Computational Conformal Geometry: Theories and Applications

Xianfeng Gu, Shing-Tung Yau
UFL, Harvard University

An important goal of classical geometry is to give a description of geometric figures that we see in nature. High power computer computation and three dimension camera allows us to apply classical and modern geometric technology to real features of objects that we see in day to day life.

For a given surface in three space, it is important to realize that there are several geometric structures that are inherit in their description. The most important one is the conformal structure. The construction of conformal structure by computer computation immediately allows us to apply modern and well developed technique of complex analysis and algebric geometry to be brought into graphic realization.

The basic technique involves solving nonlinear and linear differential equations. the differential equations are related to questions of isometric embedding of surfaces, which is a difficult subject. The equations can be elliptic , hyperbolic or mixed type. There are equations related to movement of surfaces where motions driven by curvature are important. In order to be efficient in data compression and signal processing for geometric figures. Algebraic geometry is an important tool.

Outline

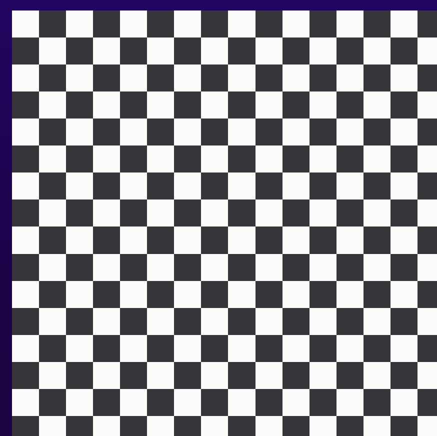
- Introduction
- Motivation
- Solution
- Applications
  - Geometric database indexing
  - Expression animation, skin deformation
- Summary

Definition: Conformal Mapping

- Scaling first fundamental from
- Angle preserving
- Similarities in the small

$$\phi : M_1 \rightarrow M_2 , g_{ij} = \lambda \phi^* \tilde{g}_{ij}$$

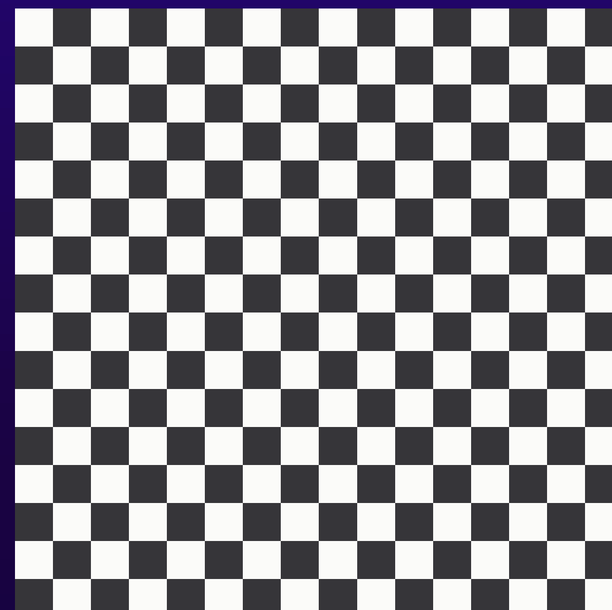
Φ



M_1 M_2

Introduction

- Surface Parametrization is a process to map a surface to a region of the plane.

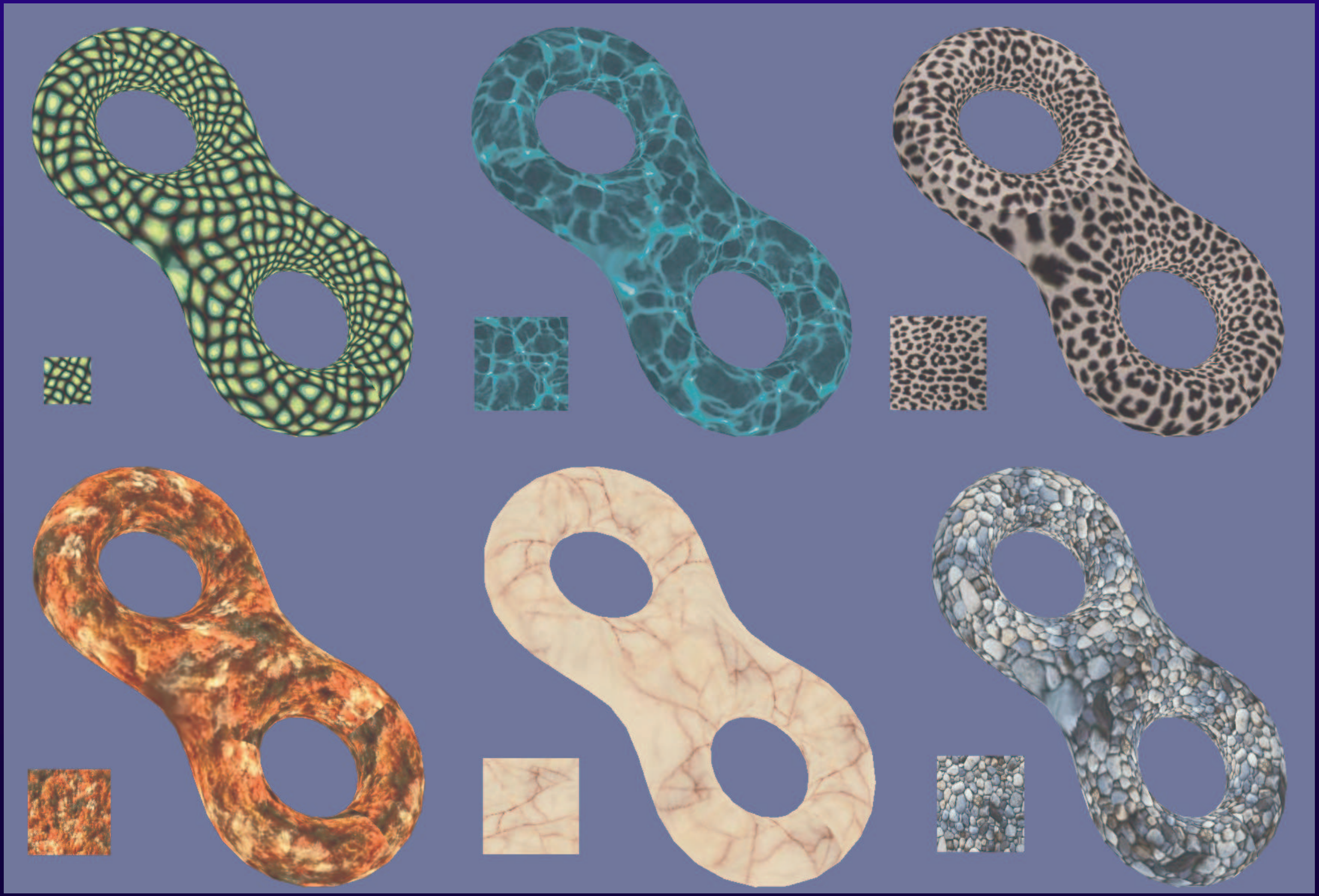


Motivation

Motivation

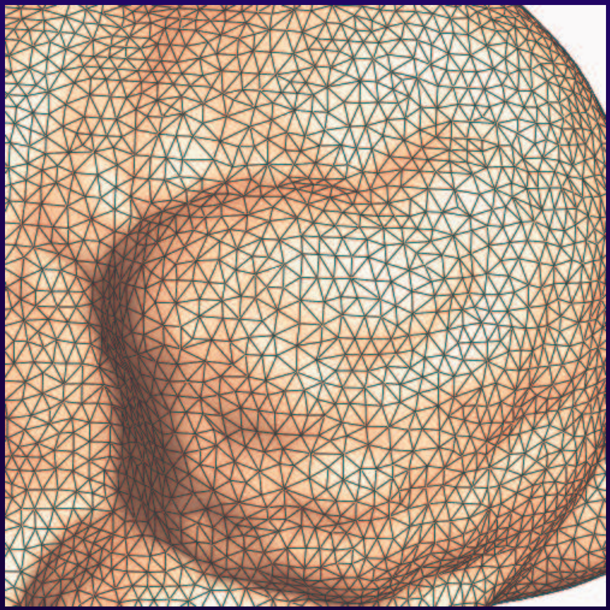
- Texture Mapping
- Geometry Remeshing
- Geometry Matching
- Medical Imaging
- Geometry Compression

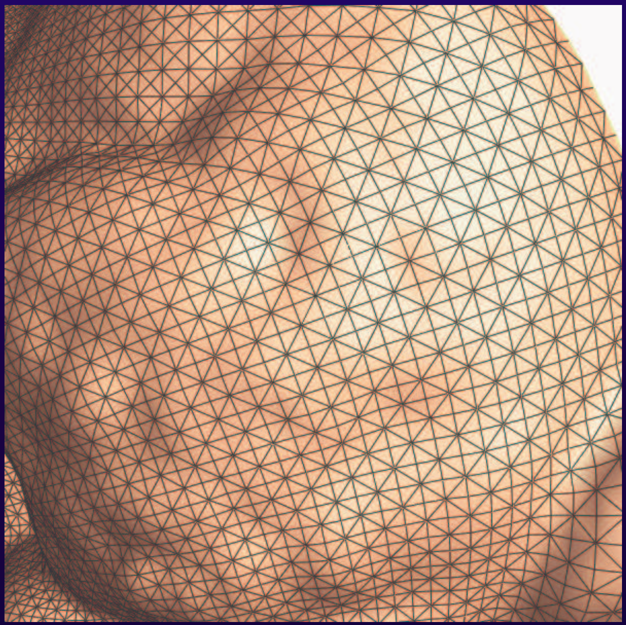
Texture Mapping – Nondistortion



Geometry Remeshing

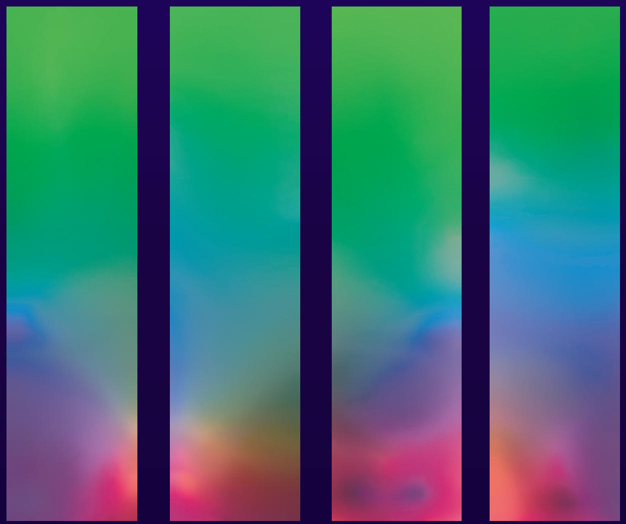
- Change irregular connectivity to regular one
- With accurate reconstructed normal

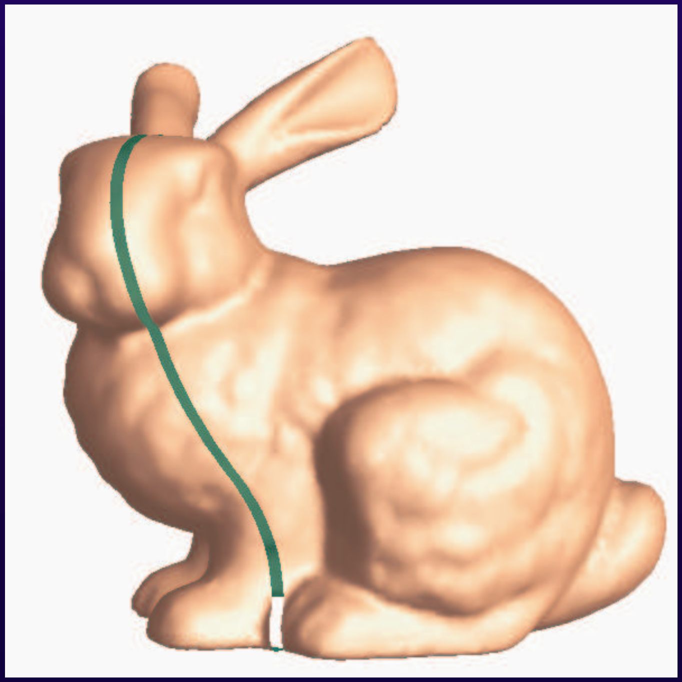




Construct conformal geometry image

- Proposed in SIGGRAPH 2002 (with Steven Gortler and Hugues Hoppe)
- Use regular grids to sample each chart
- Unify image and geometry





Geometric matching

- Parameterize surfaces to a canonical domain
- Match features by parameter
- Depends on geometry continuously

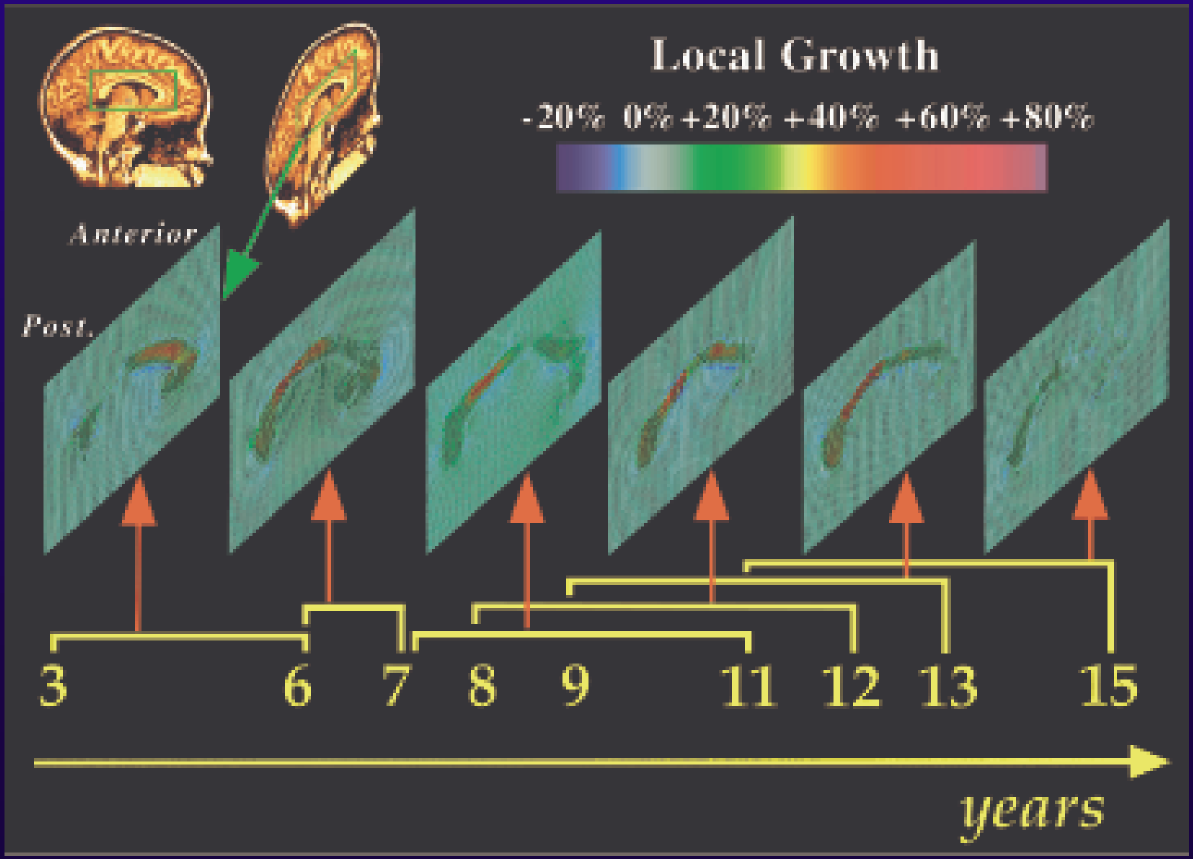
demo

Brain Conformal Mapping

- Alzheimer's Disease (AD) was as the 8th leading cause of death in U.S. for 1999; it costs U.S. society $100 billion per year[1].
- Schizophrenia affects 0.2-2% of the worldwid population; it costs $32.5 billion per year in U.S.[2]
- Brain conformal mapping can be used to the diagnosis of AD and Schizophrenia, brain development study, and surgery mapping.
- Our conformal mapping was demonstrated to be a stable solution and superior than other methods and is expected to become a standard in this file in the future.

