Explicit Approaches to Elliptic Curves and Modular Forms

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Outline of Course and this Lecture

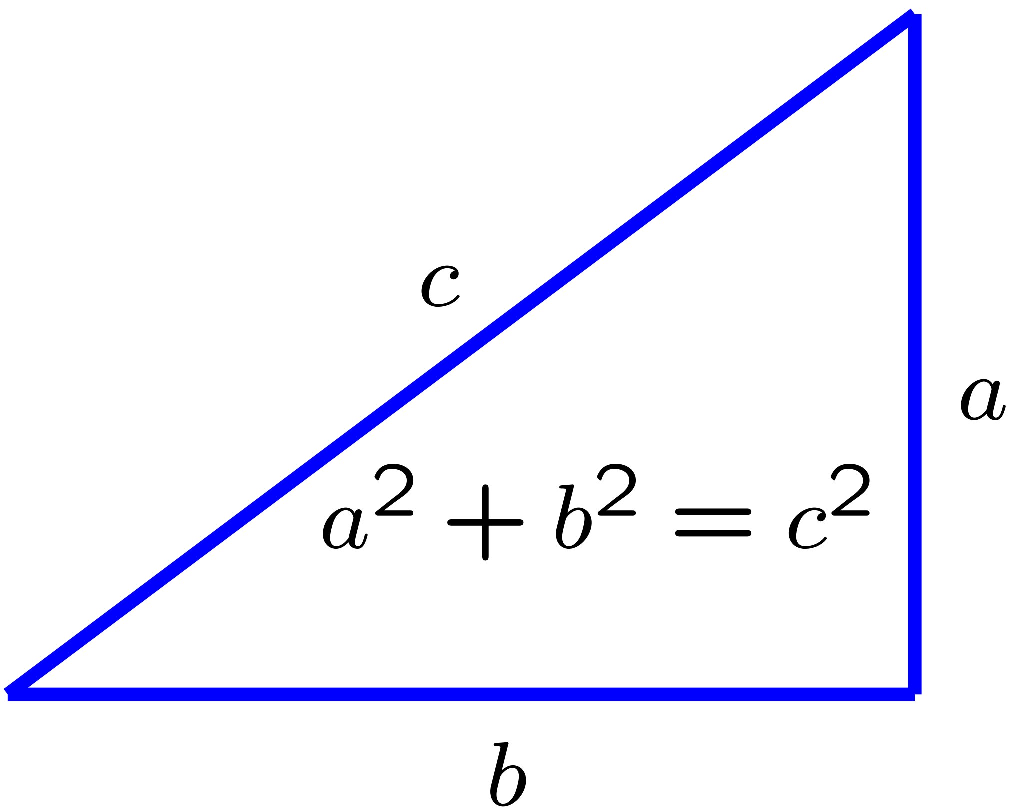
1. Pythagoras and Fermat

2. Mordell-Weil Groups and the BSD Conjecture

3. Modularity of Elliptic Curves

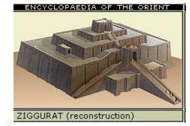
4. Computing Modular Forms

The Pythagorean Theorem

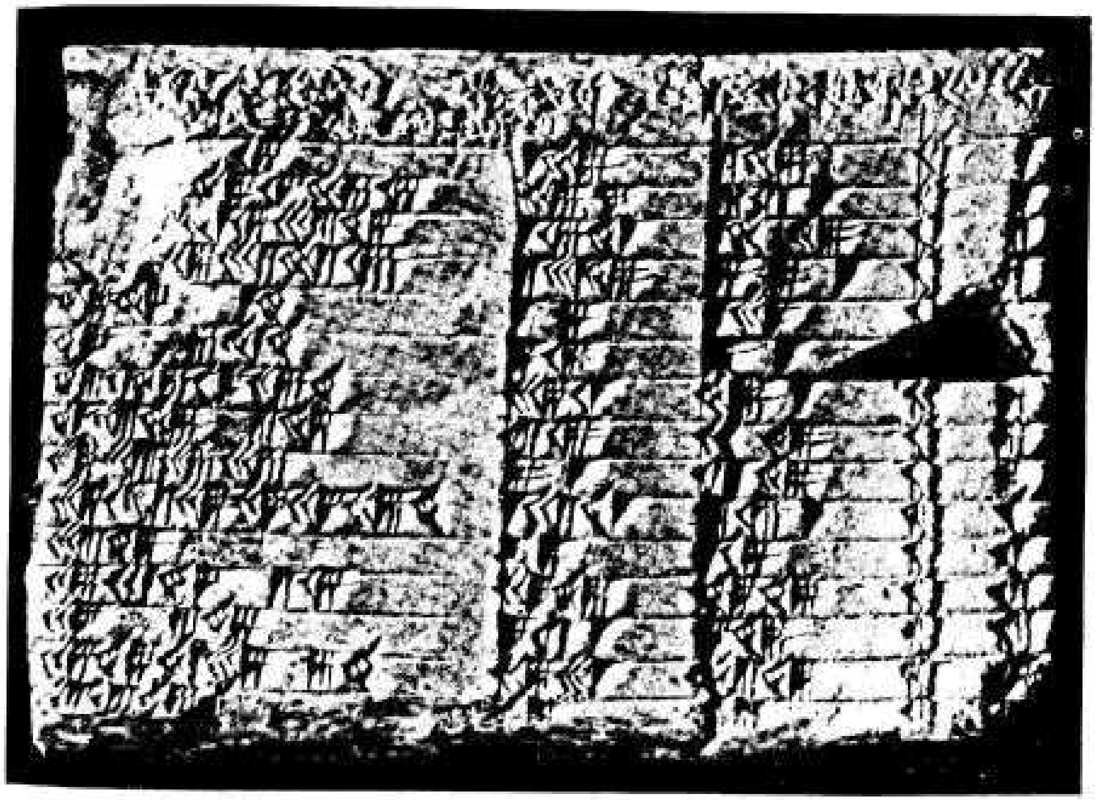


Pythagoras
Approx 569–475BC

Pythagorean Triples



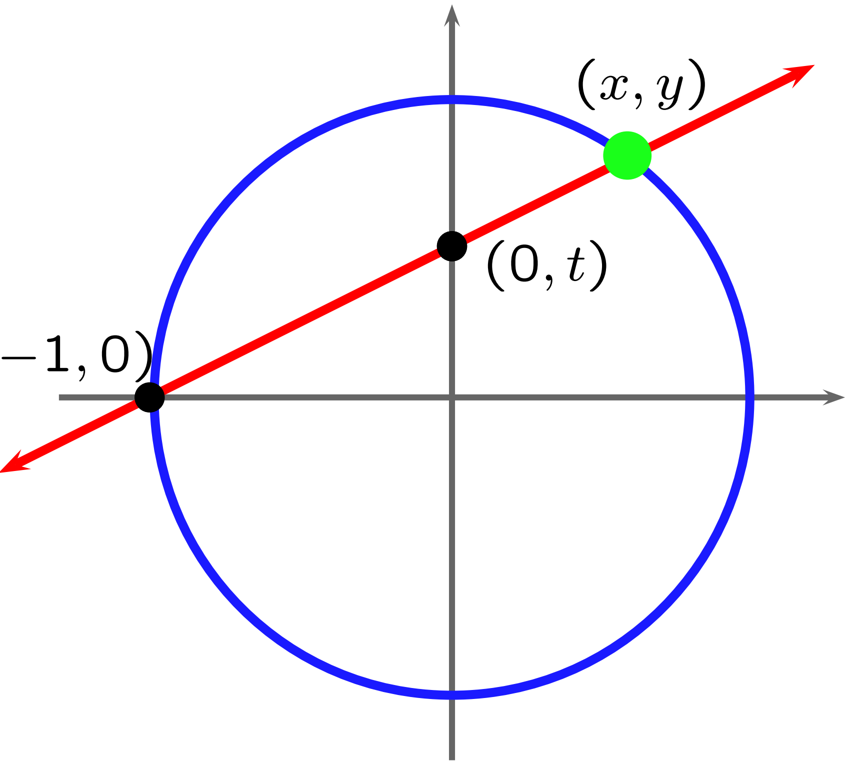
$(3,4,5)$
$(5,12,13)$
$(7,24,25)$
$(9,40,41)$
$(11,60,61)$
$(13,84,85)$
$(15,8,17)$
$(21,20,29)$
$(33,56,65)$
