An introduction to computing modular forms using modular symbols

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Abstract

In this survey paper, we explain how weight 2 modular forms on $\Gamma_0(N)$ are related to modular symbols, and how to use this relationship to explicitly compute spaces of modular forms.

Introduction

The definition of spaces of modular forms as functions on the upper half plane satisfying a certain equation is very abstract. The definition of the Hecke operators even more so. Nevertheless, one wishes to carry out explicit investigations involving these objects.

We are fortunate that we now have methods available that allow us to transform the vector space of cusp forms of given weight and level into a concrete object, which can be explicitly computed. We have the work of Atkin and Lehner, Birch and Swinnerton-Dyer, Manin, Mazur, Merel, and many others to thank for this (see, e.g., [4, 7, 14, 15]). For example, we can use the Eichler-Selberg trace formula, as extended in [10], to compute characteristic polynomials of Hecke operators. Then the method described in [23] gives a basis for certain spaces of modular forms. Alternatively, we can compute Θ-series using Brandt matrices and quaternion algebras as in [11, 17], or we can use a closely related geometric method that involves the module of enhanced supersingular elliptic curves [16]. Another related method of Birch [3] is very fast, but gives only a piece of the full space of modular forms. The power of the modular symbols approach was demonstrated by Cremona in his book [7] in which he systematically computes a large table of invariants of all elliptic curves of conductor up to 1000 (his online tables go much further).

Though the above methods are each beautiful and well suited to certain applications, we will only discuss the modular symbols method in this paper, as it has many advantages. We will primarily discuss the theory, leaving an explicit description of the objects involved for other papers. Nonetheless, there is a definite gap between the theory on the one hand, and an efficient running machine implementation on the other. To implement the algorithms hinted at below requires making absolutely everything completely explicit, then finding intelligent and efficient ways of performing the necessary manipulations. This is a nontrivial and tedious task, with room for error at every step. Fortunately, Cremona has succeeded at this in his book [7]. See also the author's MAGMA [5] package for working with modular symbols, which is part of MAGMA V2.7 (and higher), and would be useful to experiment with while reading this paper.

In this paper we will focus exclusively on the case of weight-2 modular forms for $\Gamma_0(N)$. The methods explained here extend to modular forms of integer weight greater than 2 for $\Gamma_1(N)$ (nontrivial character); for more details see [15] for the theory and [21] for the algorithms (see also [22]).

Section 1 contains a brief summary of basic facts about modular forms, Hecke operators, and integral homology. Section 2 introduces modular symbols, and describes how to compute with them. Section 3 outlines an algorithm for constructing cusp forms using modular symbols in conjunction with Atkin-Lehner theory.

This paper assumes some familiarity with algebraic curves, Riemann surfaces, and homology groups of compact surfaces. A few basic facts about modular forms are recalled, but only briefly. In particular, only a roundabout attempt is made to motivate why one might be interested in modular forms; for this, see, e.g., [4, 7, 8, 11, 18, 1]. No prior exposure to modular symbols is assumed.

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1 Modular forms and Hecke operators

The objects we will consider arise from the modular group $\mathrm{SL}_2(\mathbf{Z})$ of two-by-two integer matrices with determinant equal to one. This group acts via linear fractional transformations on the complex upper half plane $\mathfrak{h}$, and also on the extended upper half plane

$$\mathfrak{h}^* = \mathfrak{h} \cup \mathbf{P}^1(\mathbf{Q}) = \mathfrak{h} \cup \mathbf{Q} \cup \{\infty\}.$$

See [20, §1.3–1.5] for a careful description of the topology on $\mathfrak{h}^*$. A basis of neighborhoods for $\alpha \in \mathbf{Q}$ is given by the sets $\{\alpha\} \cup D$, where D is a disc in $\mathfrak{h}$ that is tangent to the real line at α. Let N be a positive integer and consider the group $\Gamma_0(N)$ of matrices $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbf{Z})$ such that $N \mid c$. The group $\Gamma_0(N)$ acts on $\mathfrak{h}^*$ by linear fractional transformations, and the quotient $\Gamma_0(N)\backslash\mathfrak{h}^*$ is a Riemann surface, which we denote by $X_0(N)$. Shimura showed in [20, §6.7] that $X_0(N)$ has a canonical structure of algebraic curve over $\mathbf{Q}$.