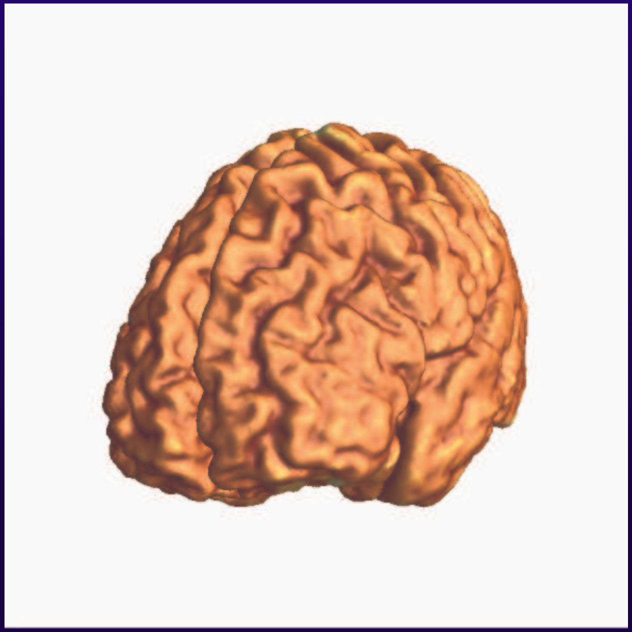
[1]Alzheimer's Disease Statistics. [2]Statistic from Brain Facts.

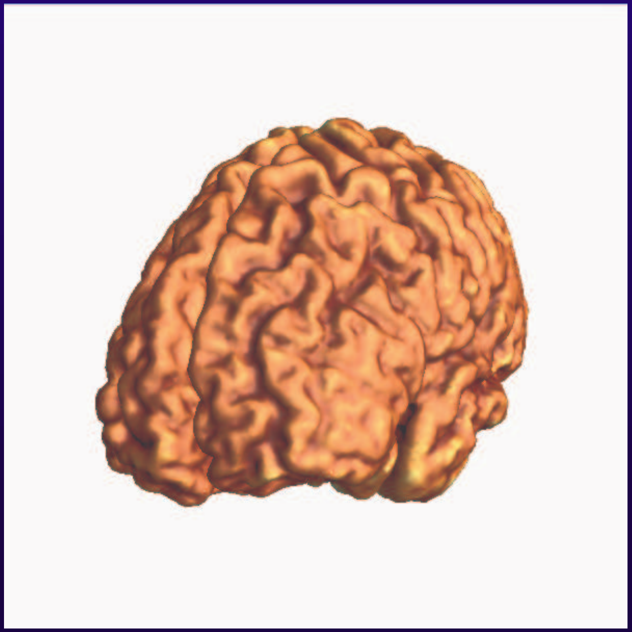
Growth patterns in the developing human brain*



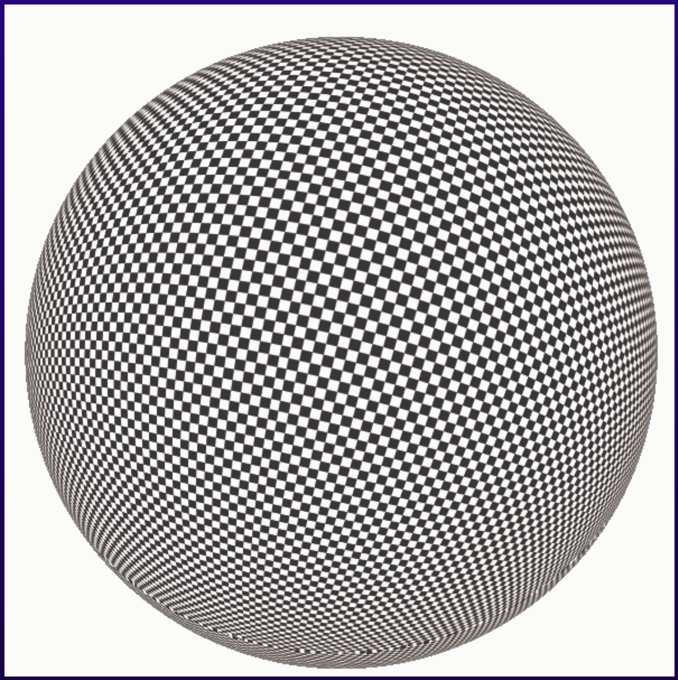
*Thompson et.al Growth patterns in the developing brain detected by using continuum mechanical tensor maps, Nature, 2000.

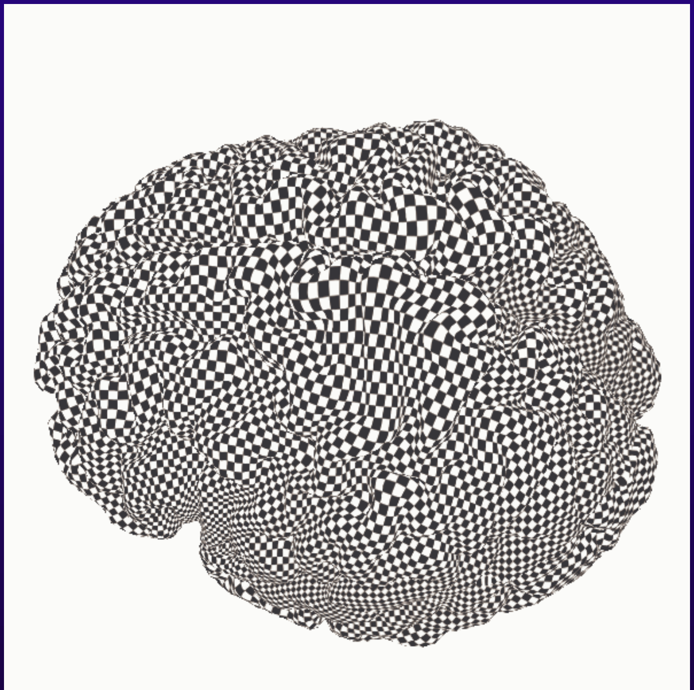
Geometry Matching-brain mapping





Conformal brain mapping





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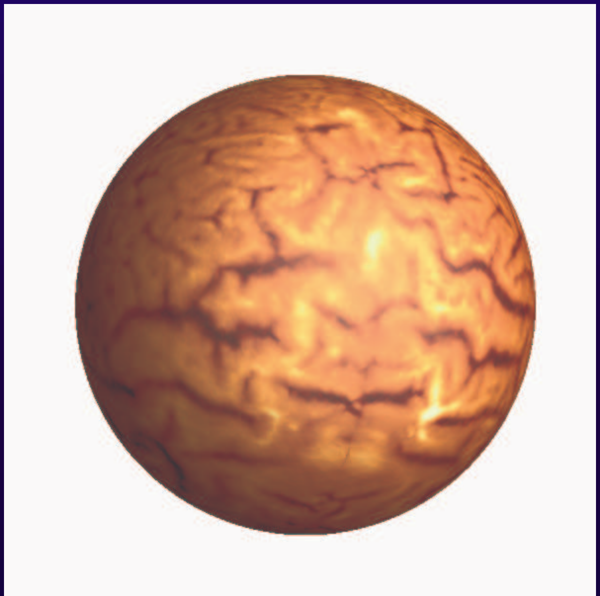
Geometric Matching-brain mapping

- Minimize L2 norm under Mobius transformation
- Least square problem



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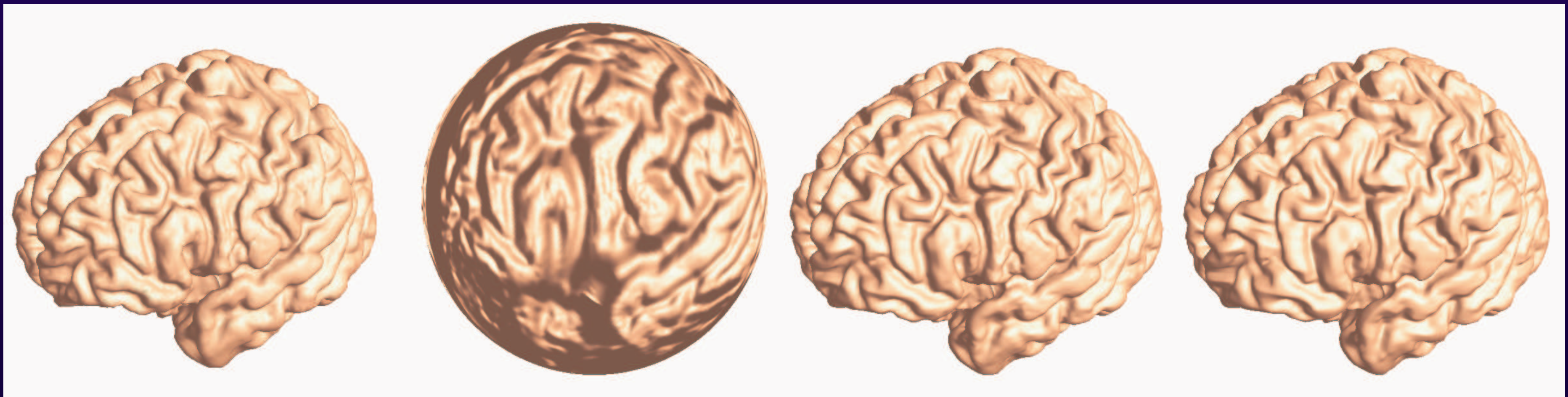
Geometry Compression

- Spherical harmonic functions
- Spectrum compression



Solution

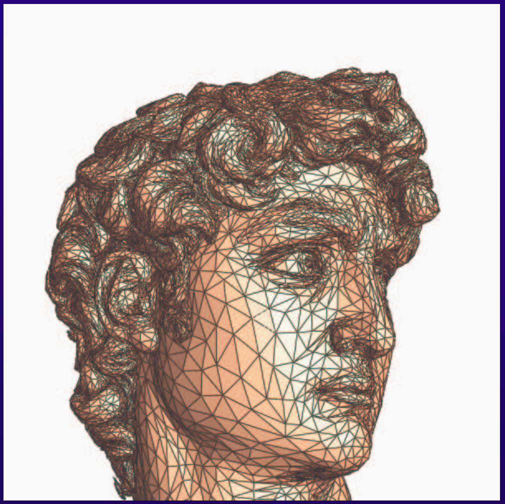
Outline

- Theoretic background
- Genus zero closed surfaces
- Nonzero genus closed surfaces
- Surfaces with boundaries
- Summary

Theoretic background – Riemann surface

- For genus zero closed surfaces, a mapping is conformal if and only if it is harmonic.
- Hodge theory: Each cohomology class has a unique harmonic one-form representative.
- Abel-Jacobi theory: The dimension of holomorphic differential group is 2 times genus.

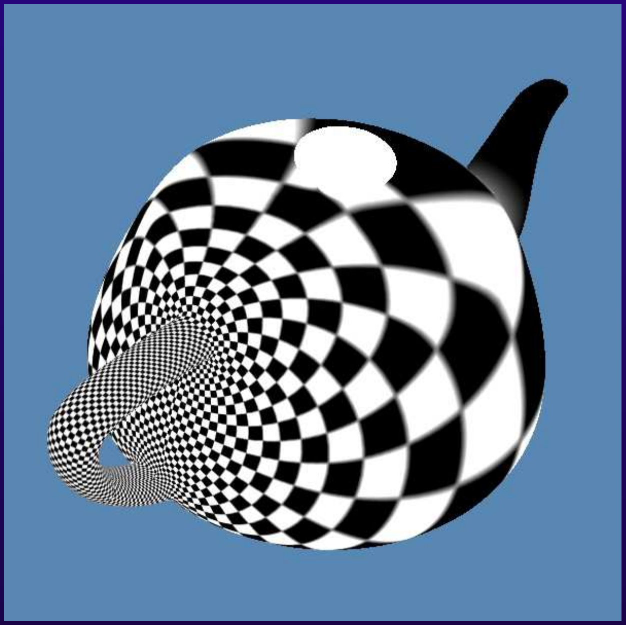
Theoretic background: Structures on mesh



- Topological structure
  - Simplicial complex
  - Homology group
- Geometric structure
  - Embedded in the Euclidean Space,induced metric
  - Curvature, geodesics
- Conformal Structure (Novel view to surfaces)
  - Riemann surfaces
  - Holomorphic differentials

Novel view

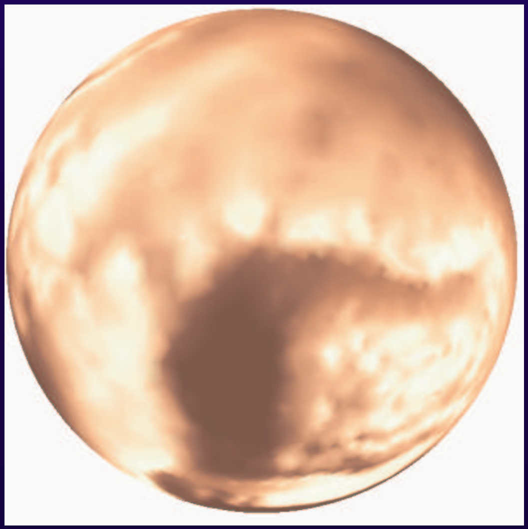
- Treat surfaces as Riemannian surfaces
- Conformal mappings
- Conformal invariants
- Conformal automorphism group



Genus zero surfaces

Genus 0 surfaces

- All conformally equivalent
- Harmonic is equivalent to conformal
- Mobius group



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Algorithm details

- Harmonic energy $f : M \rightarrow S^2$

$$E(f) = \int_M \|\nabla f\|^2 d\sigma_M$$

- Discrete harmonic energy

$$E(f) = \sum_{[u,v]\in M} k_{uv} \|f(u) - f(v)\|^2 \qquad k_{uv} = \cot\alpha + \cot\beta$$

- Discrete Laplacian

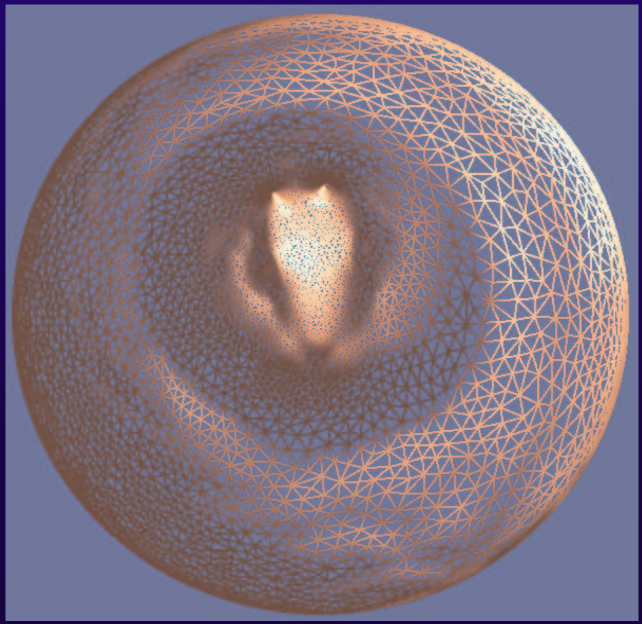
$$\Delta f(u) = \sum_{[u,v]\in M} k_{uv}(f(u) - f(v))$$

