Motives

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Motivation: What motives allow you to do

- Relate phenomena in different cohomology theories.
- "Linearize" algebraic varieties
- Import algebraic topology into algebraic geometry

Outline

- Algebraic cycles and pure motives

- Mixed motives as universal arithmetic cohomology of smooth varieties

- The triangulated category of mixed motives

- Tate motives, Galois groups and multiple zeta-values

Algebraic cycles and pure motives

For $X \in \mathbf{Sm}/k$, set

$$z^q(X) := \mathbb{Z}[\{W \subset X, \text{ closed, irreducible, } \mathrm{codim}_X W = q\}],$$

the *codimension* q *algebraic cycles on* X. Set $|\sum_i n_i W_i| = \cup_i W_i$. We have:

- A partially defined intersection product: $W \cdot W' \in z^{q+q'}(X)$ for $W \in z^q(X)$, $W' \in z^{q'}(X)$ with $\mathrm{codim}_X(|W| \cap |W'|) = q + q'$.

- A partially defined pull-back for $f : Y \to X$: $f^*(W) \in z^q(Y)$ for $W \in z^q(X)$ with $\mathrm{codim}_Y f^{-1}(|W|) = q$.

- A well-defined push-forward $f_* : z^q(Y) \to z^{q+d}(X)$ for $f : Y \to X$ proper, $d = \dim X - \dim Y$, satisfying the projection formula:

$$f_*(f^*(x) \cdot y) = x \cdot f_*(y).$$

Rational equivalence.

For $X \in \mathbf{Sm}/k$, $W, W' \in z^q(X)$, say $W \sim_{\text{rat}} W'$ if $\exists Z \in z^q(X \times \mathbb{A}^1)$ with

$$W - W' = (i_0^* - i_1^*)(Z).$$

Set $\mathrm{CH}^q(X) := z^q(X)/\sim_{\text{rat}}$.

The intersection product $\cdot$, pull-back f^* and push-forward f_* are *well-defined* on CH^*. Thus, we have the graded-ring valued functor

$$\mathrm{CH}^* : \mathbf{Sm}/k^{\mathrm{op}} \to \text{Graded Rings}$$

which is covariantly functorial for projective maps $f : Y \to X$, and satisfies the projection formula:

$$f_*(f^*(x) \cdot y) = x \cdot f_*(y).$$

Correspondences.

For $X, Y \in \mathbf{SmProj}/k$, set

$$\mathrm{Cor}_k(X,Y)^n := \mathrm{CH}^{\dim X+n}(X \times Y).$$

Composition: For $\Gamma \in \mathrm{Cor}_k(X,Y)^n$, $\Gamma' \in \mathrm{Cor}_k(Y,Z)^m$,

$$\Gamma' \circ \Gamma := p_{XZ*}(p_{XY}^*(\Gamma) \cdot p_{YZ}^*(\Gamma')) \in \mathrm{Cor}_k(X,Z)^{n+m}.$$

We have $\mathrm{Hom}_k(Y,X) \to \mathrm{Cor}_k(X,Y)^0$ by

$$f \mapsto \Gamma_f^t.$$

The category of pure Chow motives

1. $\mathbf{SmProj}/k^{\mathsf{op}} \to \mathsf{Cor}_k$: Send X to $h(X)$, f to Γ_f^t, where $\mathrm{Hom}_{\mathsf{Cor}}(h(X), h(Y)) := \mathsf{Cor}_k(X, Y) \otimes \mathbb{Q}$.

2. $\mathsf{Cor}_k \to \mathcal{M}^{\mathsf{eff}}(k)$: Add images of projectors (pseudo-abelian hull).

3. $\mathcal{M}^{\mathsf{eff}}(k) \to \mathcal{M}(k)$: Invert tensor product by the Lefschetz motive L.