$(35,12,37)$
$(39,80,89)$
$(45,28,53)$
$(55,48,73)$
$(63,16,65)$
$(65,72,97)$
$(77,36,85)$
$\vdots$



Triples of integers a, b, c such that

$$a^2 + b^2 = c^2$$

Enumerating Pythagorean Triples



$$\text{Slope} = t = \frac{y}{x+1}$$

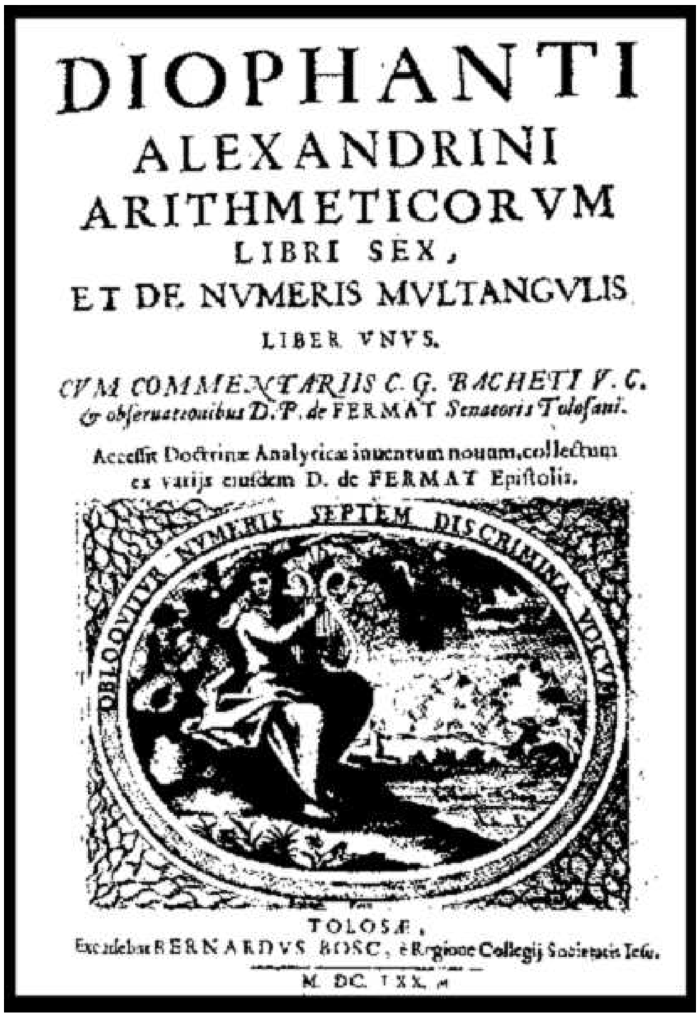
$$x = \frac{1-t^2}{1+t^2}$$

$$y = \frac{2t}{1+t^2}$$

If $t = \frac{r}{s}$, then $a = s^2 - r^2, \quad b = 2rs, \quad c = s^2 + r^2$
is a Pythagorean triple, and all primitive unordered triples arise in this way.

Fermat's "Last Theorem"

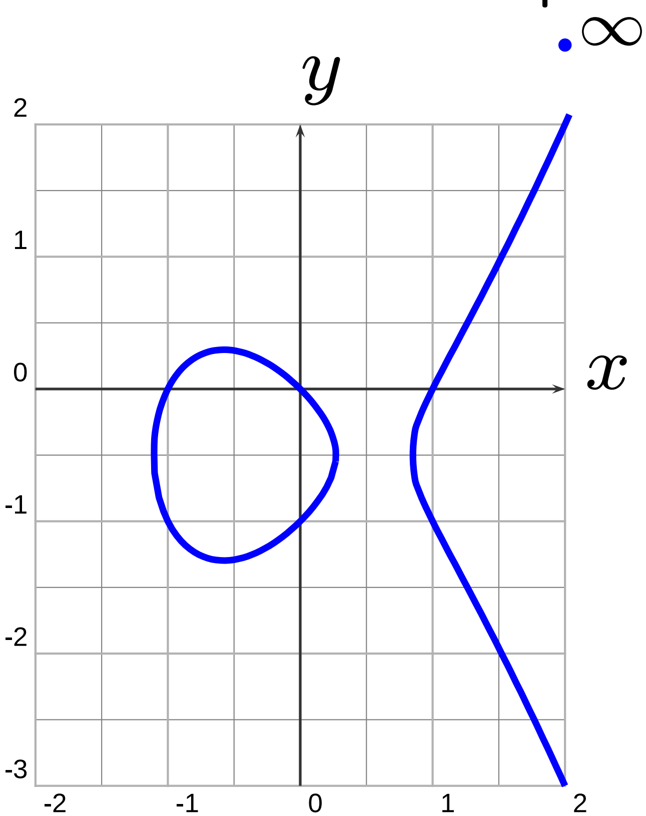
No analogue of "Pythagorean triples" with exponent 3 or higher.