A *cusp form* is a function f on $\mathfrak{h}$ such that $f(z)dz$ is a holomorphic differential on $X_0(N)$. Equivalently, a cusp form is a holomorphic function f on $\mathfrak{h}$ such that

(a) the expression $f(z)dz$ is invariant under replacing z by $\gamma(z)$ for each $\gamma \in \Gamma_0(N)$, and

(b) $f(z)$ is holomorphic at each element of $\mathbf{P}^1(\mathbf{Q})$, and moreover $f(z)$ tends to 0 as z tends to any element of $\mathbf{P}^1(\mathbf{Q})$.

The space of cusp forms on $\Gamma_0(N)$ is a finite dimensional complex vector space, of dimension equal to the genus g of $X_0(N)$. Viewed topologically, as a 2-dimensional real manifold, $X_0(N)(\mathbf{C})$ is a g-holed torus.

Condition (b) in the definition of $f(z)$ means that $f(z)$ has a Fourier expansion about each element of $\mathbf{P}^1(\mathbf{Q})$. Thus, at ∞ we have

$$\begin{aligned} f(z) &= a_1 e^{2\pi i z} + a_2 e^{2\pi i 2z} + a_3 e^{2\pi i 3z} + \cdots \\ &= a_1 q + a_2 q^2 + a_3 q^3 + \cdots, \end{aligned}$$

where, for brevity, we write $q = q(z) = e^{2\pi i z}$.

Example 1.1. Let E be the elliptic curve defined by the equation $y^2 + xy = x^3 + x^2 - 4x - 5$. For $p \neq 3, 13$, let $a_p = p + 1 - \#\tilde{E}(\mathbf{F}_p)$, where $\tilde{E}$ is the reduction of E mod p, and let $a_3 = -1$, $a_{13} = 1$. For n composite, define a_n using the relations at the end of Section 3. Then

$$\begin{aligned} f &= q + a_2 q^2 + a_3 q^3 + a_4 q^4 + a_5 q^5 + \cdots \\ &= q + q^2 - q^3 - q^4 + 2q^5 + \cdots \end{aligned}$$

is the q-expansion of a modular form on $\Gamma_0(39)$. The Shimura-Taniyama conjecture, which is now a theorem (see [6]) asserts that any q-expansion constructed as above from an elliptic curve over $\mathbf{Q}$ is a modular form.

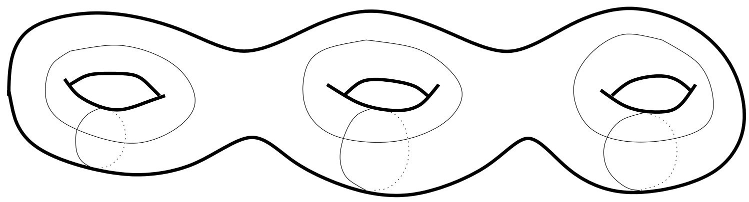
The Hecke operators are a family of *commuting* endomorphisms of $S_2(N)$, which are defined as follows. The complex points of the open subcurve $Y_0(N) = \Gamma_0(N)\backslash\mathfrak{h}$ are in bijection with pairs (E, C), where E is an elliptic curve over $\mathbf{C}$ and C is a cyclic subgroup of $E(\mathbf{C})$ of order N. If $p \nmid N$ then there are two natural maps π_1 and π_2 from $Y_0(pN)$ to $Y_0(N)$; the first, π_1, sends (E, C) to (E, C'), where C' is the unique cyclic subgroup of C of order N, and the second, π_2, sends a point $(E, C) \in Y_0(N)(\mathbf{C})$ to $(E/D, C/D)$, where D is the unique cyclic subgroup of C of order p. These maps extend in a unique way to maps from $X_0(pN)$ to $X_0(N)$:

$$\begin{tikzcd} & X_0(pN) \arrow[dl, "\pi_2"'] \arrow[dr, "\pi_1"] & \\ X_0(N) & & X_0(N). \end{tikzcd}$$

The pth *Hecke operator* T_p is $(\pi_1)_* \circ (\pi_2)^*$; it acts on most objects attached to $X_0(N)$, such as divisors and cusp forms. There is a Hecke operator T_n for every positive integer n, but we will not need to consider those with n composite.