The composition $\mathbf{SmProj}/k^{\mathsf{op}} \to \mathsf{Cor}_k \to \mathcal{M}^{\mathsf{eff}}(k) \to \mathcal{M}(k)$ yields the functor

$$h : \mathbf{SmProj}/k^{\mathsf{op}} \to \mathcal{M}(k).$$

- These are tensor categories with $h(X) \otimes h(Y) = h(X \times Y)$.
- $h(\mathbb{P}^1) = \mathbb{Q} \oplus L$ in $\mathcal{M}^{\mathsf{eff}}(k)$.
- In $\mathcal{M}(k)$, write $M(n) := M \otimes L^{\otimes -n}$. Then

$$\mathsf{Hom}_{\mathcal{M}(k)}(h(Y)(m), h(X)(n)) = \mathsf{Cor}_k(X, Y)^{n-m} \otimes \mathbb{Q}.$$

- $\mathcal{M}(k)$ is a *rigid* tensor category, with dual

$$h(X)(n)^\vee = h(X)(\mathsf{dim}\, X - n).$$

- Can use $\mathcal{M}(k)$ to give a simple proof of the Lefschetz fixed point formula and to show that the topological Euler characteristic $\chi_H(X)$ is independent of the Weil cohomology H.

- Can use other "adequate" equivalence relations $\sim$, e.g. $\sim_{\mathsf{num}}$, to form $\mathcal{M}_\sim(k)$. $\mathcal{M}_{\mathsf{num}}(k)$ is a semi-simple abelian category (Jannsen).

Mixed Motives

Bloch-Ogus cohomology. This is a *bi-graded* cohomology theory:

$$X \mapsto \oplus_{p,q} H^p(X, \Gamma(q)).$$

on $\mathbf{Sm}/k$, with

1. Gysin isomorphisms $H^p_W(X, \Gamma(q)) \cong H^{p-2d}(W, \Gamma(q-d))$ for $i : W \to X$ a closed codimension d embedding in $\mathbf{Sm}/k$.

2. Natural 1st Chern class homomorphism $c_1 : \mathrm{Pic}(X) \to H^2(X, \Gamma(1))$

3. Natural cycle classes $Z \mapsto \mathrm{cl}^q(Z) \in H^{2q}_{|Z|}(X, \Gamma(q))$ for $Z \in z^q(X)$.

4. Homotopy invariance $H^p(X, \Gamma(q)) \cong H^p(X \times \mathbb{A}^1, \Gamma(q))$.

Consequences

- Mayer-Vietoris sequence

- Projective bundle formula:

$$H^*(\mathbb{P}(E), \Gamma(*)) = \oplus_{i=0}^{r} H^*(X, \Gamma(*))\xi^i$$

  for $E \to X$ of rank $r+1$, $\xi = c_1(\mathcal{O}(1))$.

- Chern classes $c_q(E) \in H^{2q}(X, \Gamma(q))$ for vector bundles $E \to X$.

- Push-forward $f_* : H^p(Y, \Gamma(q)) \to H^{p+2d}(X, \Gamma(q+d))$ for $f : Y \to X$ projective, $d = \mathrm{codim} f$.

Examples.

- $X \mapsto \oplus_{p,q} H^p_{\text{ét}}(X, \mathbb{Q}_\ell(q))$ or $H^p_{\text{ét}}(X, \mathbb{Z}_\ell(q))$ or $H^p_{\text{ét}}(X, \mathbb{Z}/n(q))$.

- for $k \hookrightarrow \mathbb{C}$, $A \subset \mathbb{C}$, $X \mapsto \oplus_{p,q} H^p(X(\mathbb{C}), (2\pi i)^q A)$
  or $H^p(X(\mathbb{C}), (2\pi i)^q A/n)$.

- for $k \hookrightarrow \mathbb{C}$, $A \subset \mathbb{R}$, $X \mapsto \oplus_{p,q} H^p_{\mathcal{D}}(X_{\mathbb{C}}, A(q))$.

- $X \mapsto \oplus_{p,q} H^p_{\mathcal{A}}(X, \mathbb{Q}(q)) := K_{2q-p}(X)^{(q)}$.

