1.3 Integration Technique 1: Substitution and Symmetry

Homework reminder.
Quiz reminder: Friday, Jan 20 (**Ace the first quiz!**).
Office Hours: Tue 11-1.
Monday is a holiday!
Wednesday – areas between curves and *volumes*
First midterm: Wed Feb 1 at 7pm (review lecture during day!)
Quick 5 minute discussion of computers and Maxima.
Quiz format: one question on front; one on back.

Remarks:

1. The *total distance traveled* is $\int_{t_1}^{t_2} |v(t)|dt$ since $|v(t)|$ is the rate of change of $F(t) =$ distance traveled (your speedometer displays the rate of change of your odometer).
2. How to compute $\int_a^b |f(x)|dx$.
   (a) Find the zeros of $f(x)$ on $[a, b]$, and use these to break the interval up into subintervals on which $f(x)$ is always ≥ 0 or always ≤ 0.
   (b) On the intervals where $f(x) \geq 0$, compute the integral of f, and on the intervals where $f(x) \leq 0$, compute the integral of $-f$.
   (c) The sum of the above integrals on intervals is $\int |f(x)|dx$.

This section is primarily about a powerful technique for computing definite and indefinite integrals.

1.3.1 The Substitution Rule

In first quarter calculus you learned numerous methods for computing derivatives of functions. For example, the *power rule* asserts that

$$(x^a)' = a \cdot x^{a-1}.$$

We can turn this into a way to compute certain integrals:

$$\int x^a dx = \frac{1}{a+1} x^{a+1} \qquad \text{if } a \neq -1.$$

Just as with the power rule, many other rules and results that you already know yield *techniques* for integration. In general integration is potentially much trickier than differentiation, because it is often not obvious which technique to use, or even how to use it. *Integration is a more exciting than differentiation!*

Recall the *chain rule*, which asserts that

$$\frac{d}{dx} f(g(x)) = f'(g(x))g'(x).$$

We turn this into a technique for integration as follows:

Proposition 1.3.1 (Substitution Rule). *Let $u = g(x)$, we have*

$$\int f(g(x))g'(x)dx = \int f(u)du,$$

assuming that $g(x)$ is a function that is differentiable and whose range is an interval on which f is continuous.

Proof. Since f is continuous on the range of g, Theorem 1.1.5 (the fundamental theorem of Calculus) implies that there is a function F such that $F' = f$. Then

$$\begin{aligned}\int f(g(x))g'(x)dx &= \int F'(g(x))g'(x)dx \\ &= \int \left(\frac{d}{dx}F(g(x))\right)dx \\ &= F(g(x)) + C \\ &= F(u) + C = \int F'(u)du = \int f(u)du.\end{aligned}$$

□

If $u = g(x)$ then $du = g'(x)dx$, and the substitution rule simply says if you let $u = g(x)$ formally in the integral everywhere, what you naturally would hope to be true based on the notation actually is true. The substitution rule illustrates how the notation Leibniz invented for Calculus is *incredibly brilliant*. It is said that Leibniz would often spend days just trying to find the right notation for a concept. He succeeded.

As with all of Calculus, the best way to start to get your head around a new concept is to see severally clearly worked out examples. (And the best way to actually be able to use the new idea is to *do* lots of problems yourself!) In this section we present examples that illustrate how to apply the substituion rule to compute indefinite integrals.

Example 1.3.2.

$$\int x^2(x^3+5)^9 dx$$

Let $u = x^3 + 5$. Then $du = 3x^2 dx$, hence $dx = du/(3x^2)$. Now substitute it all in:

$$\int x^2(x^3+5)^9 dx = \int \frac{1}{3}u^9 = \frac{1}{30}u^{10} = \frac{1}{30}(x^3+5)^{10}.$$

There's no point in expanding this out: "only simplify for a *purpose*!"

Example 1.3.3.

$$\int \frac{e^x}{1+e^x}dx$$

Substitute $u = 1 + e^x$. Then $du = e^x dx$, and the integral above becomes

$$\int \frac{du}{u} = \ln|u| = \ln|1+e^x| = \ln(1+e^x).$$

Note that the absolute values are not needed, since $1 + e^x > 0$ for all x.

Example 1.3.4.

$$\int \frac{x^2}{\sqrt{1-x}}dx$$

Keeping in mind the power rule, we make the substitution $u = 1 - x$. Then $du = -dx$. Noting that $x = 1 - u$ by solving for x in $u = 1 - x$, we see that the above integral becomes

$$\begin{aligned}\int -\frac{(1-u)^2}{\sqrt{u}}du &= -\int \frac{1-2u+u^2}{u^{1/2}}du \\ &= -\int u^{-1/2} - 2u^{1/2} + u^{3/2}du \\ &= -\left(2u^{1/2} - \frac{4}{3}u^{3/2} + \frac{2}{5}u^{5/2}\right) \\ &= -2(1-x)^{1/2} + \frac{4}{3}(1-x)^{3/2} - \frac{2}{5}(1-x)^{5/2}.\end{aligned}$$

1.3.2 The Substitution Rule for Definite Integrals

Proposition 1.3.5 (Substitution Rule for Definite Integrals). *We have*

$$\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)du,$$

assuming that $u = g(x)$ is a function that is differentiable and whose range is an interval on which f is continuous.

Proof. If $F' = f$, then by the chain rule, $F(g(x))$ is an antiderivative of $f(g(x))g'(x)$. Thus

$$\int_a^b f(g(x))g'(x)dx = \Big[F(g(x))\Big]_a^b = F(g(b)) - F(g(a)) = \int_{g(a)}^{g(b)} f(u)du.$$

□

Example 1.3.6.

$$\int_0^{\sqrt{\pi}} x\cos(x^2)dx$$

We let $u = x^2$, so $du = 2xdx$ and $xdx = \frac{1}{2}du$ and the integral becomes

$$\frac{1}{2}\cdot\int_{(0)^2}^{(\sqrt{\pi})^2} \cos(u)du = \frac{1}{2}\cdot[\sin(u)]_0^{\pi} = \frac{1}{2}\cdot(0-0) = 0.$$

1.3.3 Symmetry

An *odd function* is a function $f(x)$ such that $f(-x) = -f(x)$, and an *even function* one for which $f(-x) = f(x)$. If f is an odd function, then for any a,

$$\int_{-a}^{a} f(x)dx = 0.$$

If f is an even function, then for any a,

$$\int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx.$$

Both statements are clear if we view integrals as computing the signed area between the graph of $f(x)$ and the x-axis.

Example 1.3.7.

$$\int_{-1}^{1} x^2dx = 2\int_0^1 x^2dx = 2\left[\frac{1}{3}x^3\right]_0^1 = \frac{2}{3}.$$