DIOPHANTI
ALEXANDRINI
ARITHMETICORVM
LIBRI SEX,
ET DE NVMERIS MVLTANGVLIS
LIBER VNVS.
CVM COMMENTARIIS C. G. BACHETI V. C.
& obseruationibus D. P. de FERMAT Senatoris Tolosani.
Accessit Doctrinæ Analyticæ inuentum nouum, collectum ex varijs eiusdem D. de FERMAT Epistolis.
OBLOQVITVR NVMERIS SEPTEM DISCRIMINA VOCVM
TOLOSÆ,
Excudebat BERNARDVS BOSC, è Regione Collegij Societatis Iesu.
M. DC. LXX.

Wiles's Proof of FLT Uses Elliptic Curves

An **elliptic curve** is a nonsingular plane cubic curve with a rational point (possibly "at infinity").



$$y^2 + y = x^3 - x$$

EXAMPLES

$$y^2 + y = x^3 - x$$

$$x^3 + y^3 = 1 \text{ (Fermat cubic)}$$

$$y^2 = x^3 + ax + b$$

$$\xcancel{3x^3 + 4y^3 + 5 = 0}$$

The Frey Elliptic Curve

Suppose Fermat's conjecture is **FALSE**. Then there is a prime $\ell \geq 5$ and coprime positive integers a, b, c with $a^\ell + b^\ell = c^\ell$.

Consider the corresponding Frey elliptic curve:

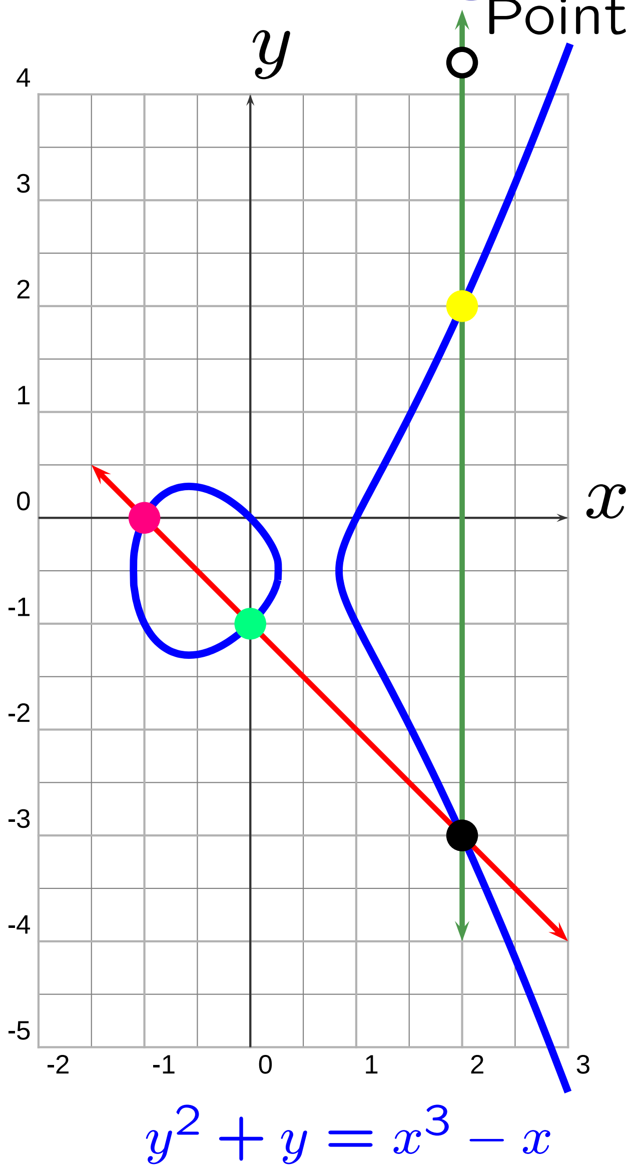
$$y^2 = x(x - a^\ell)(x + b^\ell).$$

Ribet's Theorem: This elliptic curve is not *modular*.

Wiles's Theorem: This elliptic curve is *modular*.

Conclusion: Fermat's conjecture is true.

The Group Operation

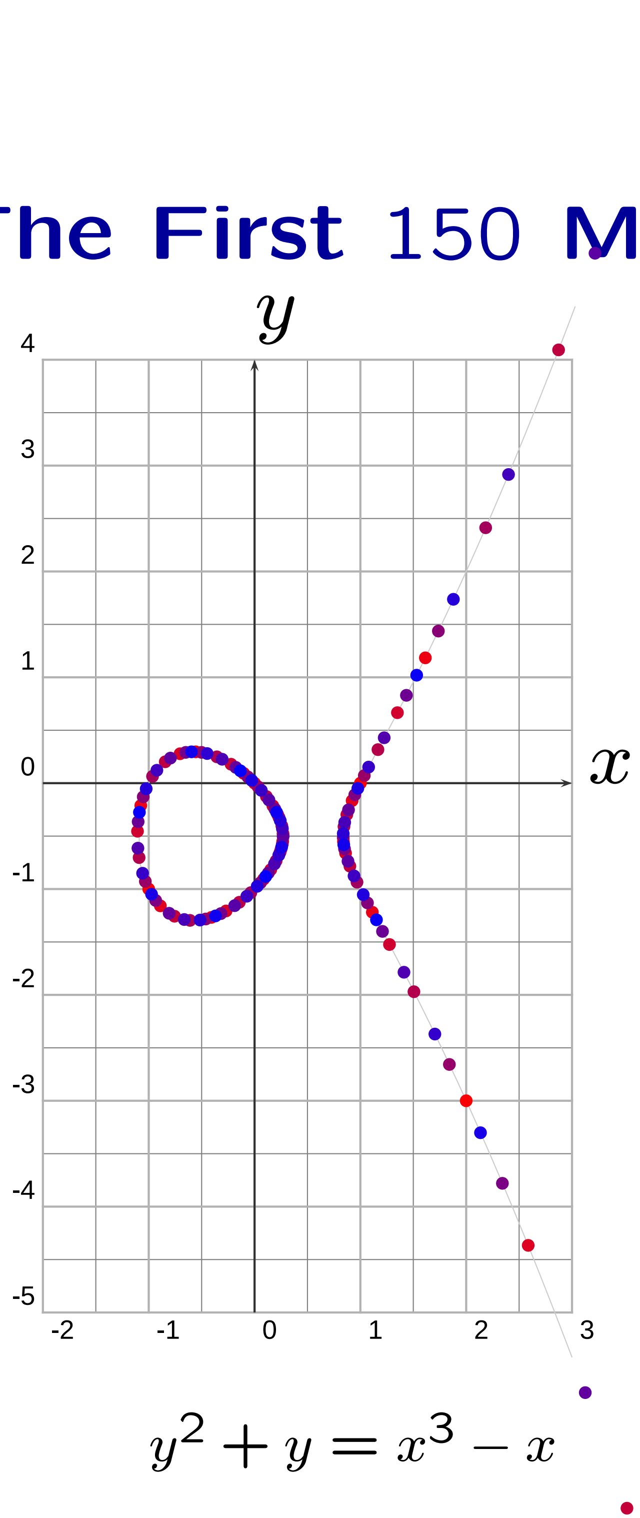


● $\oplus$ ● $=$ ●

$(-1, 0) \oplus (0, -1) = (2, 2)$

The set of points on E forms an abelian group.