Beilinson's conjectures

• There should exist an abelian rigid tensor category of *mixed motives* over k, $\mathcal{MM}(k)$, with Tate objects $\mathbb{Z}(n)$, and a functor $h : \mathbf{Sm}/k^{\mathrm{op}} \to D^b(\mathcal{MM}_k)$, satisfying

$$h(\mathrm{Spec}\, k) = \mathbb{Z}(0);\ \mathbb{Z}(n) \otimes \mathbb{Z}(m) = \mathbb{Z}(n+m);\ \mathbb{Z}(n)^\vee = \mathbb{Z}(-n),$$

• $\mathcal{MM}(k)_{\mathbb{Q}}$ should admit a faithful tensor functor

$$\omega : \mathcal{MM}(k)_{\mathbb{Q}} \to \text{finite-dim'l } \mathbb{Q}\text{-vector spaces.}$$

i.e. $\mathcal{MM}(k)_{\mathbb{Q}}$ should be a *Tannakian category*.

• Set

$$\begin{aligned} H^p_\mu(X, \mathbb{Z}(q)) &:= \mathrm{Ext}^p_{\mathcal{MM}(k)}(\mathbb{Z}(0), h(X)(q)) \\ &:= \mathrm{Hom}_{D^b(\mathcal{MM}(k))}(\mathbb{Z}(0), h(X)(q)[p]) \\ h^i(X) &:= H^i(h(X)) \end{aligned}$$

One should have

1. Natural isomorphisms $H^p_\mu(X, \mathbb{Z}(q)) \otimes \mathbb{Q} \cong K_{2q-p}(X)^{(q)}$.

2. The subcategory of semi-simple objects of $\mathcal{MM}(k)$ is $\mathcal{M}_{\mathsf{num}}(k)$ and $h^i(X)$ is in $\mathcal{M}_{\mathsf{num}}(k)$ for X smooth and projective.

3. $X \mapsto h(X)$ satisfies Bloch-Ogus axioms in the category $D^b(\mathcal{MM}_k)$.

4. For each Bloch-Ogus theory, $H^*(-, \Gamma(*))$, there is realization functor
$$\mathsf{Re}_\Gamma : \mathcal{MM}(k) \to \mathbf{Ab}.$$
Re_Γ is an exact tensor functor, sending $H^p_\mu(X, \mathbb{Z}(q))$ to $H^p(X, \Gamma(q))$.
So: $H^*_\mu(-, \mathbb{Z}(*))$ is the *universal* Bloch-Ogus theory.

Motivic complexes

Let $\Gamma_{\text{mot}}(M) := \text{Hom}_{\mathcal{MM}(k)}(\mathbb{Z}(0), M)$. The derived functor $R\Gamma_{\text{mot}}(h(X)(q))$ represents weight-q motivic cohomology:

$$H^p(R\Gamma_{\text{mot}}(h(X)(q))) = H^p_\mu(X, \mathbb{Z}(q)).$$

Even though $\mathcal{MM}(k)$ does not exist, one can try and construct the complexes $R\Gamma_{\text{mot}}(h(X)(q))$.

Beilinson and Lichtenbaum gave conjectures for the structure of these complexes (even before Beilinson had the idea of motivic cohomology).

Bloch gave the first construction of a good candidate.

Bloch's complexes:

Let $\Delta^n := \operatorname{Spec} k[t_0, \ldots, t_n]/\sum_i t_i - 1$.
A *face* of Δ^n is a subscheme F defined by $t_{i_1} = \ldots = t_{i_n} = 0$.
$n \mapsto \Delta^n$ is a cosimplicial scheme.
Let $\delta_i^n : \Delta^{n-1} \to \Delta^n$ be the coface map to $t_i = 0$.