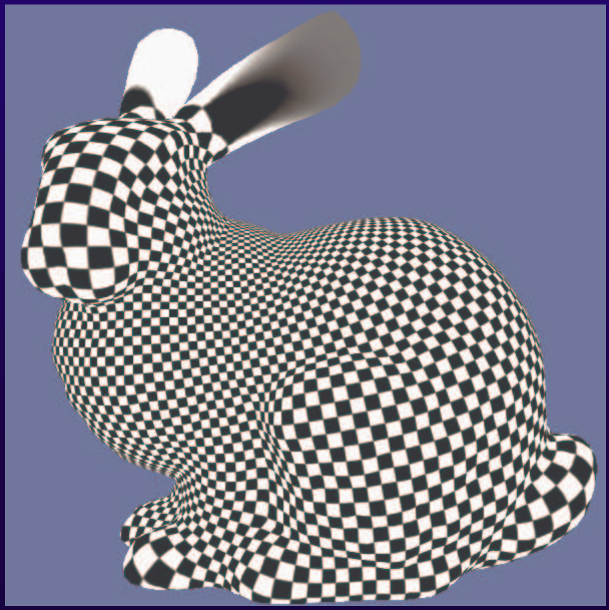
Global conformal parameterization algorithm for genus zero surface

- Use Gauss map as the initial degree one map
- Compute the gradient vector of harmonic engery on each vertex
- Project the gradient vector to the tangent space
- Update the image of each vertex along the tangential gradient vector
- Normalize the mapping by shifting the center of the mass to the sphere center

Genus zero bunny example

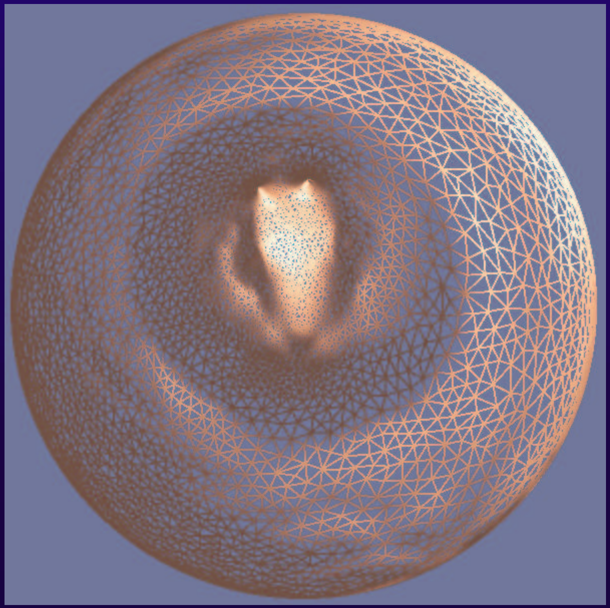
- Highly non-uniform

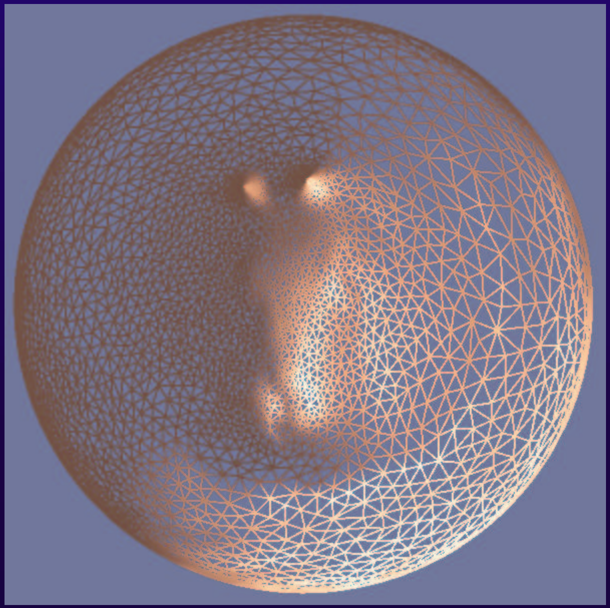




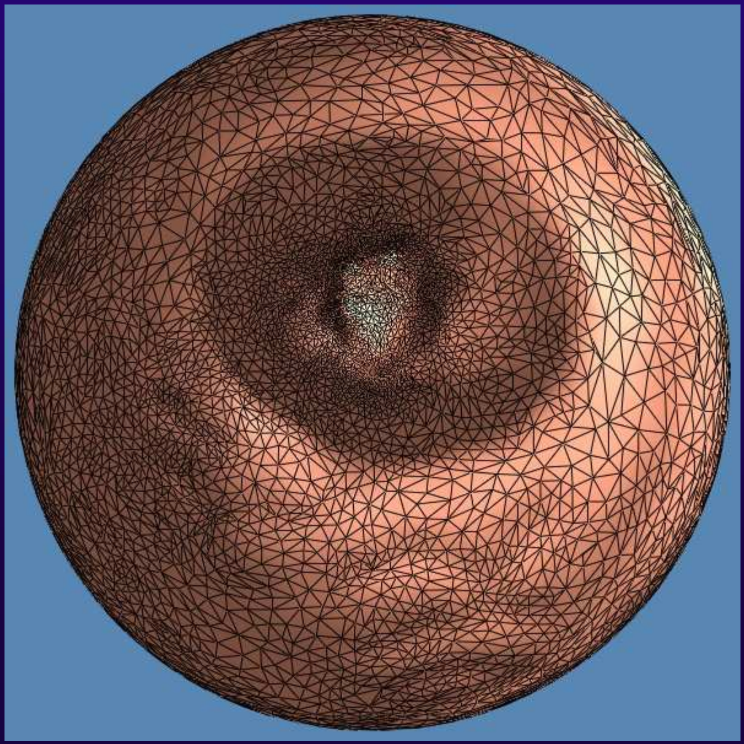
Mobius Transformation

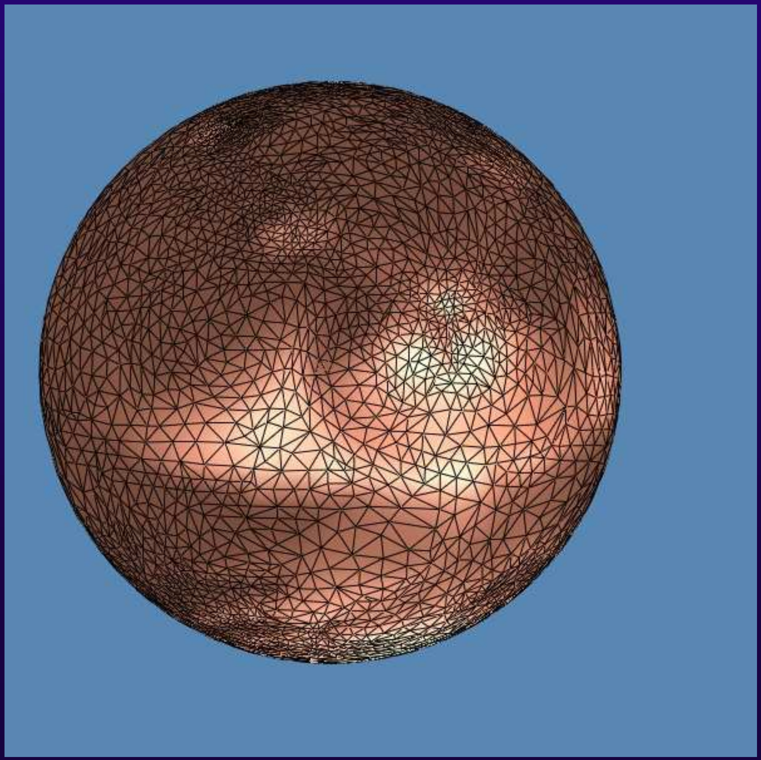
- Linear rational group on complex plane
- 6 dimensional group





Mobius Transformation





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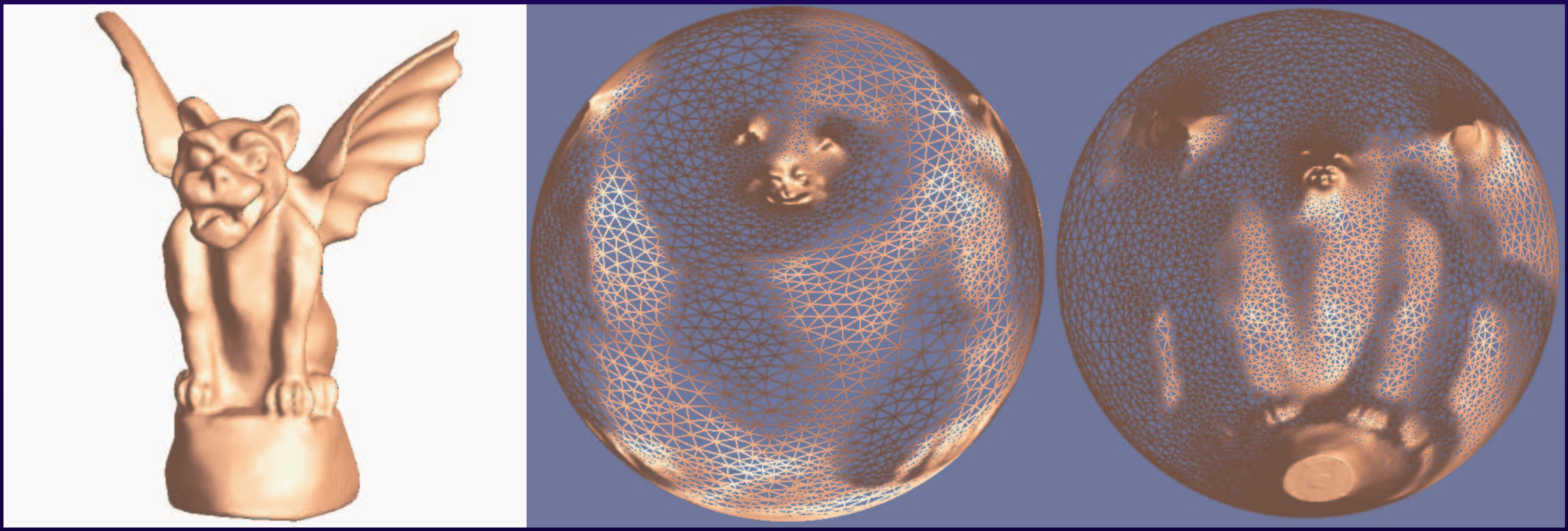
Genus zero Gargoyle example

- Spherical barycentric embedding
- Spherical conformal embedding

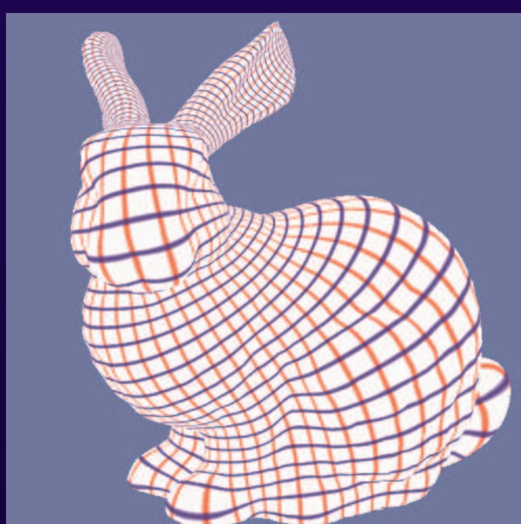


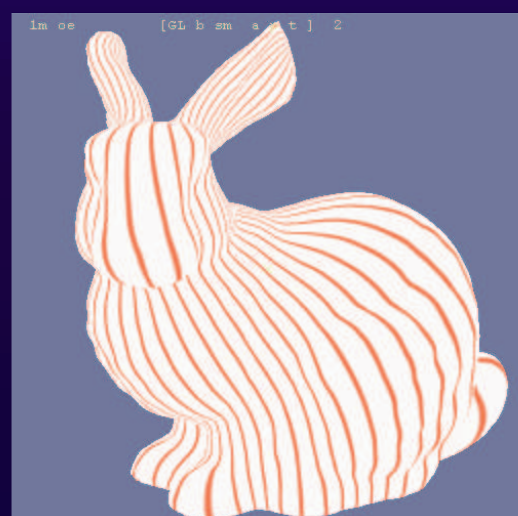
Nonzero genus surfaces

Example:Sculpture

Intuition

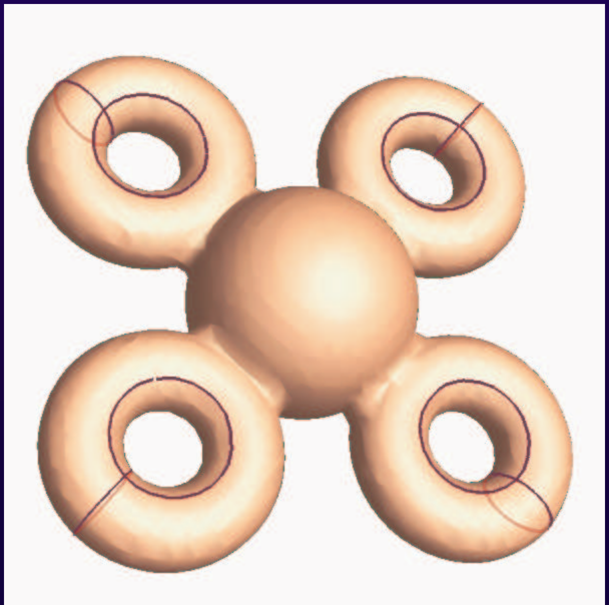
- Study the gradient fields of conformal maps
- All such gradient fields form a linear space
- The basis of such linear space is closely related to the topology of the surface





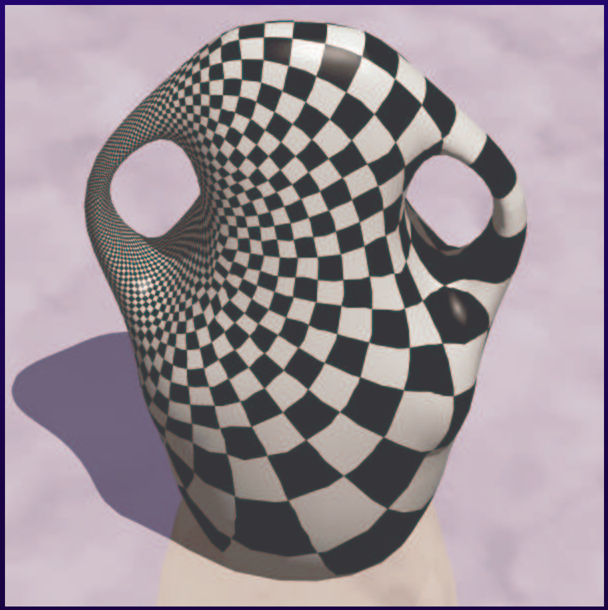
Topology: Homology group

- Curve space
- Homology Basis : A special set of curves which can be deformed to any close dcurve by merging, splitting, and duplicating operation



Conformal gradient field basis

- Dual to each handle



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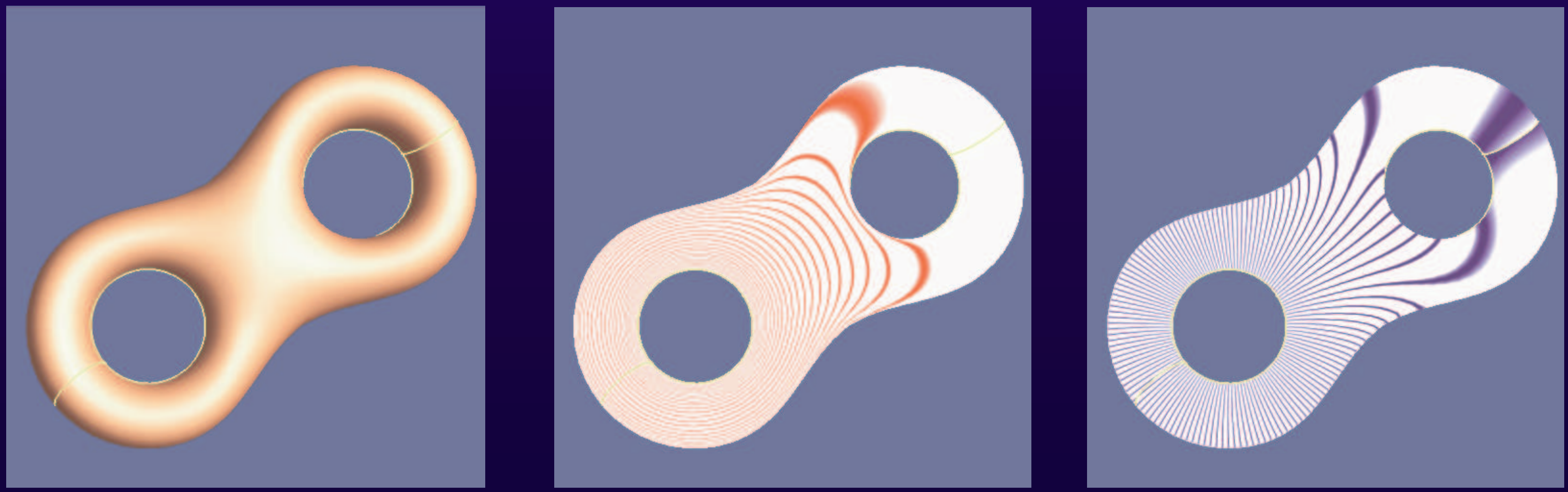
Holomorphic one-form space