The First 150 Multiples of $(0, 0)$



$$y^2 + y = x^3 - x$$

(The bluer the point, the bigger the multiple.)

Fact: The group $E(\mathbf{Q})$ is infinite cylic, generated by $(0, 0)$.

In contrast, $y^2 + y = x^3 - x^2$ has only 5 rational points!

What is going on here?

Mordell's Theorem

Theorem (Mordell). The group $E(\mathbf{Q})$ of rational points on an elliptic curve is a **finitely generated abelian group**, so

$$E(\mathbf{Q}) \cong \mathbf{Z}^r \oplus T,$$

with $T = E(\mathbf{Q})_{\text{tor}}$ finite.

Mazur classified the possibilities for T. It is conjectured that r can be arbitrary, but the biggest r ever found is (probably) 24.

The Simplest Solution Can Be Huge

Simplest solution to $y^2 = x^3 + 7823$:

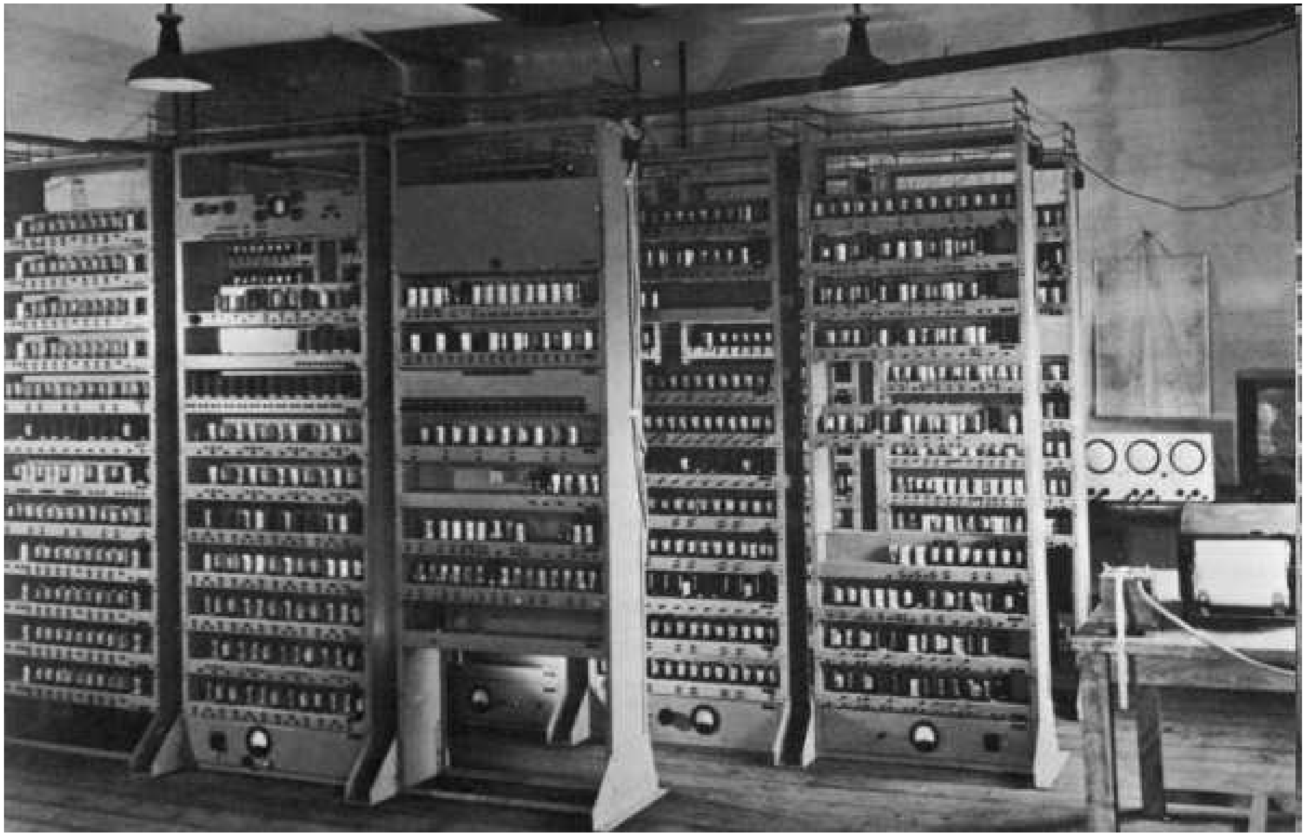
$$x = \frac{2263582143321421502100209233517777}{143560497706190989485475151904721}$$

$$y = \frac{186398152584623305624837551485596770028144776655756}{1720094998106353355821008525938727950159777043481}$$

(Found by Michael Stoll in 2002.)