Let

$$z^q(X,n) = \mathbb{Z}[\{W \subset X \times \Delta^n, \text{closed, irreducible, and for all faces } F, \operatorname{codim}_{X\times F} W \cap (X \times F) = q\}] \subset z^q(X \times \Delta^n)$$

This defines *Bloch's cycle complex* $z^q(X,*)$, with differential

$$d_n = \sum_{i=0}^{n+1} (-1)^i \delta_i^* : z^q(X,n) \to z^q(X,n-1).$$

Definition 1 *The higher Chow groups* $\mathrm{CH}^q(X,p)$ *are defined by*

$$\mathrm{CH}^q(X,p) := H_p(z^q(X,*)).$$

Set $H^p_{Bl}(X,\mathbb{Z}(q)) := \mathrm{CH}^q(X,2q-p)$.

Theorem 1

(1) For $X \in \mathbf{Sm}/k$ *there is a natural isomorphism* $\mathrm{CH}^q(X,p)_{\mathbb{Q}} \cong K_{2q-p}(X)^{(q)}$.

(2) $X \mapsto \oplus_{p,q} H^p_{Bl}(X,\mathbb{Z}(q))$ *is the universal Bloch-Ogus theory on* $\mathbf{Sm}/k$.

So, $H^p_{Bl}(X,\mathbb{Z}(q))$ is a good candidate for *motivic cohomology*.

Voevodsky's triangulated category of motives

This is a construction of a model for $D^b(\mathcal{MM}(k))$ without constructing $\mathcal{MM}(k)$.

• Form the category of *finite correspondences* SmCor(k). Objects $m(X)$ for $X \in \mathbf{Sm}/k$.

$$\mathrm{Hom}_{\mathsf{SmCor}(k)}(m(X), m(Y)) = \mathbb{Z}[\{W \subset X \times Y, \text{ closed, irreducible, } W \to X \text{ finite and surjective.}\}]$$

Composition is composition of correspondences.

Note. For finite correspondences, the intersection product is *always* defined, and the push-forward is also defined, even for non-proper schemes.

• Sending $f : X \to Y$ to $\Gamma_f \subset X \times Y$ defines

$$m : \mathbf{Sm}/k \to \mathsf{SmCor}(k).$$

Note. m is covariant, so we are constructing *homological* motives.

Form the category of bounded complexes and the homotopy category

$$\mathbf{Sm}/k \to \mathbf{Sm}\mathsf{Cor}(k) \to C^b(\mathsf{SmCor}(k)) \to K^b(\mathsf{SmCor}(k)).$$

$K^b(\mathsf{SmCor}(k))$ is a triangulated category with distinguished triangles the Cone sequences of complexes:

$$A \xrightarrow{f} B \to \mathsf{Cone}(f) \to A[1].$$

Set

$$\mathbb{Z}(1) := (m(\mathbb{P}^1)^0 \to m(\mathsf{Spec}\, k)^1)[-2]$$

• Form the category of *effective geometric motives* $DM^{\text{eff}}_{\text{gm}}(k)$ from $K^b(\text{SmCor}(k))$ by inverting the maps

1. (*homotopy invariance*) $m(X \times \mathbb{A}^1) \to m(X)$

2. (*Mayer-Vietoris*) For $U, V \subset X$ open, with $X = U \cup V$,

$$(m(U \cap V) \to m(U) \oplus m(V)) \to m(X),$$

and taking the pseudo-abelian hull.

$DM^{\text{eff}}_{\text{gm}}(k)$ has a tensor structure with $m(X) \otimes m(Y) = m(X \times Y)$.

• Form the category of *geometric motives* $DM_{\text{gm}}(k)$ from $DM^{\text{eff}}_{\text{gm}}(k)$ by inverting $\otimes\mathbb{Z}(1)$.