Example 1.2. There is a basis of $S_2(39)$ so that

$$T_2 = \begin{pmatrix} 1 & 1 & 0 \\ -2 & -3 & -2 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad T_5 = \begin{pmatrix} -4 & -2 & -6 \\ 4 & 4 & 4 \\ 0 & 0 & 2 \end{pmatrix}.$$



$$H_1(X_0(39), \mathbf{Z}) \cong \mathbf{Z} \times \mathbf{Z} \times \mathbf{Z} \times \mathbf{Z} \times \mathbf{Z} \times \mathbf{Z}$$

Figure 1: The homology of $X_0(39)$.

Notice that these matrices commute, and that 1 is an eigenvalue of T_2, and 2 is an eigenvalue of T_5.

The first homology group $H_1(X_0(N), \mathbf{Z})$ is the group of singular 1-cycles modulo homology relations. Recall that topologically $X_0(N)$ is a g-holed torus, where g is the genus of $X_0(N)$. The group $H_1(X_0(N), \mathbf{Z})$ is thus a free abelian group of rank $2g$ (see, e.g., [9, Ex. 19.30]), with two generators corresponding to each hole, as illustrated in the case $N = 39$ in Figure 1.

The Hecke operators T_p act on $H_1(X_0(N), \mathbf{Z})$, and integration defines a nondegenerate Hecke-equivariant pairing

$$\langle\ ,\ \rangle : S_2(N) \times H_1(X_0(N), \mathbf{Z}) \to \mathbf{C}.$$

Explicitly, for a path x,

$$\langle f, x \rangle = 2\pi i \int_x f(z) dz,$$

where the integral may be viewed as a complex line integral along an appropriate piece of the preimage of x in the upper half plane. The pairing is Hecke equivariant in the sense that for every prime p, we have $\langle fT_p, x \rangle = \langle f, T_p x \rangle$. As we will see, modular symbols allow us to make explicit the action of the Hecke operators on $H_1(X_0(N), \mathbf{Z})$; the above pairing then translates this into a wealth of information about cusp forms.

For a more detailed survey of the basic facts about modular curves and modular forms, we urge the reader to consult Diamond and Im's excellent survey paper [8]. For a discussion of how to draw a picture of the ring generated by the Hecke operators, see [18, §3.8].

2 Modular symbols

The modular symbols formalism provides a presentation of $H_1(X_0(N), \mathbf{Z})$ in terms of paths between elements of $\mathbf{P}^1(\mathbf{Q})$. Furthermore, a trick due to Manin gives an explicit finite list of generators and relations for the space of modular symbols.

PSfrag replacements

∞

α

β

0

$\mathbf{Q}$

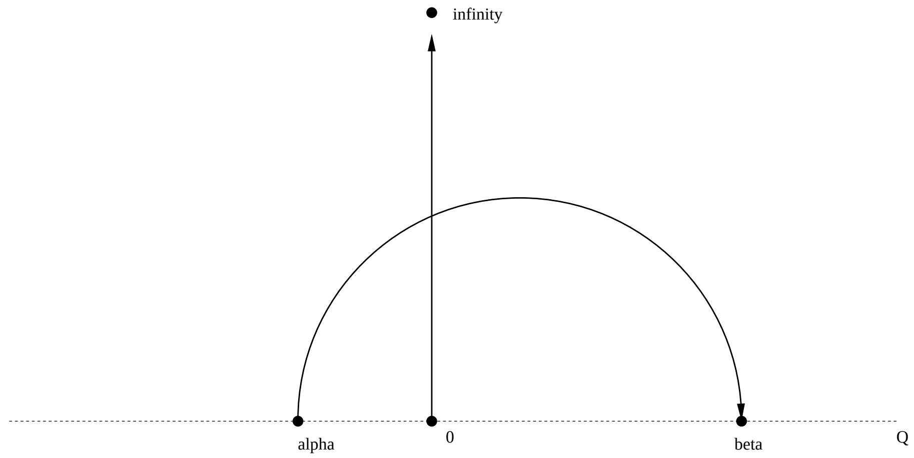


Figure 2: The modular symbols $\{\alpha, \beta\}$ and $\{0, \infty\}$.