The Central Question

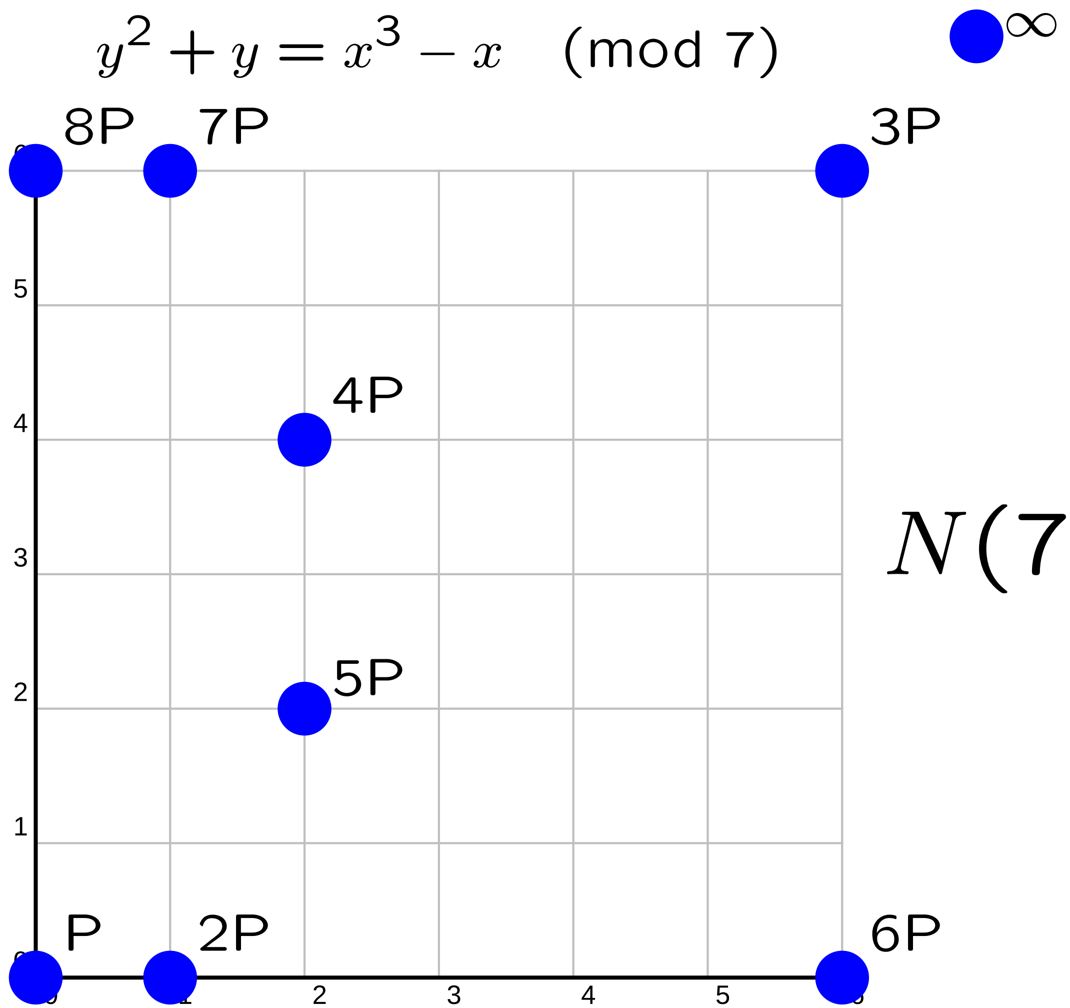
Given an elliptic curve E, what is the rank of $E(\mathbb{Q})$?



Idea!: Consider the Group Modulo p

$N(p) = \#$ of solutions $\pmod{p}$

$y^2 + y = x^3 - x \pmod{7}$

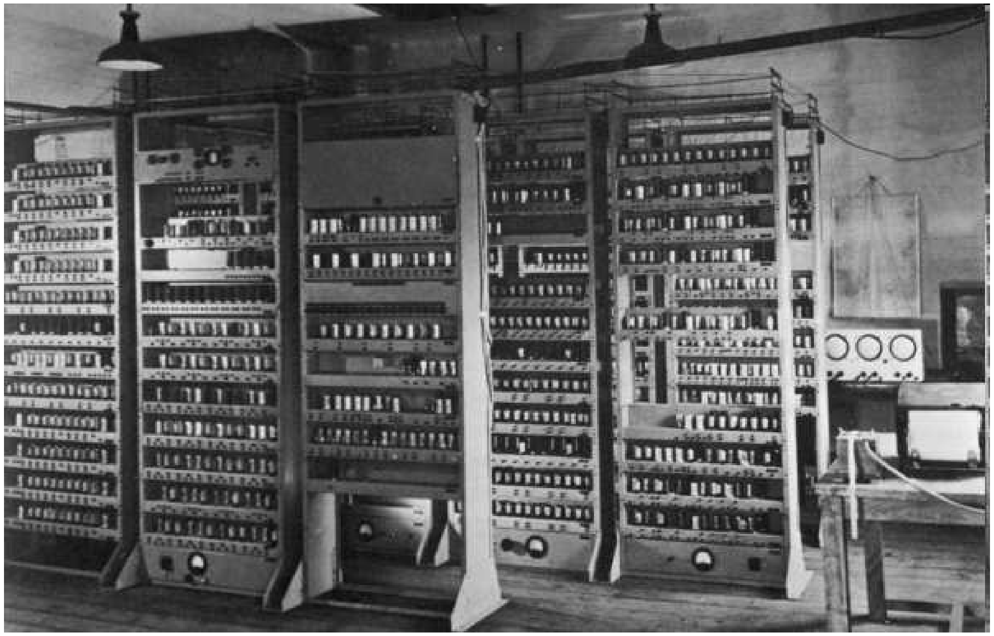


$N(7) = 9$

Counting Points

Cambridge EDSAC: The first point counting supercomputer...

Birch and Swinnerton-Dyer



Hecke Eigenvalues

Hasse

Let

$$a_p = p + 1 - N(p).$$

Hasse proved that

$$|a_p| \leq 2\sqrt{p}.$$

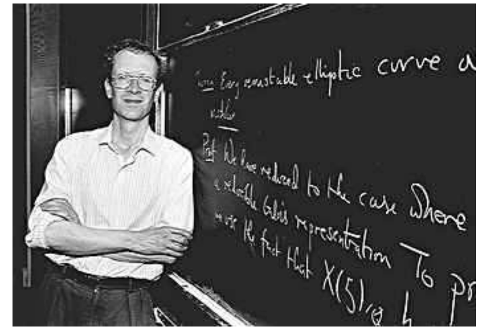
For $y^2 + y = x^3 - x$:

$$a_2 = -2, \quad a_3 = -3, \quad a_5 = -2, \quad a_7 = -1, \quad a_{11} = -5, \quad a_{13} = -2,$$

$$a_{17} = 0, \quad a_{19} = 0, \quad a_{23} = 2, \quad a_{29} = 6, \quad \ldots$$

Birch and Swinnerton-Dyer

The L-Function



Theorem (Wiles et al., Hecke) The following function extends to a holomorphic function on the whole complex plane:

$$L^*(E, s) = \prod_{p\Delta} \left(\frac{1}{1 - a_p \cdot p^{-s} + p \cdot p^{-2s}} \right).$$