- Linear functional (dual) space of homology
- Dual basis
- Conjugate 1-form



Holomorphic one-form

- A gradient field of a conformal map
- A pair of tangential vector fields (ω_x, ω_y)
- Curl is zero $curl(\omega_x) = 0, curl(\omega_y) = 0$
- Both x and y gradient fields are harmonic

$$\Delta \omega_x = 0, \Delta \omega_y = 0$$

- x,y vector fields are orthogonal at every point $\omega_y = n \times \omega_x$
- Dual to homology basis

$$\oint_{e_i} \omega_j = \delta_i^j$$

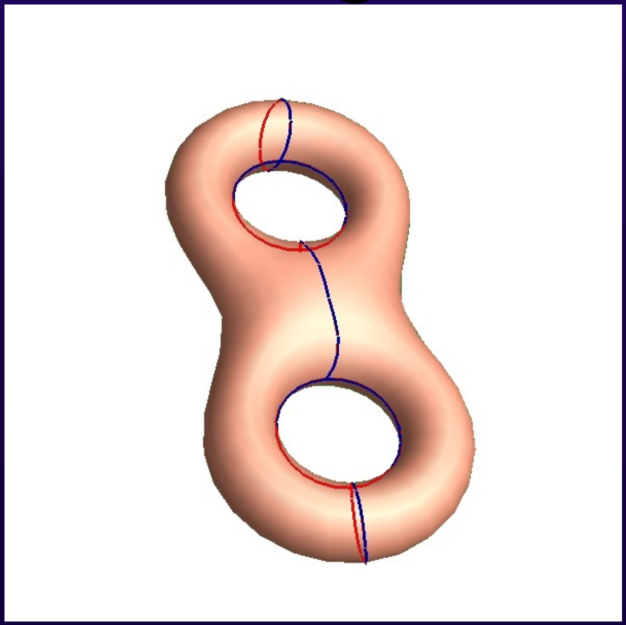
Linear system for holomorphic 1-form

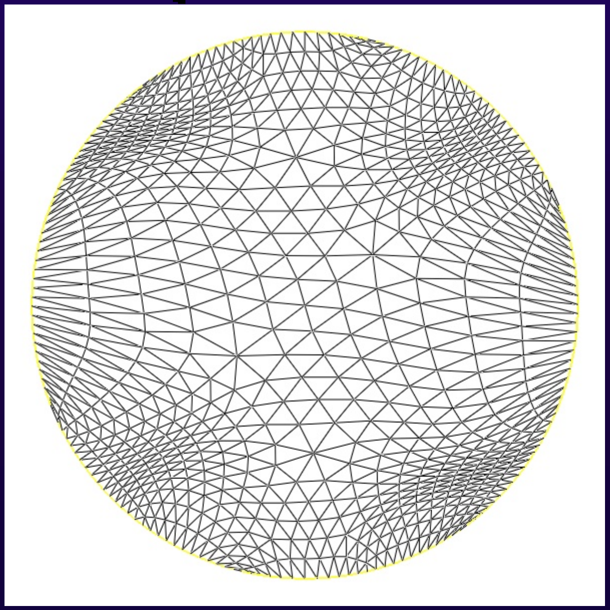
- Homology basis $\{e_1, e_2, \cdots e_{2g}\}$
- Harmonic one-form basis $\{\omega_1, \omega_2, \cdots \omega_{2g}\}$
- Holomorphic one-form basis $\{\omega_1 + i^*\omega_1, \omega_2 + i^*\omega_2, \cdots \omega_{2g} + i^*\omega_{2g}\}$

$$\begin{cases} \nabla \times \omega_i = 0 \\ \Delta \omega_i = 0 \\ \oint_{e_j} \omega_i = \delta_i^j \\ {}^*\omega_i = n \times \omega_i \end{cases}$$

Integrate on a fundamental Domain

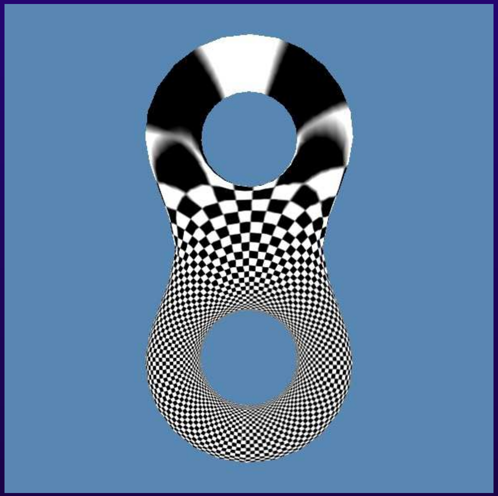
- Fix a base point, map it to the origin
- For any vertex, find an arbitrary path to the base point, integrate 1-form along the path
- The integration is path independent

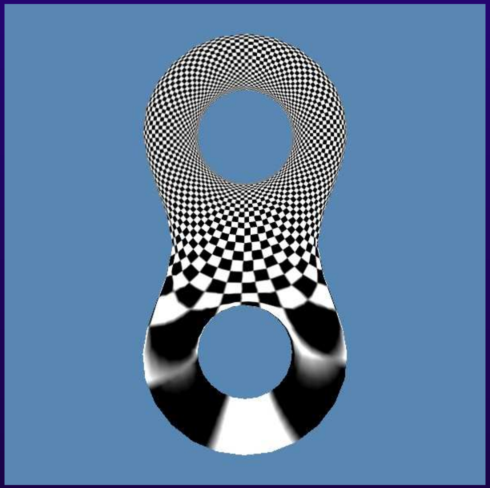




Holomorphic one-form basis

- Holomorphic 1-forms





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Holomorphic one-form space

- 2g real dimension
- Dual to homology

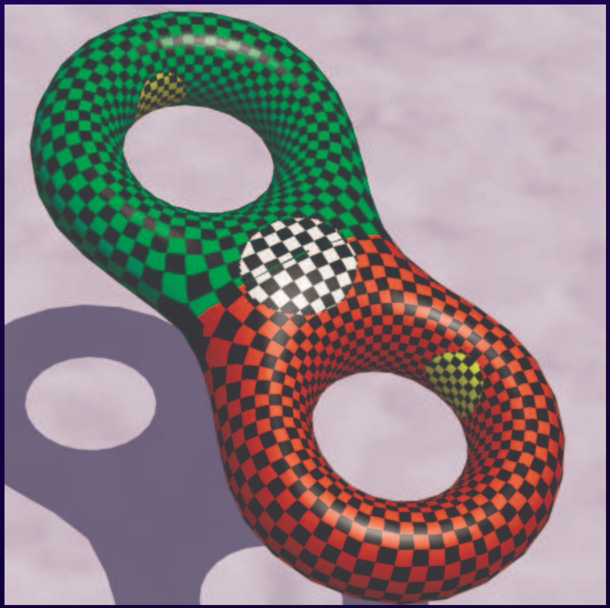
Linear combination

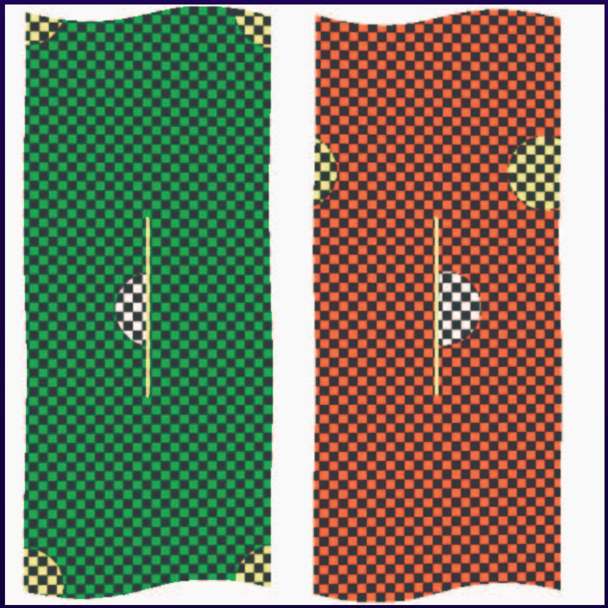
- Linearly combine holomorphic 1-form bases
- Different holomorphic one-form,different properties (conformal factor, zero points)

Conformal Atlas Structure

Global Conformal Parametrization

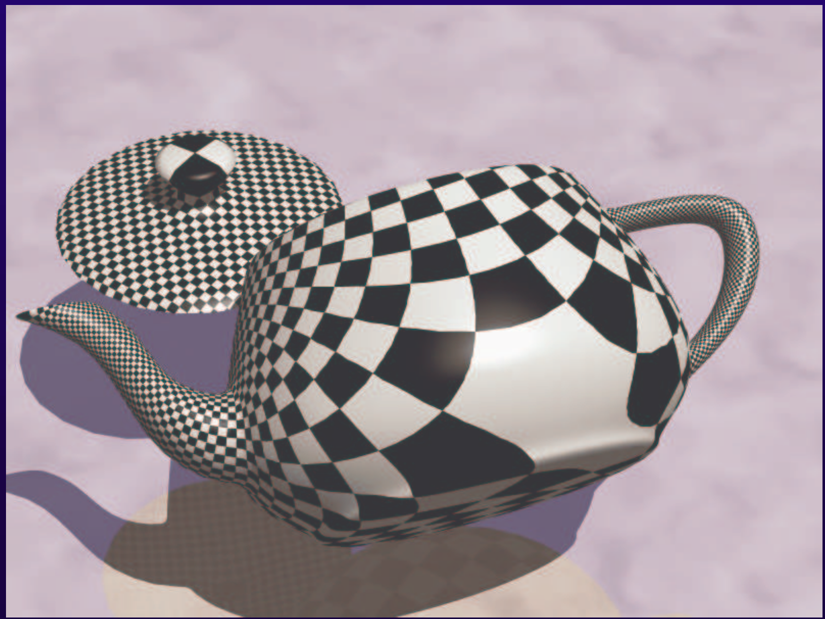
- Genus g surfaces
  - Global conformal, no boundaries
  - 2g – 2 zero points
  - each handle mapped to the plane periodically





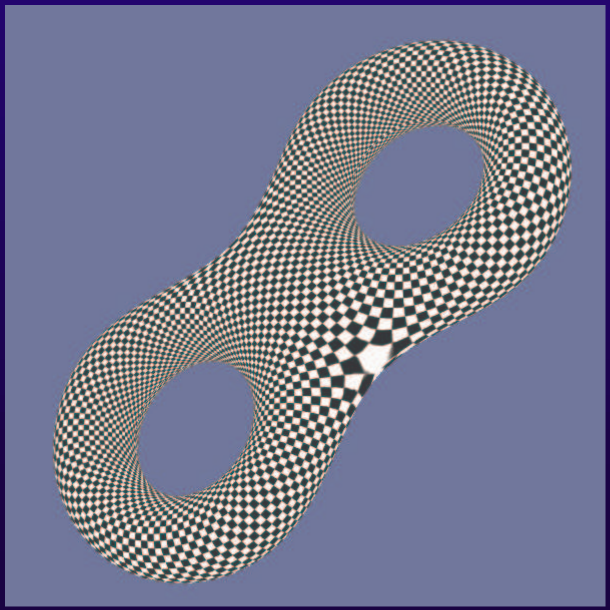
Zero points

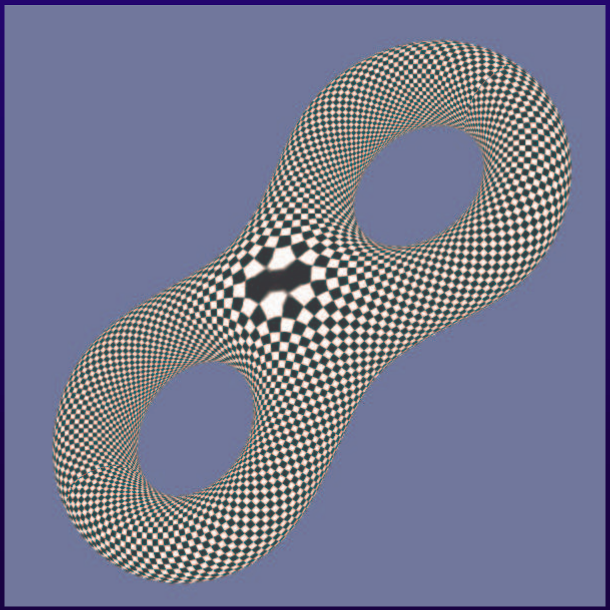
- Zero points of the tangential vector fields



Zero points

- Different holomorphic one-form, different zero points





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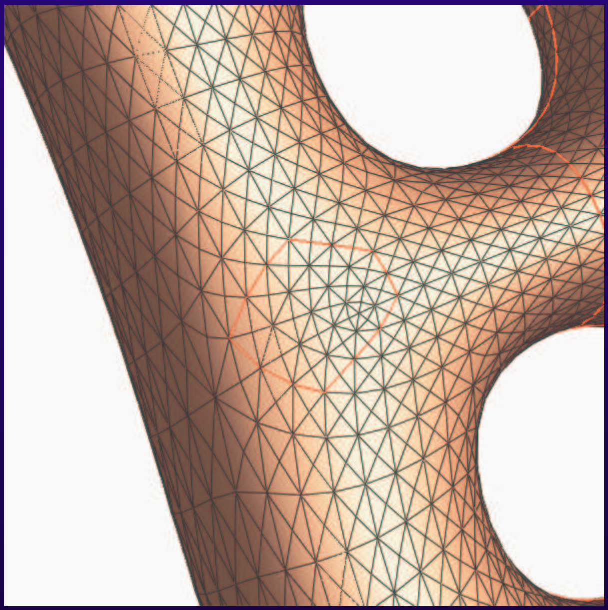
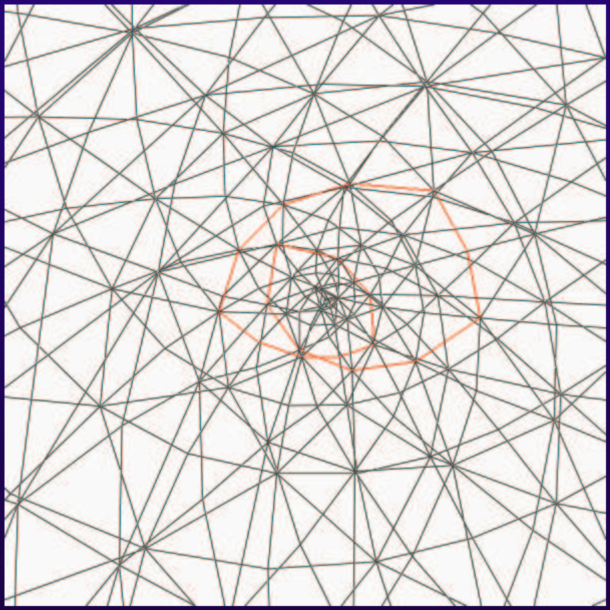
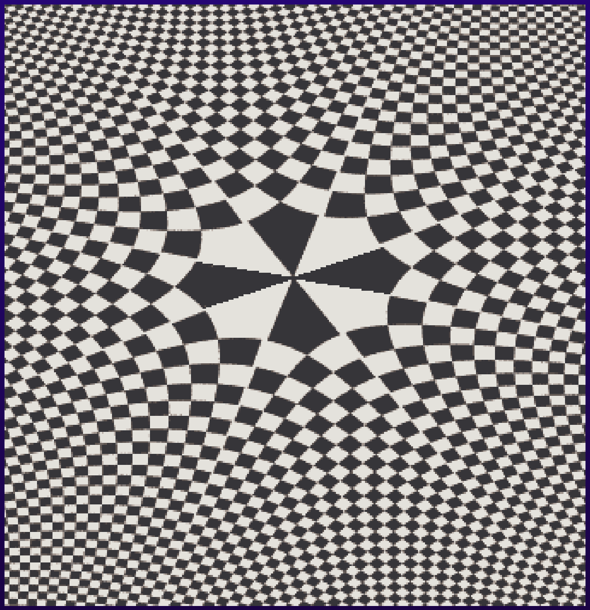
Zero Points