Categorical motivic cohomology

Definition 2 *Let $\mathbb{Z}(n) := \mathbb{Z}(1)^{\otimes n}$. For $X \in \mathbf{Sm}/k$, set*

$$H^p(X, \mathbb{Z}(q)) := \mathrm{Hom}_{DM_{\mathrm{gm}}(k)}(m(X), \mathbb{Z}(q)[p])$$

Theorem 2
(1) $DM_{\mathrm{gm}}(k)$ is a rigid triangulated tensor category with Tate objects $\mathbb{Z}(n)$.
(2) $X \mapsto m(X)^{\vee}$ satisfies the Bloch-Ogus axioms (in $DM_{\mathrm{gm}}(k)$).
(3) There are natural isomorphisms $H^p(X, \mathbb{Z}(q)) \cong H^p_{Bl}(X, \mathbb{Z}(q))$.
(4) There are realization functors for the étale theory and for the mixed Hodge theory.

Mixed Tate Motives

The triangulated category of mixed Tate motives

Let $DTM(k) \subset DM_{\mathrm{gm}}(k)_{\mathbb{Q}}$ be the full triangulated subcategory generated by the Tate objects $\mathbb{Q}(n)$. $DTM(k)$ is like the derived category of Tate MHS.

$DTM(k)$ has a *weight filtration*: Define full triangulated subcategories $W_{\leq n}DTM(k)$, $W^{>n}DTM(k)$ and $W_{=n}DTM(k)$ of $DTM(k)$:

$W^{>n}DTM(k)$ is generated by the $\mathbb{Q}(m)[a]$ with $m < -n$
$W_{\leq n}DTM(k)$ is generated by the $\mathbb{Q}(m)[a]$ with $m \geq -n$
$W_{=n}DTM(k)$ is generated by the $\mathbb{Q}(-n)[a]$.

There are exact "truncation" functors

$W^{>n} : DTM(k) \to W^{>n}DTM(k)$,
$W_{\leq n} : DTM(k) \to W_{\leq n}DTM(k)$.

There is a natural distinguished triangle

$$W_{\leq n}X \to X \to W^{>n}X \to W_{\leq n}X[1]$$

and a natural tower (the weight filtration)

$$0 = W_{\leq N-1}X \to W_{\leq N}X \to \ldots \to W_{\leq M-1}X \to W_{\leq M}X = X.$$

Let $\mathrm{gr}_n^W X$ be the cone of $W_{\leq n-1}X \to W_{\leq n}X$. The category $W_{=n}DTM(k)$ is equivalent to D^b(f.diml. $\mathbb{Q}$-v.s.), so we have the exact functor

$$\mathrm{gr}_n^W : DTM(k) \to D^b(\text{f.diml. } \mathbb{Q}\text{-v.s.}).$$

The vanishing conjectures

Suppose there were an abelian category of Tate motives over k, $TM(k)$, containing the Tate objects $\mathbb{Q}(n)$, and with $DTM(k) \cong D^b(TM(k))$. Then

$K_{2q-p}(k)^{(q)} = \mathrm{Hom}_{DTM(k)}(\mathbb{Q}(0), \mathbb{Q}(q)[p]) = \mathrm{Ext}^p_{TM(k)}(\mathbb{Q}(0), \mathbb{Q}(q)).$

Thus: $K_{2q-p}(k)^{(q)} = 0$ for $p < 0$. This is the weak form of

Conjecture 1 (Beilinson-Soulé vanishing) *For every field k, $K_{2q-p}(k)^{(q)} = 0$ for $p \leq 0$, except for the case $p = q = 0$.*

Theorem 3 *Suppose the vanishing conjecture holds for a field k. Then there is a non-degenerate t-structure on $DTM(k)$ with heart $TM(k)$ containing and generated by the Tate objects $\mathbb{Q}(n)$.*

The Tate motivic Galois group

Theorem 4 *Suppose k satisfies B-S vanishing.*

(1) The weight-filtration on $DTM(k)$ induces an exact weight-flitration on $TM(k)$.