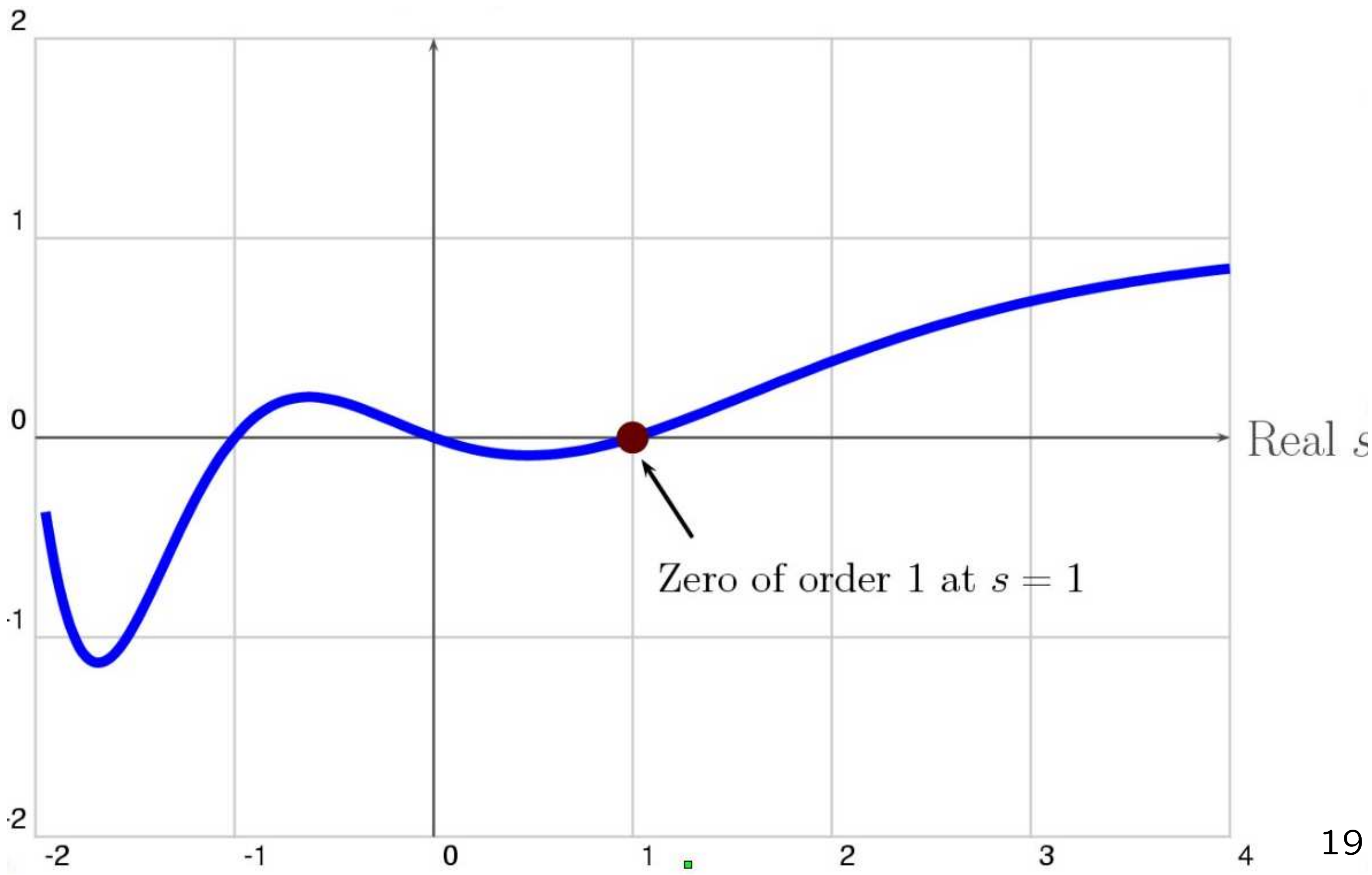
Here $a_p = p + 1 - \#E(\mathbf{F}_p)$ for all $p\Delta_E$. Note that formally,

$$L^*(E, 1) = \prod_{p\Delta} \left(\frac{1}{1 - a_p \cdot p^{-1} + p \cdot p^{-2}} \right) = \prod_{p\Delta} \left(\frac{p}{p - a_p + 1} \right) = \prod_{p\Delta} \frac{p}{N_p}$$

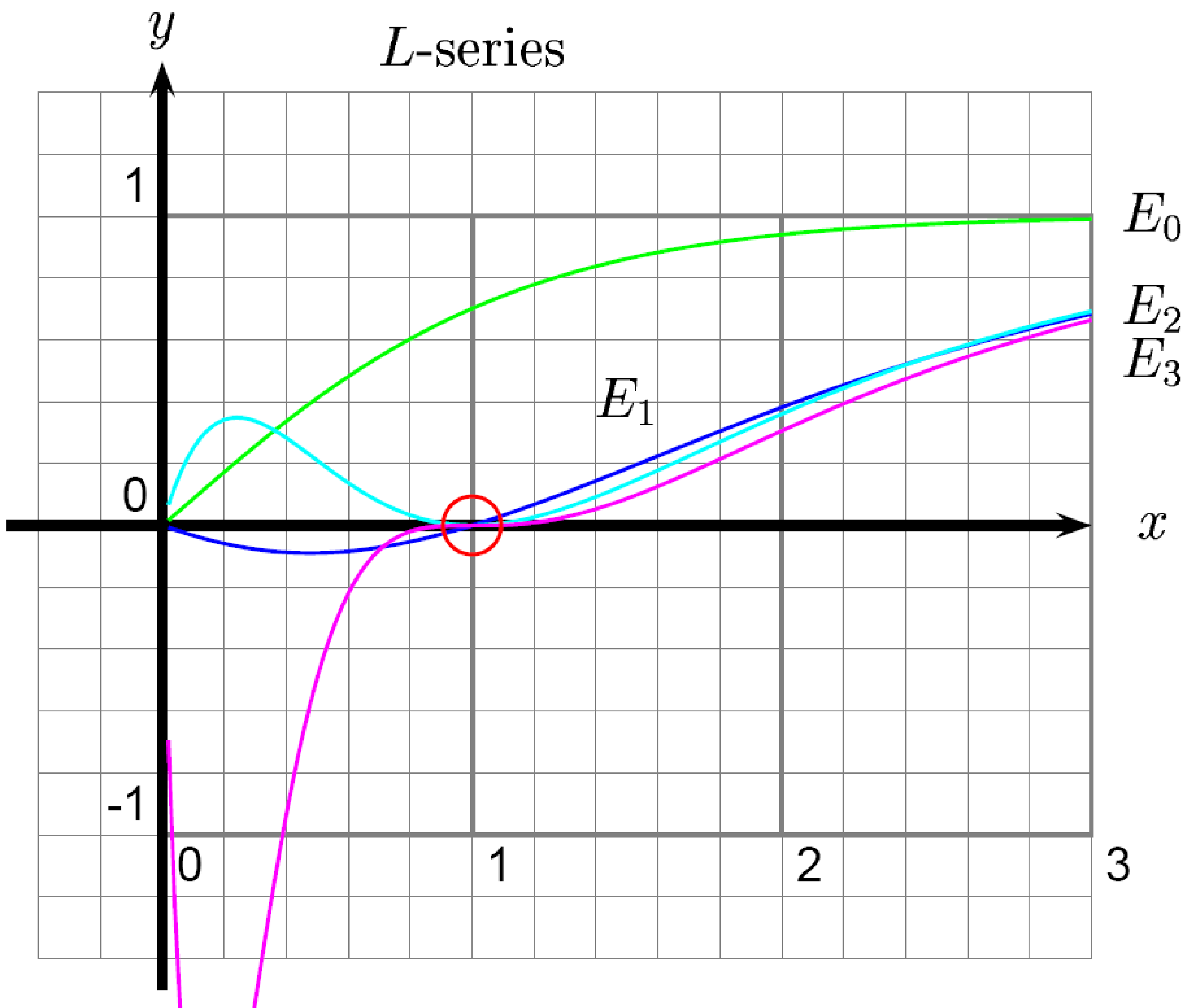
Standard extension to $L(E, s)$ at bad primes.