Locally it behaves like $\omega = z^2$

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Global conformal atlas

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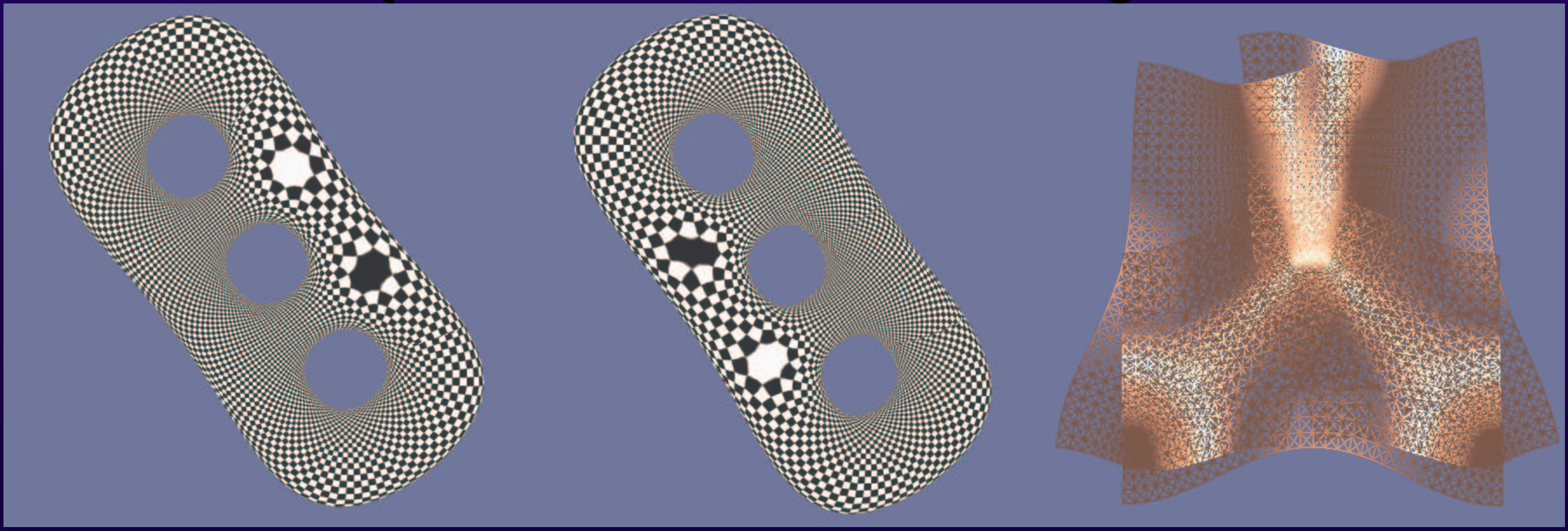
Handle Separation

- Locate zero points
- In parameter domain, find connecting curves
- Lift the planar curve to the original surface



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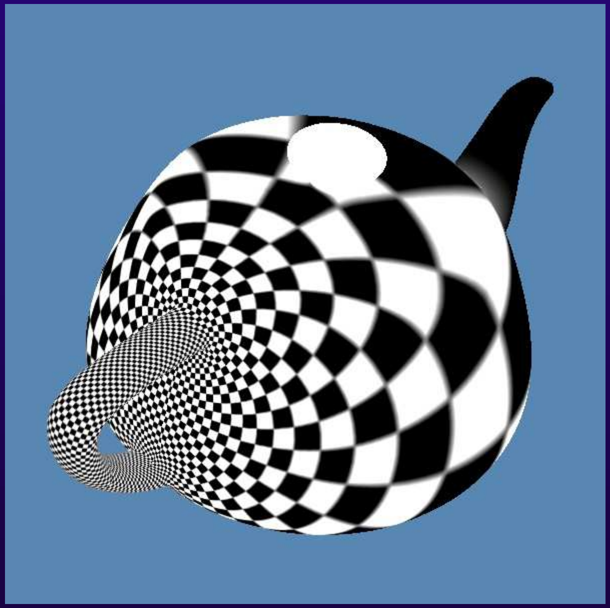
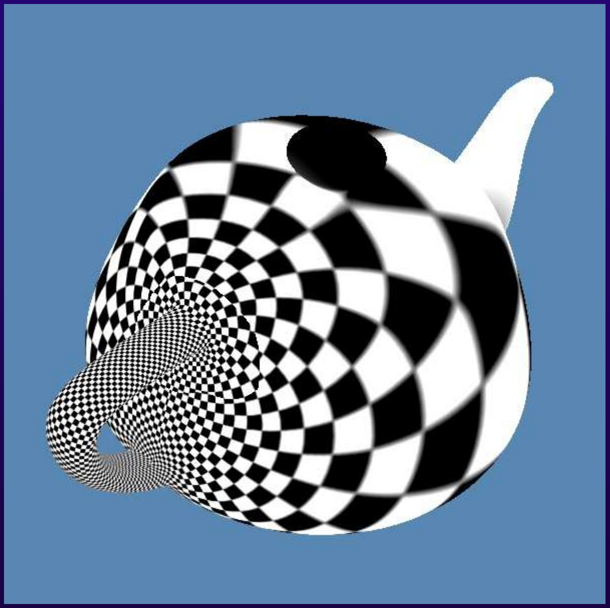
Properties: Homology Basis Independent

Properties: Homology Basis Independent

- Homology basis independent





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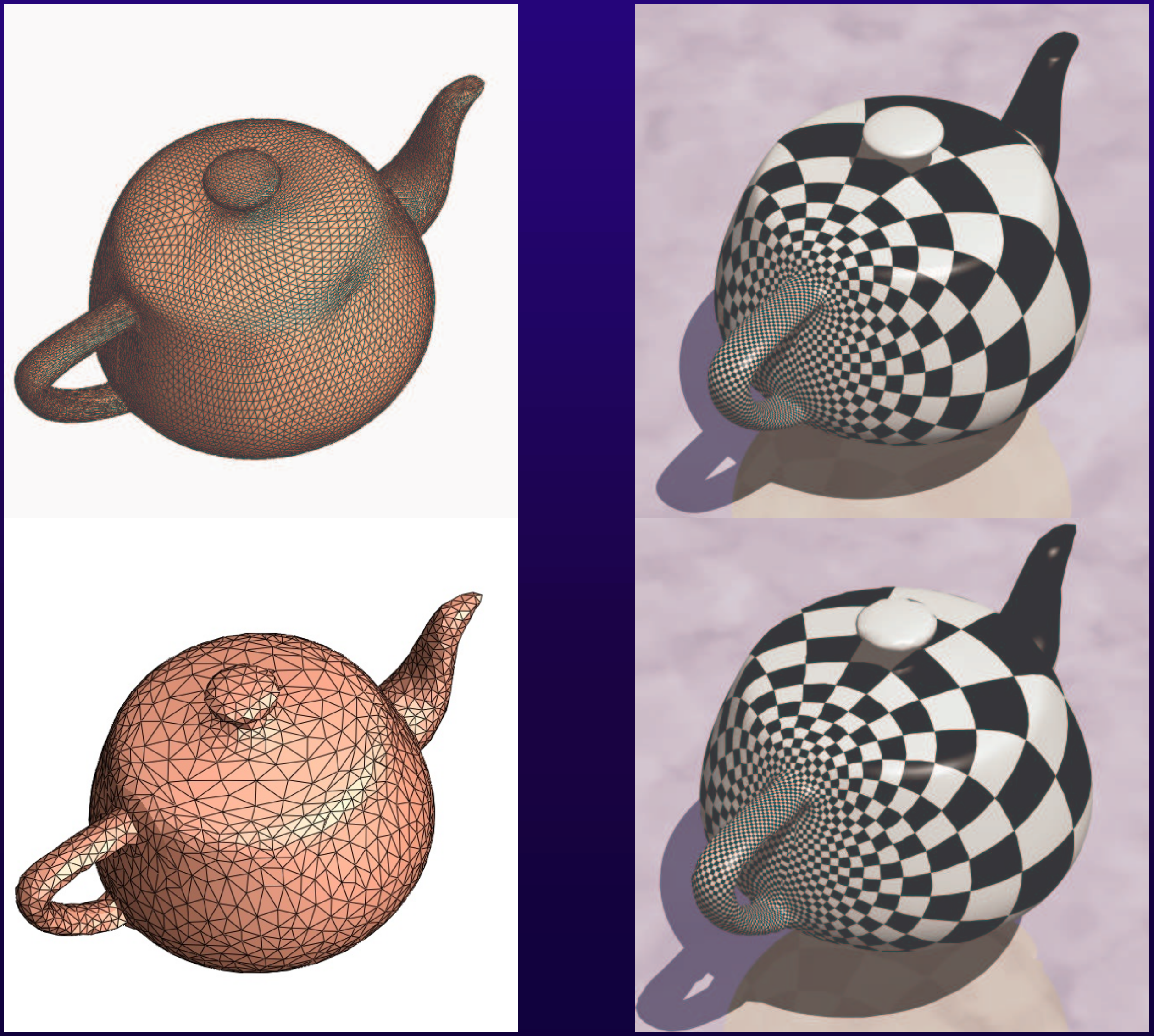
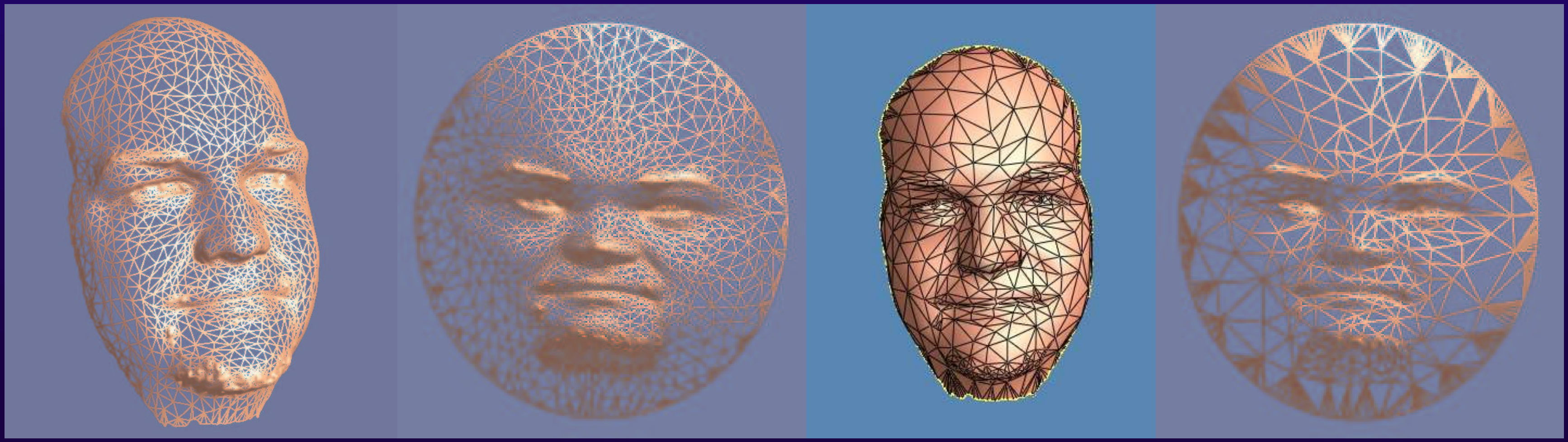
Properties:Triangulation & Resolution Independent



Conformal mapping properties

- Intrinsic to geometry
- Depends on metric continuously



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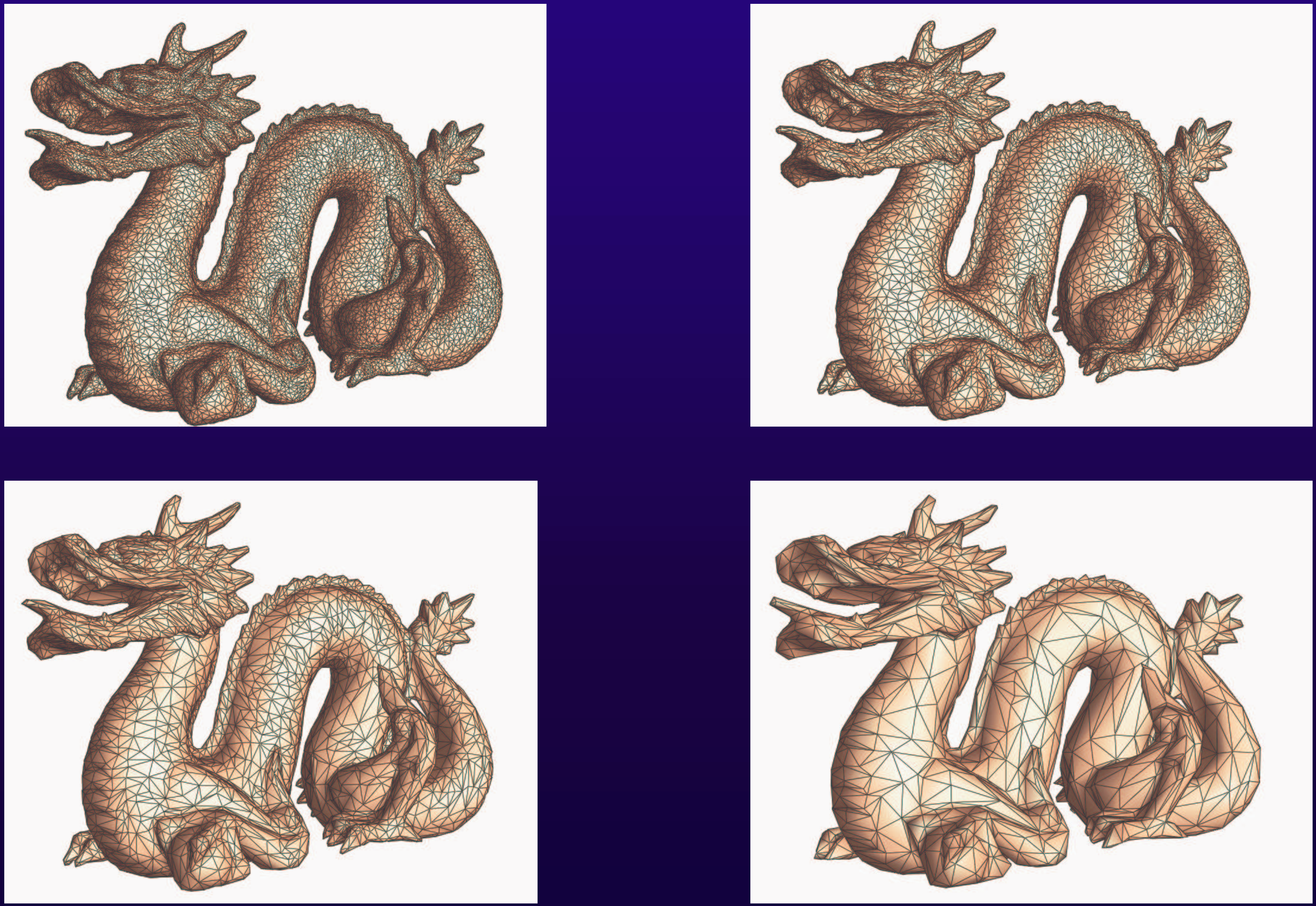
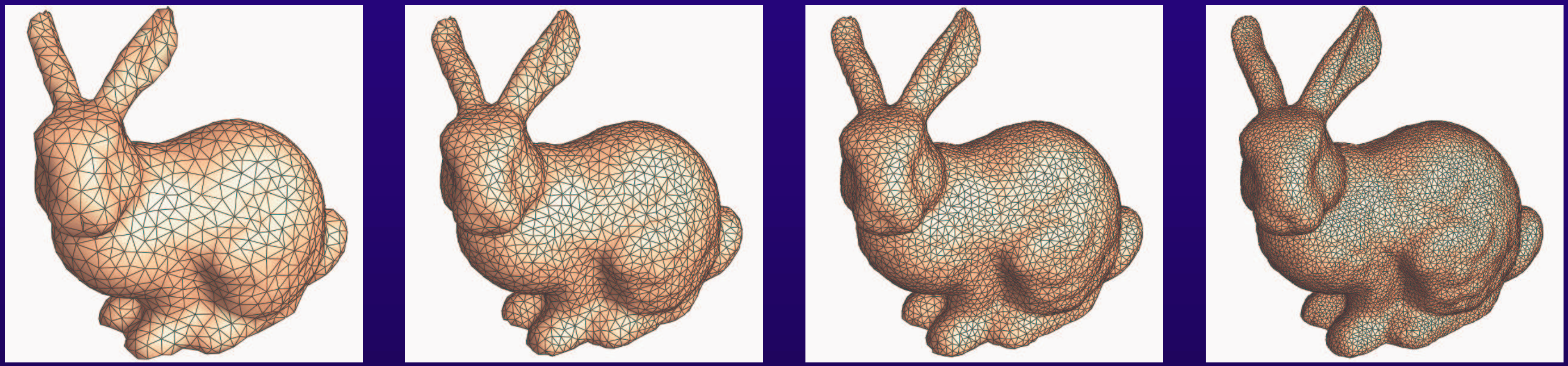
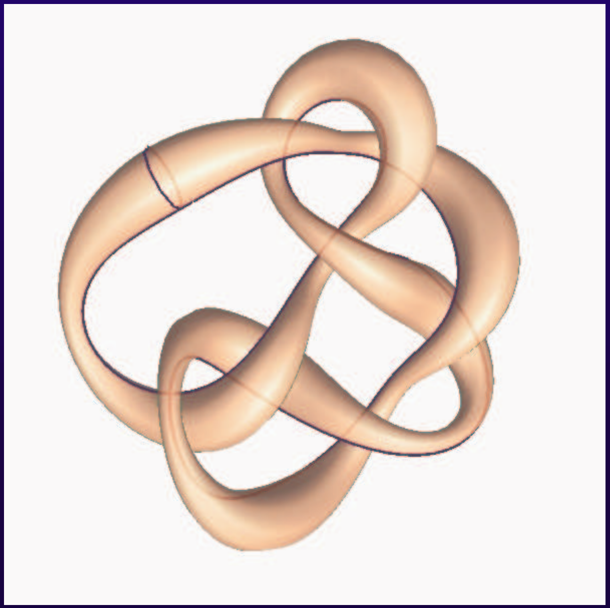
Progressive Mesh