The *modular symbol* defined by a pair $\alpha, \beta \in \mathbf{P}^1(\mathbf{Q})$ is denoted $\{\alpha, \beta\}$. As illustrated in Figure 2, this modular symbol should be viewed as the homology class, relative to the cusps, of a geodesic path from α to β in $\mathfrak{h}^*$. The homology group relative to the cusps is a slight enlargement of the usual homology group, in that we allow paths with endpoints in $\mathbf{P}^1(\mathbf{Q})$ instead of restricting to closed loops.

Motivated by this picture, we declare that modular symbols satisfy the following homology relations: if $\alpha, \beta, \gamma \in \mathbf{Q} \cup \{\infty\}$, then

$$\{\alpha, \beta\} + \{\beta, \gamma\} + \{\gamma, \alpha\} = 0.$$

Furthermore, the space of modular symbols is torsion free, so, e.g., $\{\alpha, \alpha\} = 0$ and $\{\alpha, \beta\} = -\{\beta, \alpha\}$.

Denote by $\boldsymbol{\mathcal{M}}_2$ the free abelian group with basis the set of symbols $\{\alpha, \beta\}$ modulo the three-term homology relations above and modulo any torsion. There is a left action of $\mathrm{GL}_2(\mathbf{Q})$ on $\boldsymbol{\mathcal{M}}_2$, whereby a matrix g acts by

$$g\{\alpha, \beta\} = \{g(\alpha), g(\beta)\},$$

and g acts on α and β by a linear fractional transformation. The space $\boldsymbol{\mathcal{M}}_2(N)$ of *modular symbols for* $\Gamma_0(N)$ is the quotient of $\boldsymbol{\mathcal{M}}_2$ by the submodule generated by the infinitely many elements of the form $x - g(x)$, for x in $\boldsymbol{\mathcal{M}}_2$ and g in $\Gamma_0(N)$, and modulo any torsion. A *modular symbol for* $\Gamma_0(N)$ is an element of this space. We frequently denote the equivalence class that defines a modular symbol by giving a representative element.

Example 2.1. Since $\gamma = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \in \Gamma_0(N)$, we have $\{\infty, 0\} = \{\gamma(\infty), \gamma(0)\} = \{\infty, 1\}$. Thus $0 = \{\infty, 1\} - \{\infty, 0\} = \{\infty, 1\} + \{0, \infty\} = \{0, \infty\} + \{\infty, 1\} = \{0, 1\}$.

In [13], Manin proved that there is a natural injection $H_1(X_0(N), \mathbf{Z}) \hookrightarrow \boldsymbol{\mathcal{M}}_2(N)$. The image of $H_1(X_0(N), \mathbf{Z})$ in $\boldsymbol{\mathcal{M}}_2(N)$ is as follows. Let $\boldsymbol{\mathcal{B}}_2(N)$ denote the free abelian group whose basis is the finite set $\Gamma_0(N)\backslash\mathbf{P}^1(\mathbf{Q})$. The *boundary map* $\delta : \boldsymbol{\mathcal{M}}_2(N) \to \boldsymbol{\mathcal{B}}_2(N)$ sends $\{\alpha, \beta\}$ to $[\beta] - [\alpha]$, where $[\beta]$ denotes the basis element of $\boldsymbol{\mathcal{B}}_2(N)$ corresponding to $\beta \in \mathbf{P}^1(\mathbf{Q})$. The kernel $\boldsymbol{\mathcal{S}}_2(N)$ of δ is the subspace of *cuspidal* modular symbols. An element of $\boldsymbol{\mathcal{S}}_2(N)$ can be thought of as a linear combination of paths in $\mathfrak{h}^*$ whose endpoints are cusps, and whose images in $X_0(N)$ are a linear combination of loops. We thus obtain a map $\varphi : \boldsymbol{\mathcal{S}}_2(N) \to H_1(X_0(N), \mathbf{Z})$.

Theorem 2.2. *The map φ given above defines a canonical isomorphism*

$$\boldsymbol{\mathcal{S}}_2(N) \cong H_1(X_0(N), \mathbf{Z}).$$

Example 2.3. We illustrate modular symbols in the case when $N = 11$. Using, e.g., the author's package in [5], one finds that $\boldsymbol{\mathcal{M}}_2(11)$ is the free abelian group of rank 3 generated by $\{-1/7, 0\}$, $\{-1/5, 0\}$, and $\{\infty, 0\}$. The integral homology $H_1(X_0(N), \mathbf{Z})$ corresponds to the abelian subgroup generated by $\{-1/7, 0\}$ and $\{-1/5, 0\}$. See [7, Appendix to Ch. II] for a more detailed description of the computation of $\boldsymbol{\mathcal{M}}_2(11)$.