(2) The functors gr_n^W induce a faithful exact tensor functor

$$\omega := \oplus_n \mathrm{gr}_n^W : TM(k) \to \text{f. dim'l graded } \mathbb{Q}\text{-vector spaces.}$$

Corollary 1 *Suppose k satisfies B-S vanishing. Then there is a graded pro-unipotent algebraic group $\mathcal{U}_k^{\mathrm{mot}}$ over $\mathbb{Q}$, and an equivalence of $TM(k) \otimes_{\mathbb{Q}} K$ with the graded representations of $\mathcal{U}_k^{\mathrm{mot}}(K)$, for all fields $K \supset \mathbb{Q}$.*

In fact $\mathcal{U}_k^{\mathrm{mot}} = \mathrm{Aut}(\omega)$.

Let $\mathcal{L}_k^{\text{mot}}$ be the Lie algebra of $\mathcal{U}_k^{\text{mot}}$. For each field $K \supset \mathbb{Q}$, $\mathcal{L}_k^{\text{mot}}(K)$ is a graded pro-Lie algebra over K and

$$\text{GrRep}_K \mathcal{L}_k^{\text{mot}}(K) \cong TM(k) \otimes_{\mathbb{Q}} K.$$

Example. Let k be a number field. Then k satisfies B-S vanishing. $\mathcal{L}_k^{\text{mot}}$ is the free pro-Lie algebra on $\prod_{n\geq 1} H^1(k, \mathbb{Q}(n))^\vee$, with $H^1(k, \mathbb{Q}(n))^\vee$ in degree $-n$. Note that

$$H^1(k, \mathbb{Q}(n)) = \begin{cases} \mathbb{Q}^{r_1+r_2} & n > 1 \text{ odd} \\ \mathbb{Q}^{r_2} & n > 1 \text{ even} \\ k^\times \otimes \mathbb{Q} & n = 1. \end{cases}$$

Definition 3 *Let k be a number field. S a set of primes. Set $\mathcal{L}_{\mathcal{O}_{k,S}}^{\text{mot}} := \mathcal{L}_k^{\text{mot}} / < k_S^\times \otimes \mathbb{Q}^\vee >$. Set $TM(\mathcal{O}_{k,S}) :=$ GrRep $\mathcal{L}_{\mathcal{O}_{k,S}}^{\text{mot}}$.*

Relations with $G_k := Gal(\bar{k}/k)$

k: a number field
q. uni-Rep$_\ell(G_k) :=$ finite dim'l $\mathbb{Q}_\ell$ filtered rep'n of G_k, with nth graded quotient being $\chi_{\mathrm{cycl}}^{\otimes -n}$.

Let $\mathcal{L}_{k,\ell}$ be the associated graded Lie algebra, $\mathcal{L}_{k,S,\ell}$ the quotient corresponding to the unramified outside S representations.

The $\mathbb{Q}_\ell$-étale realization of $DM_{\mathrm{gm}}(k)$ gives a functor $\mathrm{Re}_{\text{ét},\ell} : TM(k) \to$ q. uni-Rep(G_k), so

$$\mathrm{Re}^*_{\text{ét},\ell} : \mathcal{L}_{k,S,\ell} \to \mathcal{L}^{\mathrm{mot}}_{\mathcal{O}_{k,S\cup\ell}}(\mathbb{Q}_\ell).$$

Example. $k = \mathbb{Q}$, $S = \emptyset$. $\mathcal{L}^{\mathrm{mot}}_{\mathbb{Z}}$ is the free Lie algebra on generators $s_3, s_5, \ldots$, $\mathcal{L}^{\mathrm{mot}}_{\mathbb{Z},\ell}$ has one additional generator $s_1^{(\ell)}$ in degree -1.

Application

Consider the action of $G_{\mathbb{Q}}$ on $\pi_1^{\mathrm{geom}}(\mathbb{P}^1 \setminus \{0, 1, \infty\}) = \widehat{F}_2$ via the split exact sequence

$$1 \to \pi_1^{\mathrm{geom}}(\mathbb{P}^1 \setminus \{0, 1, \infty\}) \to \pi_1(\mathbb{P}^1_{\mathbb{Q}} \setminus \{0, 1, \infty\}) \to G_{\mathbb{Q}} \to 1.$$

It is known that this action is pro-unipotent.