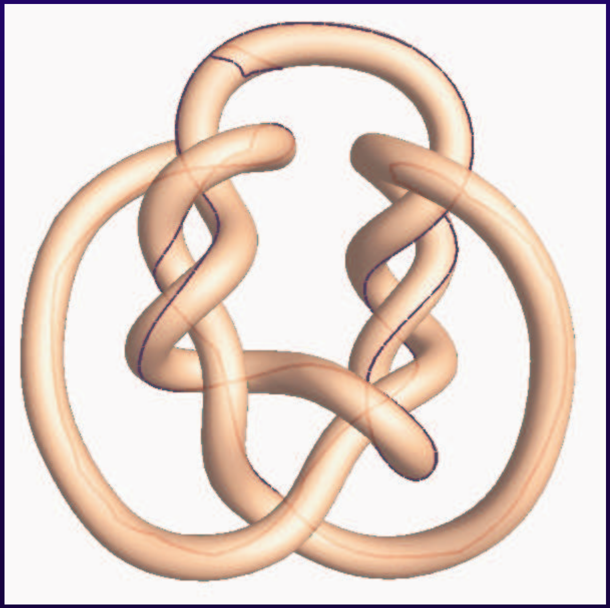
Properties:Triangulation & Resolution Independent



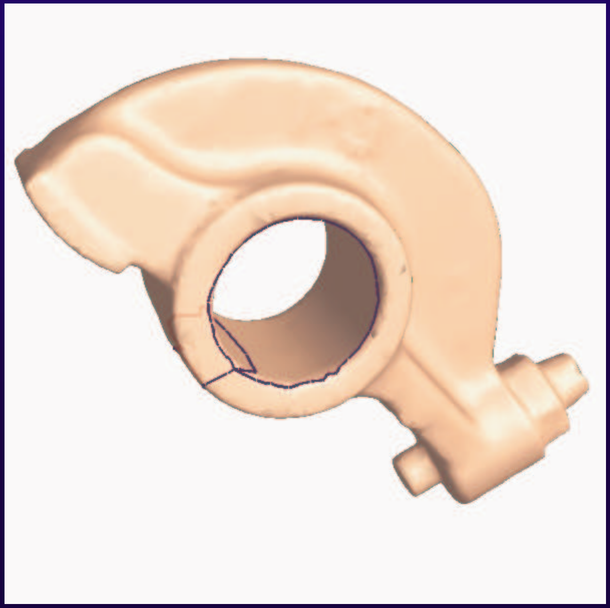
Results: Knot

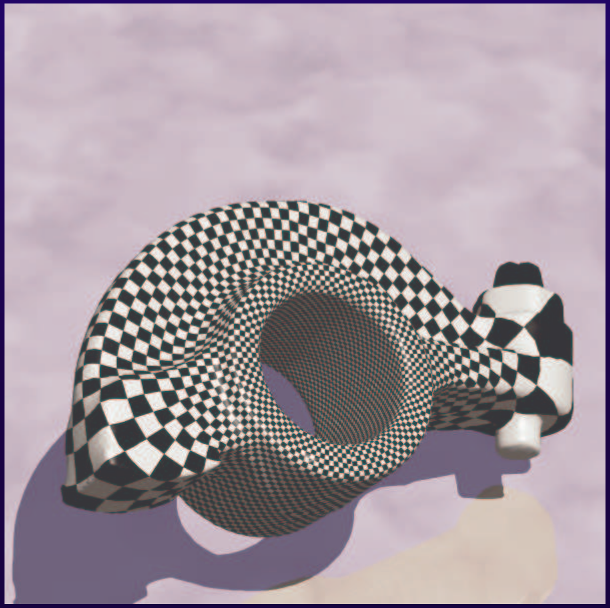


Example:knot

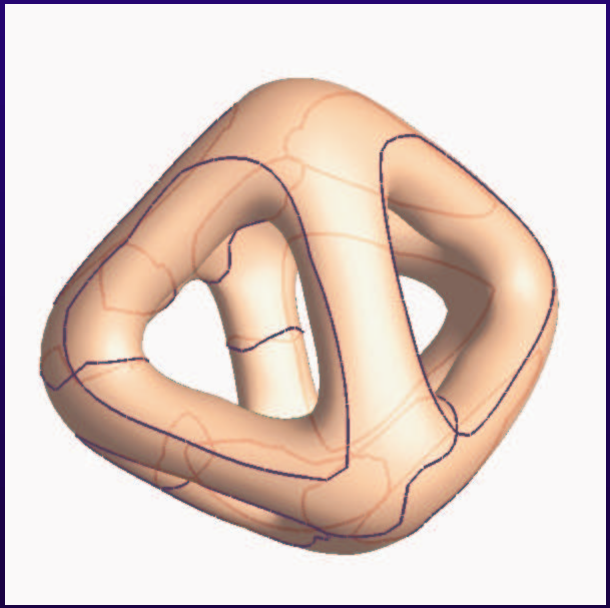


Example:Rocker





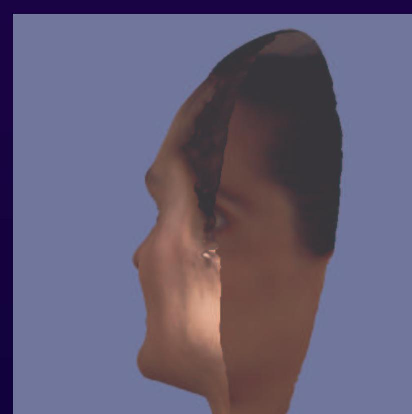
Example:Teri Surface



Surfaces with boundaries

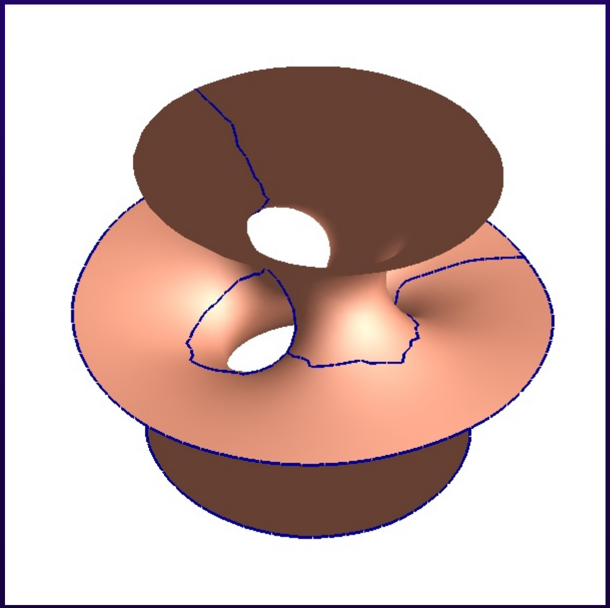
Surfaces with boundaries

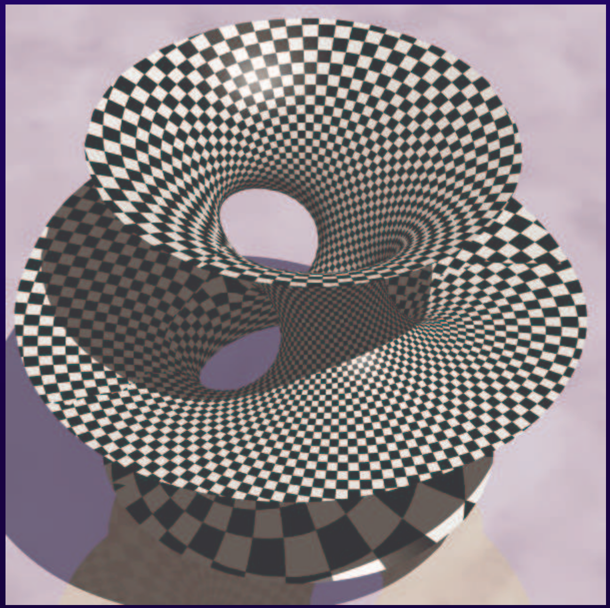
- Copy the surface, invert the orientation
- Glue two copies together along the boundaries
- Treat the double covering as a closed surface
- Keep symmetry


Example:Minimal Surface

- Genus one, 3 boundaries
- Genus four

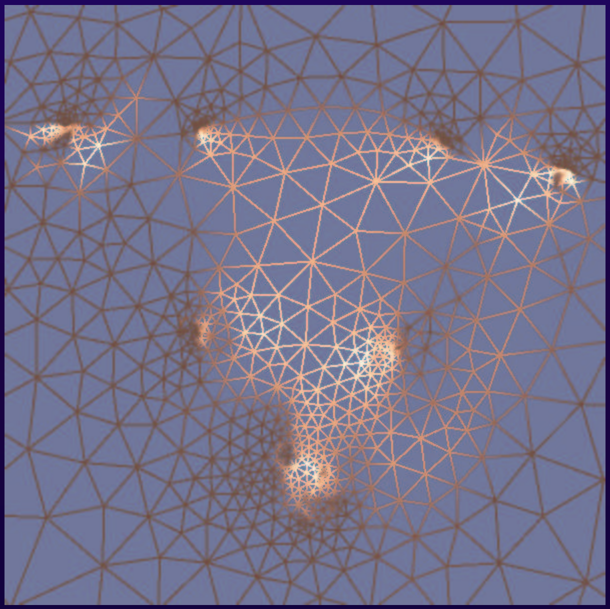




Topology modification

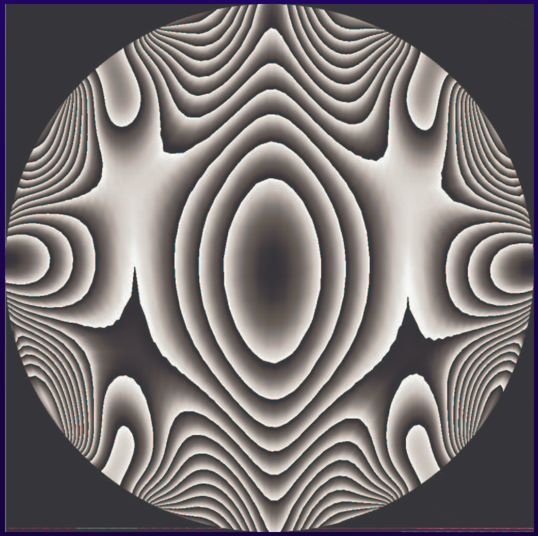
Non-uniformity

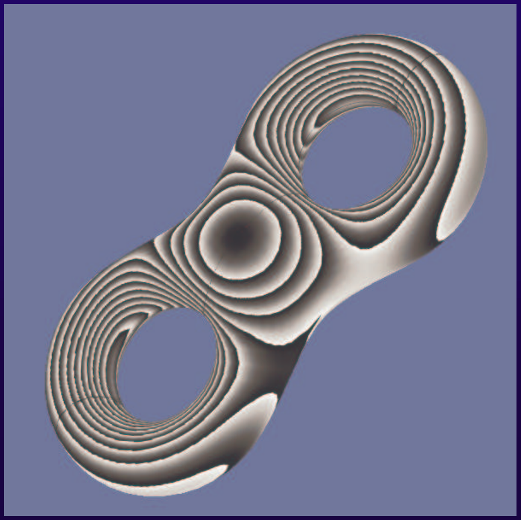
- Extruding parts are denser
- Non-uniform