2.1 Manin's trick

In this section, we describe a trick of Manin that shows that the space of modular symbols can be computed.

By reducing modulo N, one sees that the group $\Gamma_0(N)$ has finite index in $\mathrm{SL}_2(\mathbf{Z})$. Let $r_0, r_1, \ldots, r_m$ be distinct right coset representatives for $\Gamma_0(N)$ in $\mathrm{SL}_2(\mathbf{Z})$, so that

$$\mathrm{SL}_2(\mathbf{Z}) = \Gamma_0(N)r_o \cup \Gamma_0(N)r_1 \cup \cdots \cup \Gamma_0(N)r_m,$$

where the union is disjoint. For example, when N is prime, a list of coset representatives is

$$\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}, \begin{pmatrix}1 & 0\\ 2 & 1\end{pmatrix}, \begin{pmatrix}1 & 0\\ 3 & 1\end{pmatrix}, \ldots, \begin{pmatrix}1 & 0\\ N-1 & 1\end{pmatrix}, \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}.$$

In general, the right cosets of $\Gamma_0(N)$ in $\mathrm{SL}_2(\mathbf{Z})$ are in bijection with the elements of $\mathbf{P}^1(\mathbf{Z}/N\mathbf{Z})$, the bijection sending a coset representative $\left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right)$ to the class of $(c : d)$ in $\mathbf{P}^1(\mathbf{Z}/N\mathbf{Z})$ (see [7, §2.2] for complete details).

The following trick of Manin (see [13, §1.5] and [7, §2.1.6]) allows us to write every modular symbol as a $\mathbf{Z}$-linear combination of symbols of the form $r_i\{0, \infty\}$. In particular, the finitely many symbols $r_i\{0, \infty\}$ generate $\boldsymbol{\mathcal{M}}_2(N)$.

Because of the relation $\{\alpha, \beta\} = \{0, \beta\} - \{0, \alpha\}$, it suffices to consider modular symbols of the form $\{0, b/a\}$, where the rational number b/a is in lowest terms. Expand b/a as a continued fraction and consider the successive convergents in lowest terms:

$$\frac{b_{-2}}{a_{-2}} = \frac{0}{1}, \quad \frac{b_{-1}}{a_{-1}} = \frac{1}{0}, \quad \frac{b_0}{a_0} = \frac{b_0}{1}, \ldots, \quad \frac{b_{n-1}}{a_{n-1}}, \quad \frac{b_n}{a_n} = \frac{b}{a}$$

where the first two are added formally. Then

$$b_k a_{k-1} - b_{k-1} a_k = (-1)^{k-1},$$

so that

$$g_k = \begin{pmatrix} b_k & (-1)^{k-1} b_{k-1} \\ a_k & (-1)^{k-1} a_{k-1} \end{pmatrix} \in \mathrm{SL}_2(\mathbf{Z}).$$

Hence

$$\left\{ \frac{b_{k-1}}{a_{k-1}}, \frac{b_k}{a_k} \right\} = g_k\{0, \infty\} = r_i\{0, \infty\},$$

for some i, is of the required special form.

Example 2.4. Let $N = 11$, and consider the modular symbol $\{0, 4/7\}$. We have

$$\frac{4}{7} = 0 + \frac{1}{1 + \frac{1}{1+\frac{1}{3}}},$$

so the partial convergents are

$$\frac{b_{-2}}{a_{-2}} = \frac{0}{1}, \quad \frac{b_{-1}}{a_{-1}} = \frac{1}{0}, \quad \frac{b_0}{a_0} = \frac{0}{1}, \quad \frac{b_1}{a_1} = \frac{1}{1}, \quad \frac{b_2}{a_2} = \frac{1}{2}, \quad \frac{b_3}{a_3} = \frac{4}{7}.$$

