integral. If you have this picture in your mind, then the Riemann sum has *done its job*. If you understand this, you're more likely to know what integral to write down; if you don't, then you might not.

Remark 1.4.1. By the linearity property of integration, our sought for area is the difference

$$\int_a^b f(x)dx - \int_a^b g(x)dx,$$

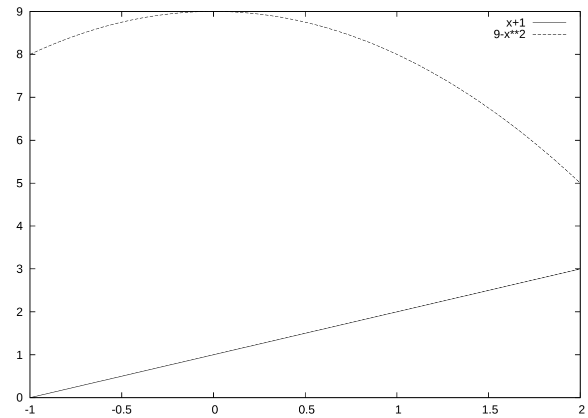
of two signed areas.

1.4.1 Examples

Example 1.4.2. Find the area enclosed by $y = x + 1$, $y = 9 - x^2$, $x = -1$, $x = 2$.

$$\text{Area} = \int_{-1}^{2} \Big[(9 - x^2) - (x + 1)\Big] dx$$

Figure 1.4.1: What is the enclosed area?



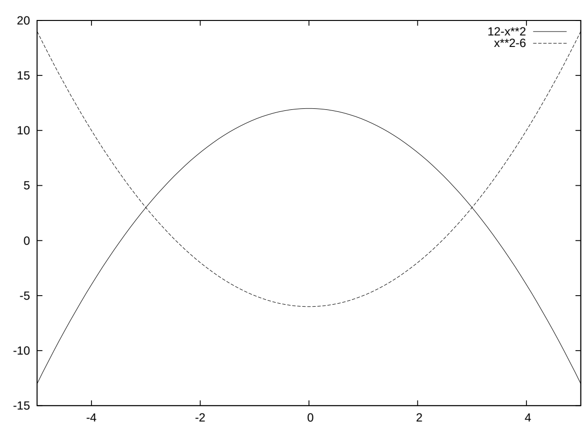
We have reduced the problem to a computation:

$$\int_{-1}^{2} [(9-x^2)-(x+1)]dx = \int_{-1}^{2} (8-x-x^2)dx = \left[8x - \frac{1}{2}x^2 - \frac{1}{3}x^3\right]_{-1}^{2} = \frac{39}{2}.$$

The above example illustrates the simplest case. In practice more interesting situations often arise. The next example illustrates finding the boundary points a, b when they are not explicitly given.

Example 1.4.3. Find area enclosed by the two parabolas $y = 12 - x^2$ and $y = x^2 - 6$.

Figure 1.4.2: What is the enclosed area?



Problem: We didn't tell you what the boundary points a, b are. We have to figure that out. How? We must find *exactly* where the two curves intersect, by setting the two curves equal and finding the solution. We have

$$x^2 - 6 = 12 - x^2,$$

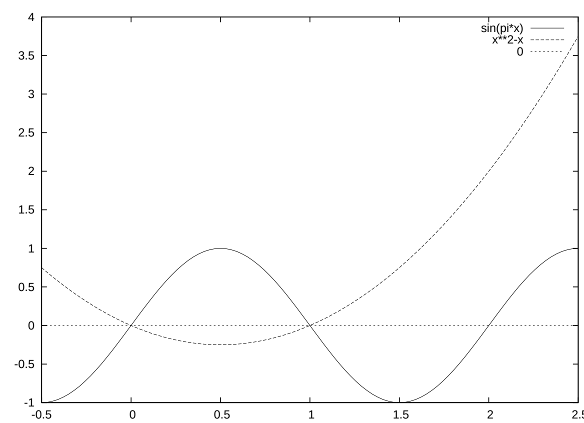
so $0 = 2x^2 - 18 = 2(x^2 - 9) = 2(x-3)(x+3)$, hence the intersect points are at $a = -3$ and $b = 3$. We thus find the area by computing

$$\int_{-3}^{3} \left[12 - x^2 - (x^2 - 6)\right] dx = \int_{-3}^{3} (18 - 2x^2) dx = 4 \int_{0}^{3} (9 - x^2) dx = 4 \cdot 18 = 72.$$

Example 1.4.4. A common way in which you might be tested to see if you *really* understand what is going on, is to be asked to find the area between two graphs $x = f(y)$ and $x = g(y)$. If the two grahs are vertical, subtract off the right-most curve. Or, just "switch x and y" everywhere (i.e., relfect about $y = x$). The area is unchanged.

Example 1.4.5. Find the area (*not signed area!*) enclosed by $y = \sin(\pi x)$, $y = x^2 - x$, and $x = 2$.

Figure 1.4.3: Find the area



Write $x^2 - x = (x - 1/2)^2 - 1/4$, so that we can obtain the graph of the parabola by shifting the standard graph. The area comes in two pieces, and the upper and lower curve switch in the middle. Technically, what we're doing is integrating the *absolute value* of the difference. The area is

$$\int_{0}^{1} \sin(\pi x) - (x^2 - x) dx - \int_{1}^{2} (x^2 - x) - \sin(\pi x) dx = \frac{4}{\pi} + 1$$

Something to take away from this is that in order to solve this sort of problem, you need some facility with graphing functions. If you aren't comfortable with this, review.

1.5 Computing Volumes of Surfaces of Revolution

Everybody knows that given a solid box, volume is

$$\text{volume} = \text{length} \times \text{width} \times \text{height}.$$

More generally, the volume of cylinder is $V = \pi r^2 h$ (cross sectional area times height). Even more generally, if the base of a prism has area A, the volume of the prism is $V = Ah$.

But what if our solid object looks like a complicated blob? How would we compute the volume? We'll do something that by now should seem familiar, which is to chop the object into small pieces and take the limit of approximations.

[[Picture of solid sliced vertically into a bunch of vertical thin solid discs.]]

Assume that we have a function

$$A(x) = \text{cross sectional area at } x.$$

The volume of our potentially complicated blob is approximately $\sum A(x_i)\Delta x$. Thus

$$\begin{aligned}\text{volume of blob} &= \lim_{n\to\infty}\sum_{i=1}^{n} A(x_i)\Delta x\\ &= \int_a^b A(x)dx\end{aligned}$$

Example 1.5.1. Find the volume of the pyramid with height H and square base with sides of length L.

Figure 1.5.1: How Big is Pharaoh's Place?

For convenience look at pyramid on its side, with the tip of the pyramid at the origin. We need to figure out the cross sectional area as a function of x, for $0 \leq x \leq H$. The function that gives the distance $s(x)$ from the x axis to the edge is a line, with $s(0) = 0$ and $s(H) = L/2$. The equation of this line is thus $s(x) = \frac{L}{2H}x$. Thus the cross sectional area is

$$A(x) = (2s(x))^2 = \frac{x^2L^2}{H^2}.$$

The volume is then

$$\int_0^H A(x)dx = \int_0^H \frac{x^2L^2}{H^2}dx = \left[\frac{x^3L^2}{3H^2}\right]_0^H = \frac{H^3L^2}{3H^2} = \frac{1}{3}HL^2.$$