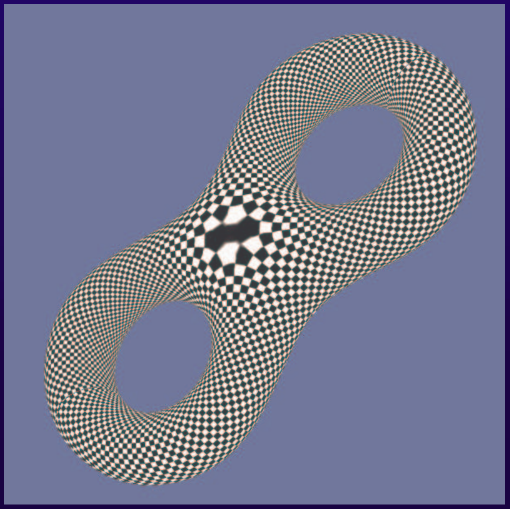


Conformal Factor

- Level Sets of conformal factor function





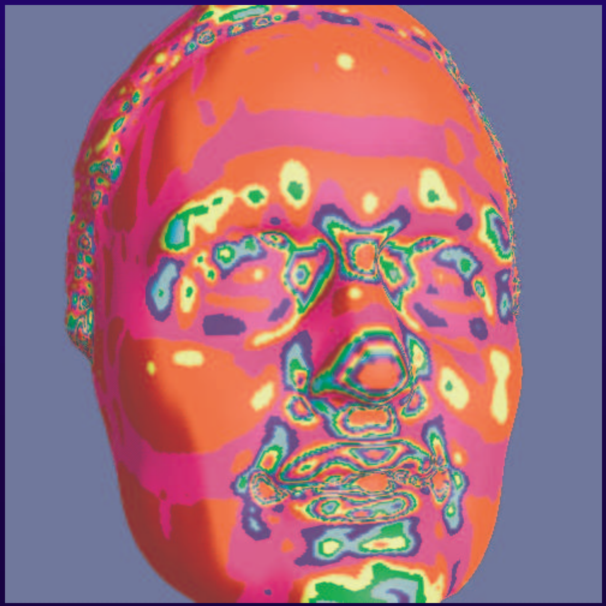


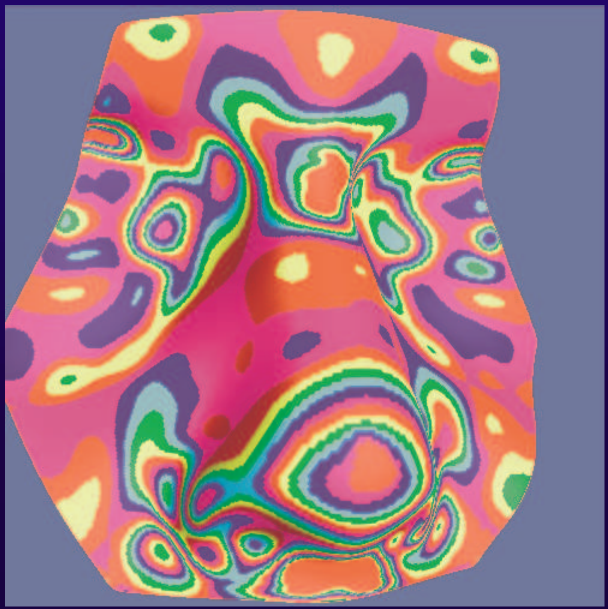
Conformal factor

- Denser for high curvature regions

Geometric Matching

- Level set of Gaussian curvature
- Gradient of gaussian curvature





Facial Recognition using Conformal Invariants: Period Matrix Computation

$$\sqrt{-1}\begin{pmatrix} 1.436033 & 0.258267 & 0.274593 & 0.472174 \\ 0.258267 & 1.072796 & 0.251679 & 0.241556 \\ 0.274593 & 0.251680 & 1.352364 & 0.437323 \\ 0.472173 & 0.241556 & 0.437323 & 1.678392 \end{pmatrix}$$

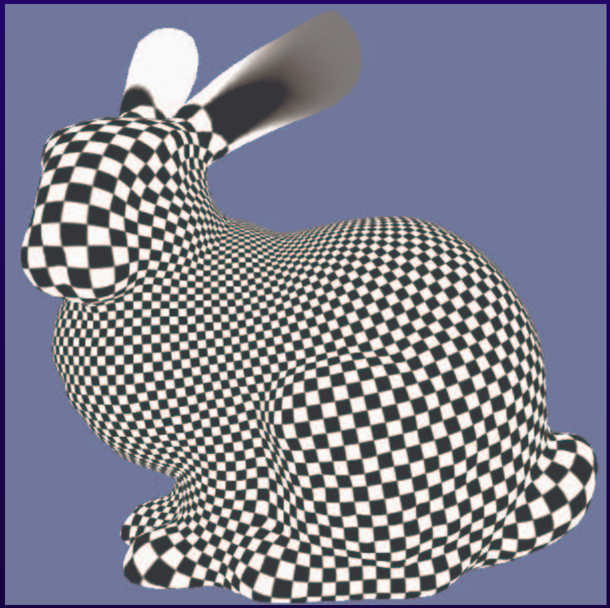
$$D = \begin{pmatrix} 2.4504 & 0 & 0 & 0 \\ 0 & 0.8916 & 0 & 0 \\ 0 & 0 & 1.1195 & 0 \\ 0 & 0 & 0 & 1.0782 \end{pmatrix}, V = \begin{pmatrix} -0.5092 & -0.3363 & 0.7491 & -0.2578 \\ -0.2960 & 0.8204 & -0.0013 & -0.4892 \\ -0.4599 & -0.4237 & -0.6509 & -0.4305 \\ -0.6646 & 0.1854 & -0.1230 & 0.7134 \end{pmatrix}$$

$$\sqrt{-1}\begin{pmatrix} 0.890499 & 0.229891 & 0.247912 & 0.429846 \\ 0.229891 & 0.699143 & 0.229518 & 0.269387 \\ 0.247913 & 0.229518 & 0.876034 & 0.446700 \\ 0.429846 & 0.269388 & 0.446700 & 1.433510 \end{pmatrix}$$

$$D = \begin{pmatrix} 2.0471 & 0 & 0 & 0 \\ 0 & 0.5158 & 0 & 0 \\ 0 & 0 & 0.6335 & 0 \\ 0 & 0 & 0 & 0.7027 \end{pmatrix}, V = \begin{pmatrix} 0.4256 & 0.3765 & -0.6500 & -0.5046 \\ 0.2937 & -0.7958 & 0.1298 & -0.5134 \\ 0.4300 & 0.4464 & 0.7409 & -0.2587 \\ 0.7401 & -0.1600 & -0.1082 & 0.6442 \end{pmatrix}$$

Topology Modification

- Punch small holes at the end of the extruding region

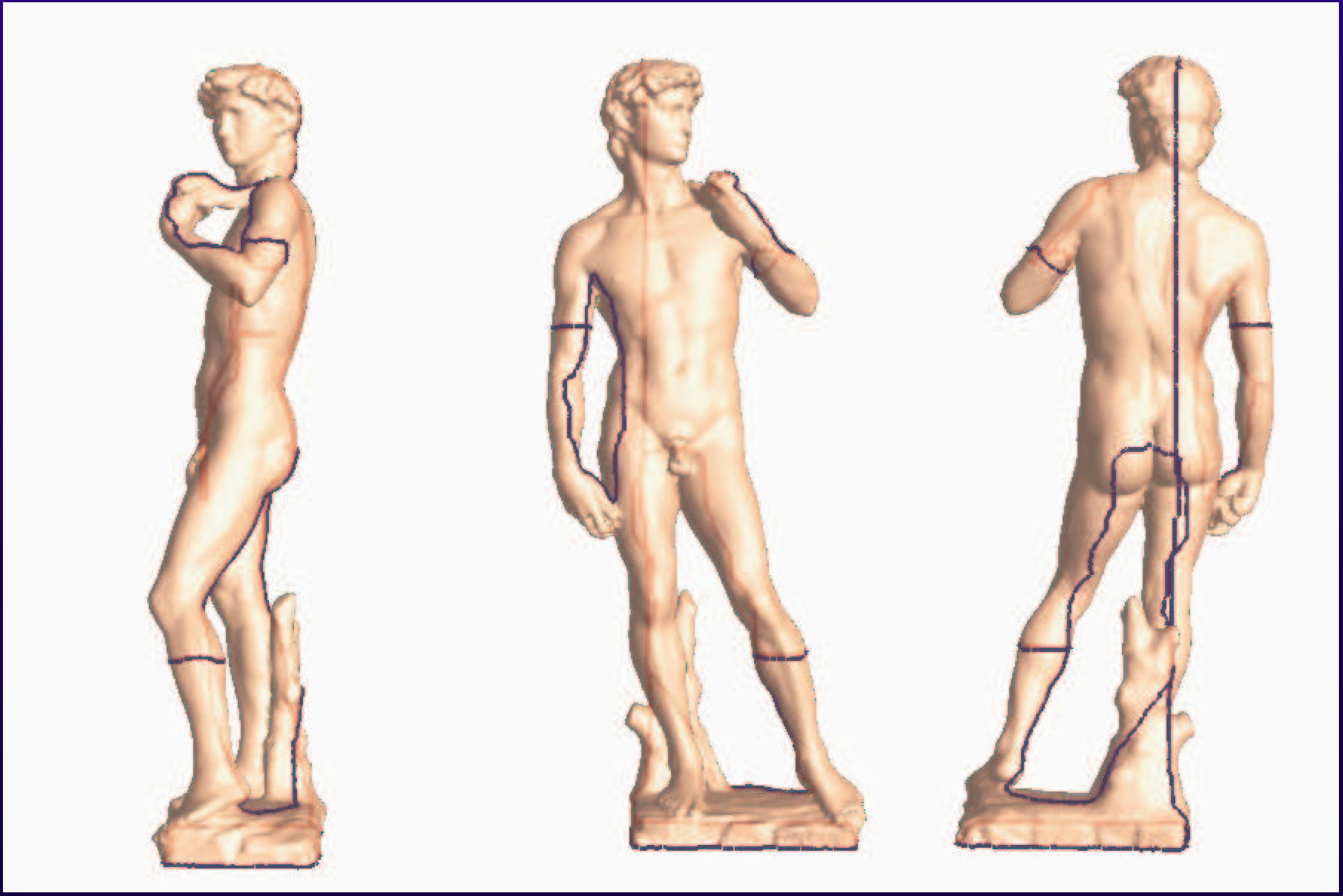


demo

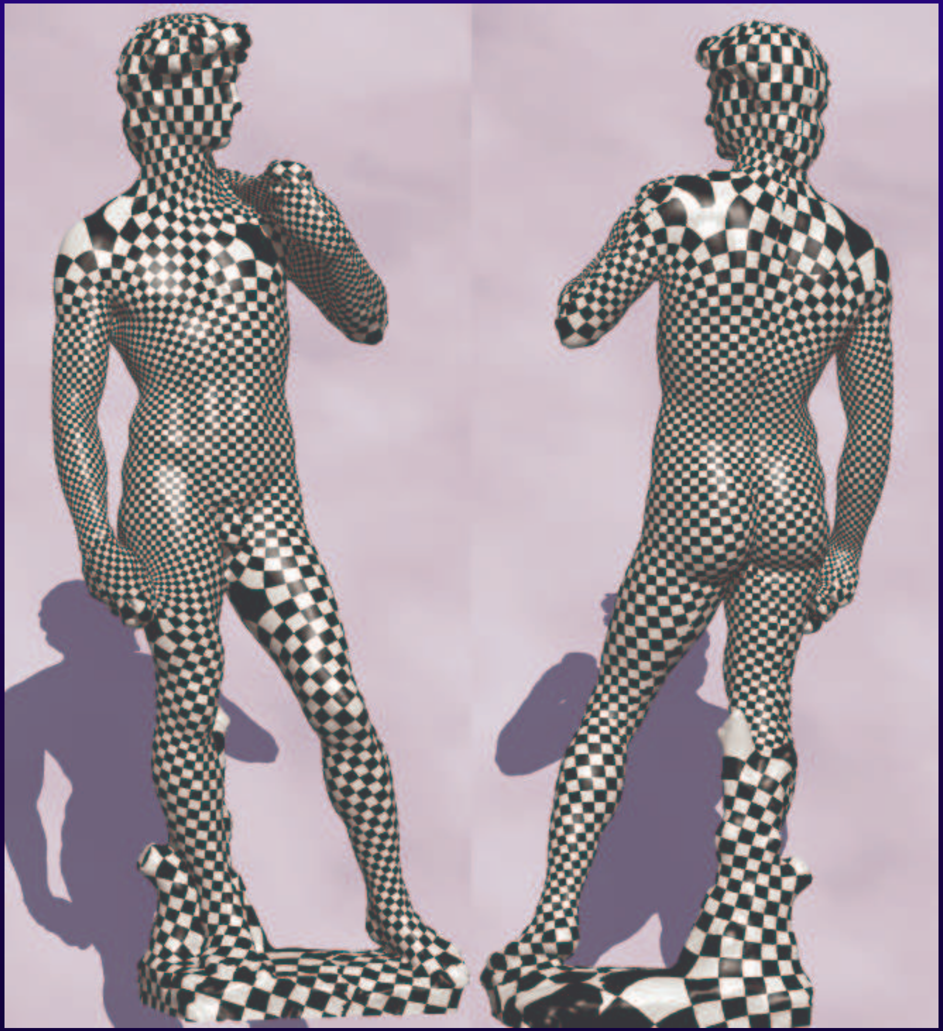
Hand Model

David

demo



David

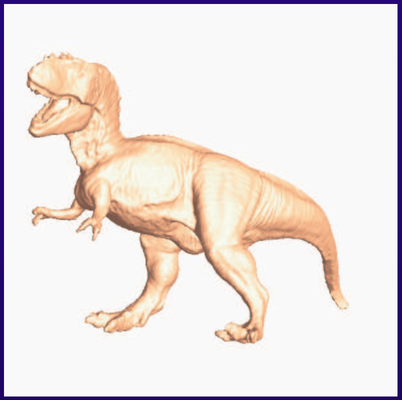


demo

Applications

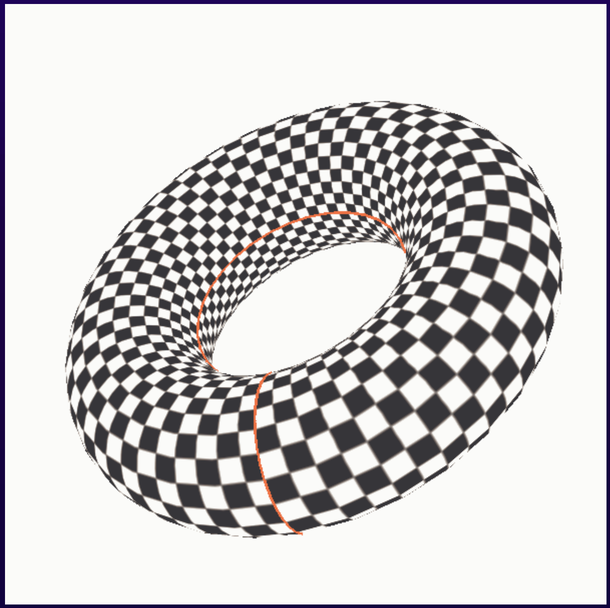
Geometry classification

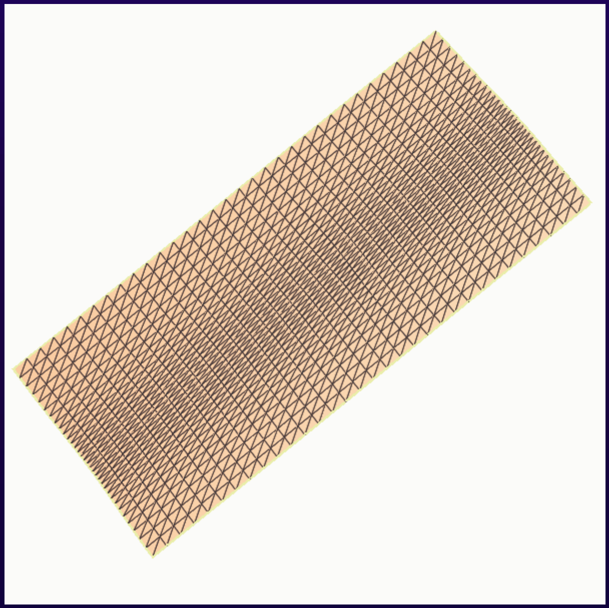





Conformal invariants - Periods

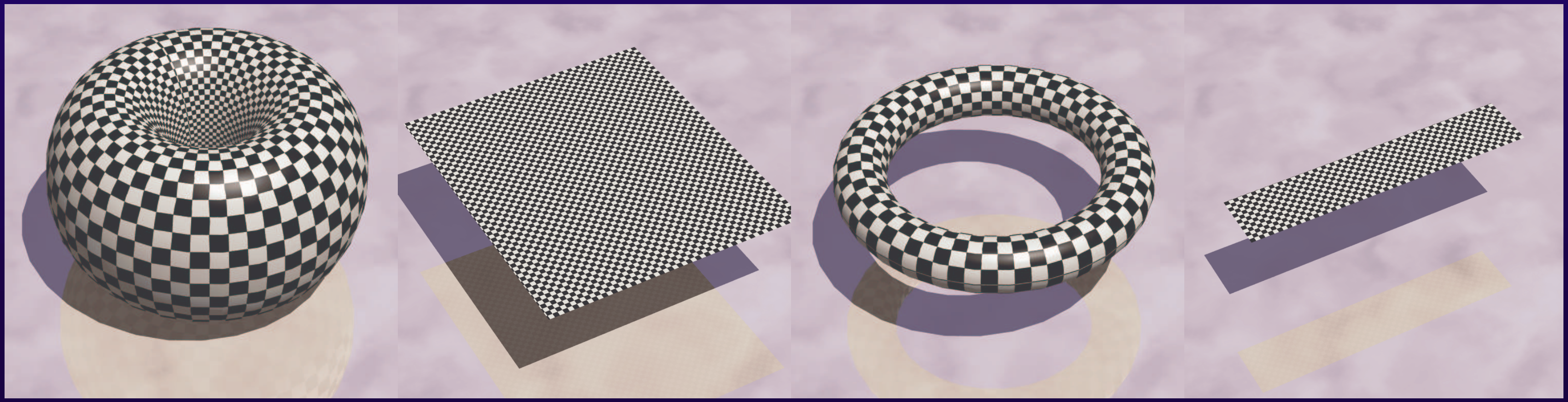
- Torus is conformally mapped to a parallelogram
- The shape factors of the parallelogram are conformal invariants