Thus, noting as in Example 2.1 that $\{0, 1\} = 0$, we compute

$$\begin{aligned} \{0, 4/7\} &= \{0, \infty\} + \{\infty, 0\} + \{0, 1\} + \{1, 1/2\} + \{1/2, 4/7\} \\ &= \begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix} \{0, \infty\} + \begin{pmatrix} 4 & 1 \\ 7 & 2 \end{pmatrix} \{0, \infty\} \\ &= \begin{pmatrix} 1 & 0 \\ 9 & 1 \end{pmatrix} \{0, \infty\} + \begin{pmatrix} 1 & 0 \\ 9 & 1 \end{pmatrix} \{0, \infty\} \\ &= 2 \cdot \left[\begin{pmatrix} 1 & 0 \\ 9 & 1 \end{pmatrix} \{0, \infty\} \right] \end{aligned}$$

2.2 Manin symbols

As above, fix coset representatives $r_0, \ldots, r_m$ for $\Gamma_0(N)$ in $\mathrm{SL}_2(\mathbf{Z})$. Denote the modular symbol $r_i\{0, \infty\}$ by $[r_i]$. The symbols $[r_0], \ldots, [r_m]$ are called *Manin symbols*, and they are equipped with a right action of $\mathrm{SL}_2(\mathbf{Z})$, which is given by $[r_i]g = [r_j]$, where $\Gamma_0(N) r_j = \Gamma_0(N) r_i g$. Recall that $\mathrm{SL}_2(\mathbf{Z})$ is generated by the two matrices $\sigma = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ and $\tau = \left(\begin{smallmatrix} 1 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ (see Theorem 2 of [19, VII.1.2]).

Theorem 2.5 (Manin). *The Manin symbols $[r_0], \ldots, [r_m]$ satisfy the following relations:*

$$[r_i] + [r_i]\sigma = 0$$
$$[r_i] + [r_i]\tau + [r_i]\tau^2 = 0.$$

Furthermore, these relations generate all relations (modulo torsion relations).

This theorem, which is proved in [13, §1.7], provides a finite presentation for the space of modular symbols.

2.3 Hecke operators on modular symbols

When p is a prime not dividing N, define

$$T_p\{\alpha,\beta\} = \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}\{\alpha,\beta\} + \sum_{r \bmod p} \begin{pmatrix} 1 & r \\ 0 & p \end{pmatrix}\{\alpha,\beta\}.$$

As mentioned before, this definition is compatible with the integration pairing $\langle\, ,\, \rangle$ of Section 1, in the sense that $\langle f T_p, x\rangle = \langle f, T_p x\rangle$. When $p \mid N$, the definition is the same, except that the matrix $\left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right)$ is dropped.

For example, when $N = 11$ we have

$$\begin{aligned} T_2\{0, 1/5\} &= \{0, 2/5\} + \{0, 1/10\} + \{1/2, 3/5\} \\ &= -2\{0, 1/5\}. \end{aligned}$$

In [15], L. Merel gives a description of the action of T_p directly on Manin symbols $[r_i]$ (see also, [7, §2.4]). For example, when $p = 2$ and N is odd, we have

$$T_2([r_i]) = [r_i]\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} + [r_i]\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} + [r_i]\begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix} + [r_i]\begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}.$$

3 Computing the space of modular forms

In this section we describe how to use modular symbols to construct a basis of $S_2(N)$ consisting of modular forms that are eigenvectors for every element of the ring $\mathbf{T}'$ generated by the Hecke operator T_p, with $p \nmid N$. Such eigenvectors are called *eigenforms*.

Suppose M is a positive integer that divides N. As explained in [12, VIII.1–2], for each divisor d of N/M there is a natural *degeneracy map* $\beta_{M,d} : S_2(M) \to S_2(N)$ given by $\beta_{M,d}(f(q)) = f(q^d)$. The *new subspace* of $S_2(N)$, denoted $S_2(N)^{\text{new}}$, is the complementary $\mathbf{T}$-submodule of the $\mathbf{T}$-module generated by the images of all maps $\beta_{M,d}$, with M and d as above. (It is a nontrivial fact that this complement is well defined; one possible proof uses the Petersson inner product.)