Real Graph of the L-Series of

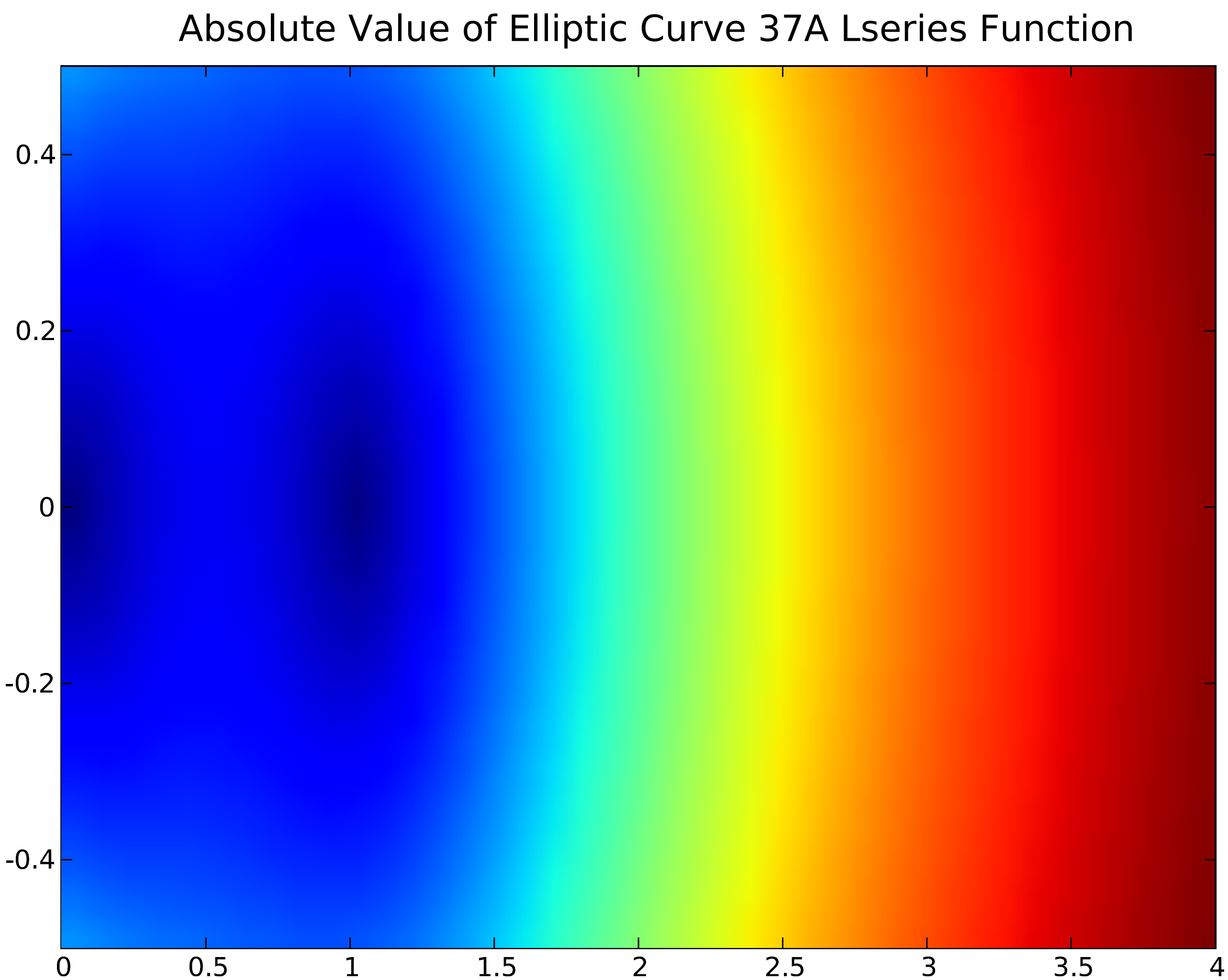
$$y^2 + y = x^3 - x$$

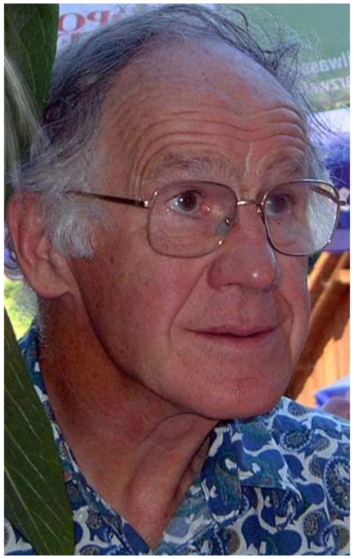


More Graphs of Elliptic Curve L-functions



Absolute Value of L-series on Complex Plane for $y^2 + y = x^3 - x$





Conjectures Proliferated

"The subject of this lecture is rather a special one. I want to describe some computations undertaken by myself and Swinnerton-Dyer on EDSAC, by which we have calculated the zeta-functions of certain elliptic curves. As a result of these computations we have found an analogue for an elliptic curve of the Tamagawa number of an algebraic group; and conjectures have proliferated. [...] though the associated theory is both abstract and technically complicated, the objects about which I intend to talk are usually simply defined and often machine computable; **experimentally we have detected certain relations between different invariants**, but we have been unable to approach proofs of these relations, which must lie very deep." – Birch 1965

The Birch and Swinnerton-Dyer Conjecture

Conjecture: Let E be any elliptic curve over $\mathbf{Q}$. The order of vanishing of $L(E,s)$ as $s=1$ equals the rank of $E(\mathbf{Q})$.

The Kolyvagin and Gross-Zagier Theorem

Theorem: If the ordering of vanishing $\mathrm{ord}_{s=1} L(E, s)$ is ≤ 1, then the conjecture is true for E.

Elliptic Curves are "Modular"

An elliptic curve is **modular** if the numbers a_p are coefficients of a "modular form". Equivalently, if $L(E, s)$ extends to a complex analytic function on $\mathbf{C}$ (with functional equation).