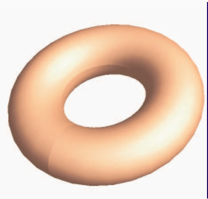
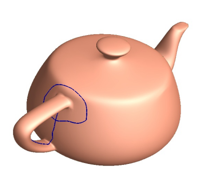
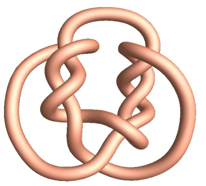
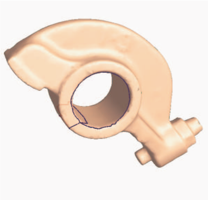


Conformal invariants - periods

- Topologically equivalent
- Not Conformally equivalent

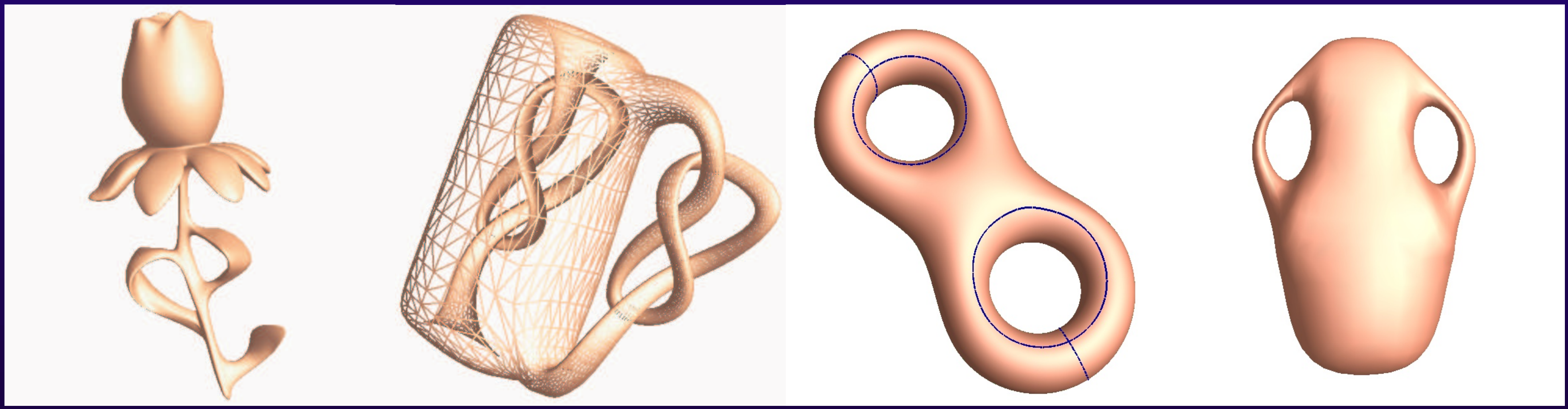


Conformal Invariants - periods

Snap shot	mesh	angle	Length ratio	Vertice s	faces
	Torus	89.987	2.2916	1089	2048
	Teapot	89.95	3.0264	17024	34048
	knot	85.1	31.150	5808	11616
	rocker	85.432	4.9928	3750	7500

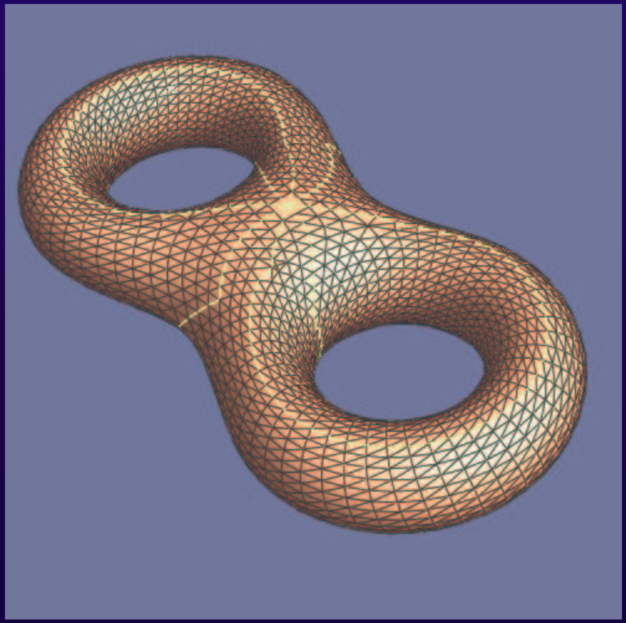
Conformal invariants – period matrix

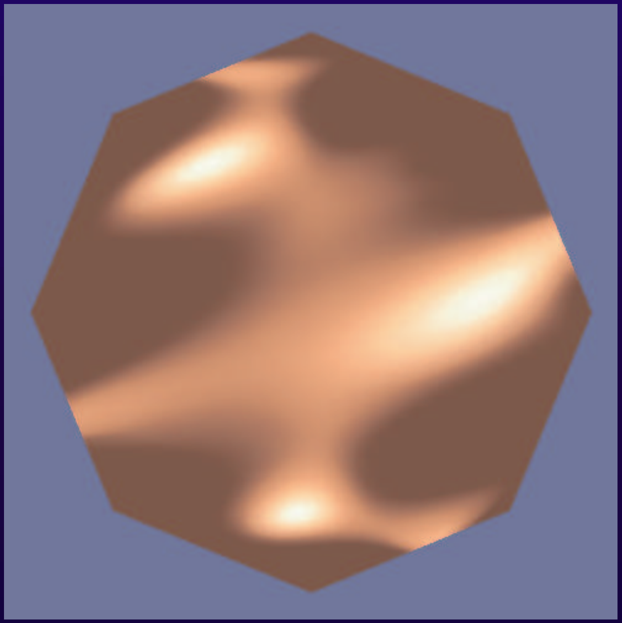
- High genus case – period matrix



Embedding in Hyperbolic Space

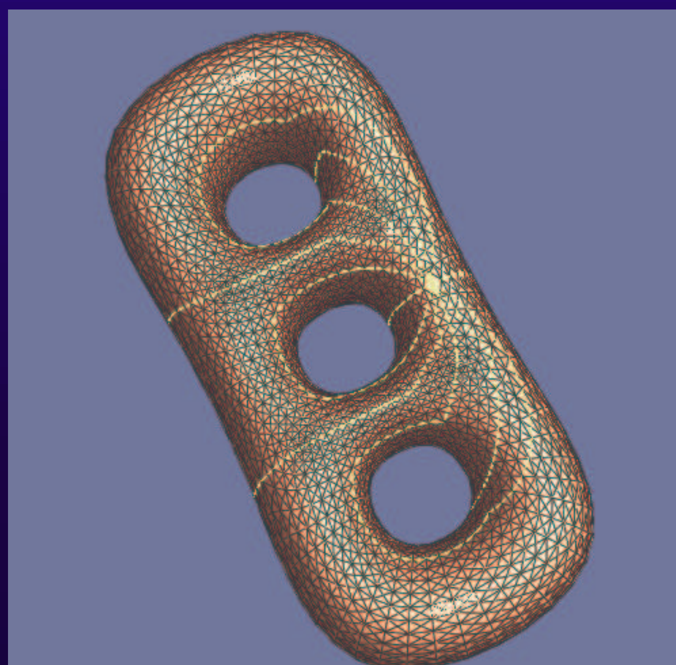
- Cut along a set of canonical homology basis
- Get a fundamental domain



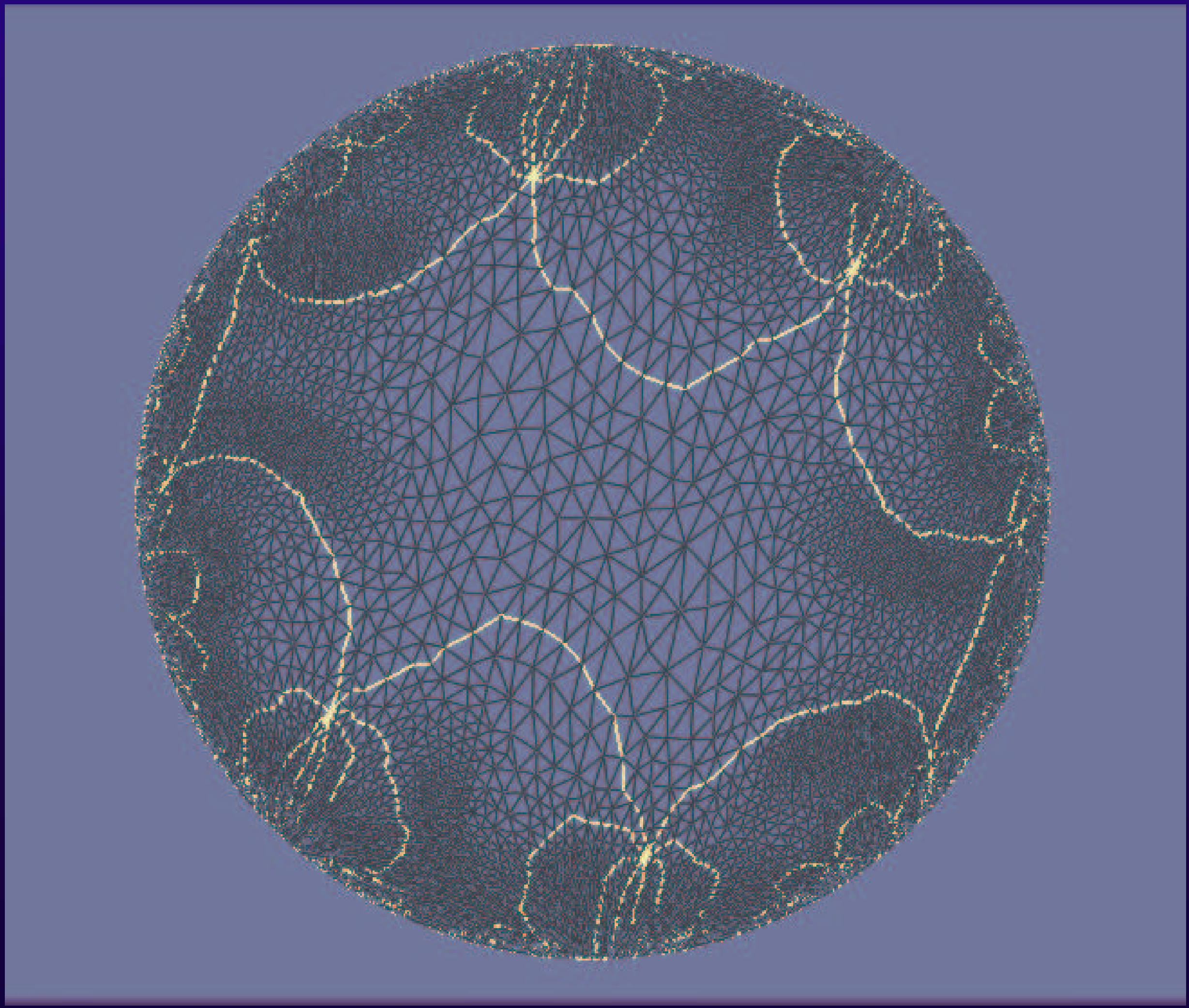


Embedding in Hyperbolic Space

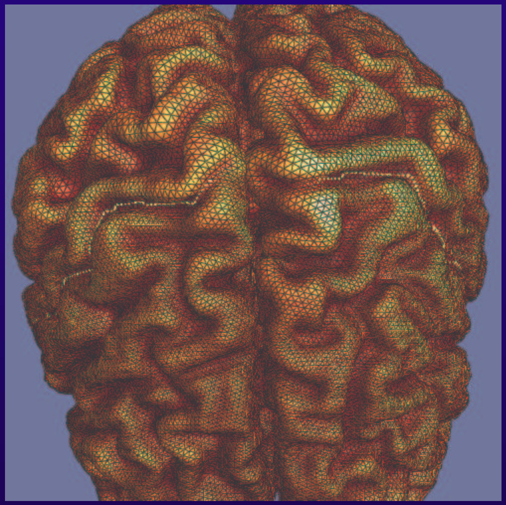
- Canonical Homology Basis

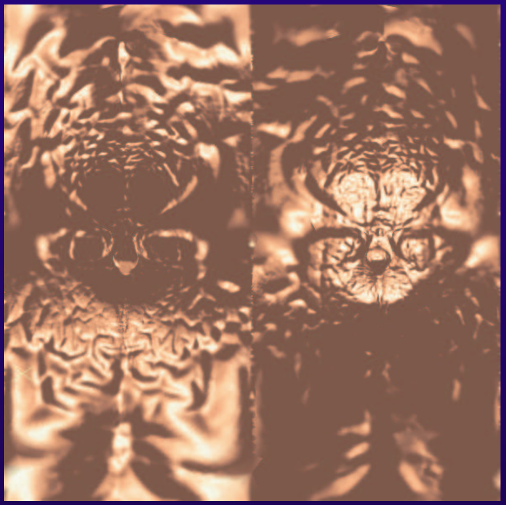


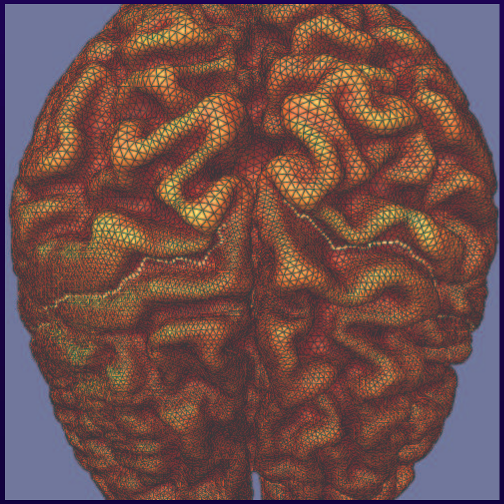
Embedding in Hyperbolic Space

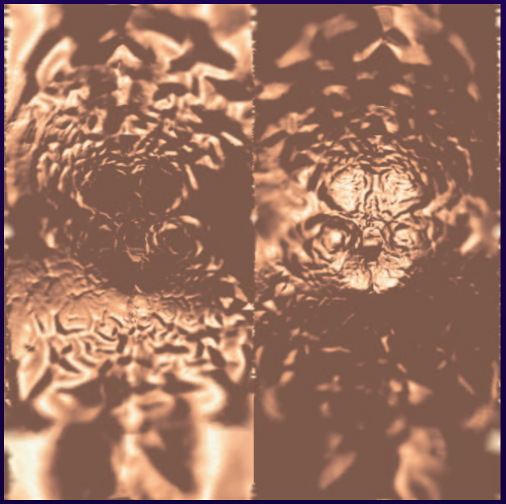


Matching Landmarks









Summary

- Compute conformal structures of surfaces
  - for general surfaces with arbitrary topologies
  - Intrinsic to geometry, independent of triangulations, insensitive to resolutions
  - Conformal invariants
  - Holomorphic differential group

Future Research

- Surface classifications based on conformal invariants – geometric database
- Surface isometric deformation – expression, skin deformation
- Theoretic problems
  - Genus g surface has 6g-6 conformal equivalent classes
  - How to parameterize these 6g-6 conformal classes

Thank you !