The theory of Atkin and Lehner [2] asserts that, as a $\mathbf{T}'$-module, $S_2(N)$ decomposes as follows:

$$S_2(N) \quad = \bigoplus_{M|N,\ d|N/M} \beta_{M,d}(S_2(M)^{\text{new}}).$$

To compute $S_2(N)$ it thus suffices to compute $S_2(M)^{\text{new}}$ for each positive divisor M of N.

We now turn to the problem of computing $S_2(N)^{\text{new}}$. Atkin and Lehner [2] also proved that $S_2(N)^{\text{new}}$ is spanned by eigenforms, each of which occurs with multiplicity one in $S_2(N)^{\text{new}}$. Moreover, if $f \in S_2(N)^{\text{new}}$ is an eigenform then the coefficient of q in the q-expansion of f is nonzero, so it is possible to normalize f so that coefficient of q is 1. With f so normalized, if $T_p(f) = a_p f$, then the

pth Fourier coefficient of f is a_p. If $f = \sum_{n=1}^{\infty} a_n q^n$ is a normalized eigenvector for all T_p, then the a_n, with n composite, are determined by the a_p, with p prime, by the following formulas: $a_{nm} = a_n a_m$ when n and m are relatively prime, and $a_{p^r} = a_{p^{r-1}} a_p - p a_{p^{r-2}}$ for $p \nmid N$ prime. When $p \mid N$, $a_{p^r} = a_p^r$. We conclude that in order to compute $S_2(N)^{\mathrm{new}}$, it suffices to compute all systems of eigenvalues $\{a_2, a_3, a_5, \ldots\}$ of the Hecke operators $T_2, T_3, T_5, \ldots$ acting on $S_2(N)^{\mathrm{new}}$. Given a system of eigenvalues, the corresponding eigenform is $f = \sum_{n=1}^{\infty} a_n q^n$, where the a_n, for n composite, are determined by the recurrence given above.

In light of the pairing $\langle\, , \rangle$ introduced in Section 1, computing the above systems of eigenvalues $\{a_2, a_3, a_5, \ldots\}$ amounts to computing the systems of eigenvalues of the Hecke operators T_p on the subspace V of $\boldsymbol{\mathcal{S}}_2(N)$ that corresponds to the new subspace of $S_2(N)$. For each proper divisor M of N and each divisor d of N/M, let $\phi_{M,d} : \boldsymbol{\mathcal{S}}_2(N) \to \boldsymbol{\mathcal{S}}_2(M)$ be the map sending x to $\left(\begin{smallmatrix} d & 0 \\ 0 & 1 \end{smallmatrix}\right) x$. Then V is the intersection of the kernels of all maps $\phi_{M,d}$.

The computation of the systems of eigenvalues of a collection of commuting diagonalizable endomorphisms involves standard linear algebra techniques, such as computation of characteristic polynomials and kernels of matrices. There are, however, several tricks that greatly speed up this process, some of which are described in [21, §3.5.4].

Example 3.1. All forms in $S_2(39)$ are new. Up to Galois conjugacy, the eigenvalues of the Hecke operators T_2, T_3, T_5, and T_7 on $\boldsymbol{\mathcal{S}}_2(39)$ are $\{1, -1, 2, -4\}$ and $\{a, 1, -2a-2, 2a+2\}$, where $a^2 + 2a - 1 = 0$. Each of these eigenvalues occur in $\boldsymbol{\mathcal{S}}_2(39)$ with multiplicity two; for example, the characteristic polynomial of T_2 on $\boldsymbol{\mathcal{S}}_2(39)$ is $(x-1)^2 \cdot (x^2 + 2x - 1)^2$. Thus $S_2(39)$ is spanned by

$$f_1 = q + q^2 - q^3 - q^4 + 2q^5 - q^6 - 4q^7 + \cdots,$$
$$f_2 = q + aq^2 + q^3 + (-2a-1)q^4 + (-2a-2)q^5 + aq^6 + (2a+2)q^7 + \cdots,$$

and the Galois conjugate of f_2.

3.1 Summary

To compute the q-expansion, to some precision, of each eigenforms in $S_2(N)$, we use the degeneracy maps so that we only have to solve the problem for $S_2(N)^{\mathrm{new}}$. Here, using modular symbols, we compute all systems of eigenvalues $\{a_2, a_3, a_5, \ldots\}$, then write down each of the corresponding eigenforms $f = q + a_2 q^2 + q_3 q^3 + \cdots$.

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