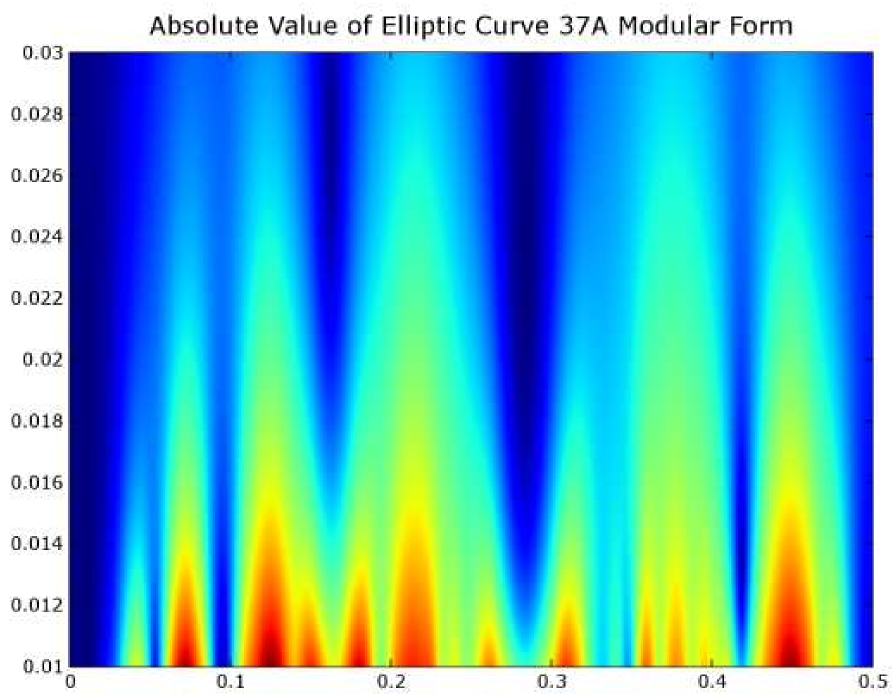
Theorem (Wiles et al.): *Every elliptic curve over the rational numbers is modular.*

Wiles at the Institute for Advanced Study

Modular Forms

The definition of modular forms as holomorphic functions satisfying a certain equation is very abstract.

For today, I will skip the abstract definition, and instead give you an explicit "engineer's recipe" for producing modular forms. In the meantime, here's a picture:



Computing Modular Forms: Motivation

Motivation: Data about modular forms is **extremely** useful to many research mathematicians (e.g., number theorists, cryptographers). This data is like the astronomer's telescope images.

One of my longterm research goals is to compute modular forms on a **huge** scale, and make the resulting database widely available. I have done this on a smaller scale during the last 5 years — see `http://modular.ucsd.edu/Tables/`

What to Compute: Newforms

For each positive integer N there is a finite list of **newforms** of level N. E.g., for $N = 37$ the newforms are

$$f_1 = q - 2q^2 - 3q^3 + 2q^4 - 2q^5 + 6q^6 - q^7 + \cdots$$
$$f_2 = q + q^3 - 2q^4 - q^7 + \cdots,$$

where $q = e^{2\pi i z}$.

The newforms of level N determine all the modular forms of level N (like a basis in linear algebra). The coefficients are algebraic integers. *Goal: compute these newforms.*

Bad idea – write down many elliptic curves and compute the numbers a_p by counting points over finite fields. No good – this misses most of the interesting newforms, and gets newforms of all kinds of random levels, but you don't know if you get everything of a given level.

An Engineer's Recipe for Newforms

Fix our positive integer N. For simplicity assume that N is prime.

1. Form the $N+1$ dimensional $\mathbf{Q}$-vector space V with basis the symbols $[0],\ldots,[N-1],[\infty]$.

2. Let R be the suspace of V spanned by the following vectors, for $x = 0,\ldots,N-1,\infty$:

$$[x] - [N-x]$$
$$[x] + [x.S]$$
$$[x] + [x.T] + [x.T^2]$$

   $S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$, $T = \left(\begin{smallmatrix} 0 & -1 \\ 1 & -1 \end{smallmatrix}\right)$, and $x.\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = (ax+c)/(bx+d)$.

3. Compute the quotient vector space $M = V/R$. This involves "intelligent" **sparse Gauss elimination** on a matrix with $N+1$ columns.

4. Compute the matrix T_2 on M given by

$$[x] \mapsto [x.\left(\begin{smallmatrix} 1 & 0 \\ 0 & 2 \end{smallmatrix}\right)] + [x.\left(\begin{smallmatrix} 2 & 0 \\ 0 & 1 \end{smallmatrix}\right)] + [x.\left(\begin{smallmatrix} 2 & 1 \\ 0 & 1 \end{smallmatrix}\right)] + [x.\left(\begin{smallmatrix} 1 & 0 \\ 1 & 2 \end{smallmatrix}\right)].$$

   This matrix is unfortunately not sparse. Similar recipe for matrices T_n for any n.

5. Compute the **characteristic polynomial** f of T_2.

6. **Factor** $f = \prod g_i^{e_i}$. Assume all $e_i = 1$ (if not, use a random linear combination of the T_n.)

7. Compute the **kernels** $K_i = \ker(g_i(T_2))$. The **eigenvalues** of T_3, T_5, etc., acting on an **eigenvector** in K_i give the coefficients a_p of the newforms of level N.

Implementation

- I implemented code for computing modular forms that's included with **MAGMA** (non-free, closed source): `http://magma.maths.usyd.edu.au/magma/`.

- I want something better, so I'm implementing modular symbols algorithms as part of **SAGE**: `http://modular.ucsd.edu/sage/`.

- I'm finishing a **book** on these algorithms that will be published by the American Mathematical Society.

The Modular Forms Database Project

- Create a database of all newforms of level N for each $N < 100000$. This will require many gigabytes to store. (50GB?)

- So far this has only been done for $N < 7000$ (and is incomplete), so 100000 is a **major challenge**.

- Involves sparse linear algebra over $\mathbf{Q}$ on spaces of dimension up to 200000 and dense linear algebra on spaces of dimension up to 25000.

- Easy to parallelize – run one process for each N.

- Will be very useful to number theorists and cryptographers.

- John Cremona has done something similar but only for the newforms corresponding to elliptic curves (he's at around 120000 right now), so this should be do-able.

Goals for Math 168

- **[Elliptic Curves]** Definition, group structure, applications to cryptography, L-series, the Birch and Swinnerton-Dyer conjecture (a million dollar Clay Math prize problem).

- **[Modular Forms]** Definition (of modular forms of weight 2), connection with elliptic curves and Andrew Wiles's celebrated proof of Fermat's Last Theorem, how to use modular symbols to compute modular forms.

- **[Research]** Get everyone in 168a involved in some aspect of my research program: algorithms needed for SAGE, making data available online, efficient linear algebra, etc.