Conjecture 2 (Deligne-Goncharov) *The image of Lie*$(G_{\mathbb{Q}})$ *in Lie*$(Aut(\pi_1^{geom}(\mathbb{P}^1 \setminus \{0, 1, \infty\})))$ *is free, generated by certain elements* $\tilde{s}_{2n+1}$ *of weight* $2n+1$, $n = 1, 2, \ldots$.

Theorem 5 (Hain-Matsumoto) *The image of Lie*$(G_{\mathbb{Q}})$ *in Lie*$(Aut(\pi_1^{geom}(\mathbb{P}^1 \setminus \{0, 1, \infty\})))$ *is generated by the* $\tilde{s}_{2n+1}$.

Idea: Factor the action of Lie$(G_{\mathbb{Q}})$ through $\mathfrak{L}_{\mathbb{Z}}^{\mathrm{mot}}$.

Multiple zeta-values

Let $\mathcal{L}_{\mathsf{Hdg}}$ be the $\mathbb{C}$-Lie algebra governing Tate MHS. Since

$$\mathrm{Ext}^1_{MHS}(\mathbb{Q}, \mathbb{Q}(n)) = \mathbb{C}/(2\pi i)^n\mathbb{Q},$$
$$\mathrm{Ext}^p_{MHS}(\mathbb{Q}, \mathbb{Q}(n)) = 0;\ p \geq 2$$

$\mathcal{L}_{\mathsf{Hdg}}$ is the free graded pro-Lie algebra on $\prod_n(\mathbb{C}/(2\pi i)^n\mathbb{Q})^\vee$.

The Hodge realization gives the map of co-Lie algebras

$$\mathrm{Re}_{\mathsf{Hdg}} : (\mathcal{L}^{\mathsf{mot}}_{\mathbb{Z}})^\vee \to \mathcal{L}^\vee_{\mathsf{Hdg}},$$

so $\mathrm{Re}_{\mathsf{Hdg}}(s^\vee_{2n+1})$ is a complex number (mod $(2\pi i)^{2n+1}\mathbb{Q}$). In fact:

$$\mathrm{Re}_{\mathsf{Hdg}}(s^\vee_{2n+1}) = \zeta(2n+1) \bmod (2\pi i)^{2n+1}\mathbb{Q}.$$

The element $s_{2n+1}^{\vee}$ is just a generator for $H^1(\text{Spec}\,\mathbb{Q}, \mathbb{Q}(n+1))$, i.e, an extension

$$0 \to \mathbb{Q}(0) \to E_{2n+1} \to \mathbb{Q}(n+1) \to 0,$$

and $\text{Re}_{\text{Hdg}}(s_{2n+1}^{\vee})$ is the *period* of this extension. One can construct more complicated "framed objects" in $TM(\mathbb{Z})$ and get other periods.

Using the degeneration divisors in $\overline{\mathcal{M}}_{0,n}$, Goncharov and Manin have constructed framed mixed Tate motives $Z(k_1, \ldots, k_r)$, $k_r \geq 2$, with

$$\text{Per}(Z(k_1, \ldots, k_r)) = \zeta(k_1, \ldots, k_r)$$

where $\zeta(k_1, \ldots, k_r)$ is the multiple zeta-value

$$\zeta(k_1, \ldots, k_r) := \sum_{1 \leq n_1 < \ldots < n_r} \frac{1}{n_1^{k_1} \cdots n_r^{k_r}}.$$

This leads to:

Theorem 6 (Terasoma) *Let L_n be the $\mathbb{Q}$-subspace of $\mathbb{C}$ generated by the $\zeta(k_1, \ldots, k_r)$ with $n = \sum_i k_i$. Then*

$$\dim_{\mathbb{Q}} L_n \leq d_n,$$

where d_n is defined by $d_0 = 1$, $d_1 = 0$, $d_2 = 1$ and $d_{i+3} = d_{i+1} + d_i$.

Proof. The $\zeta(k_1, \ldots, k_r)$ with $n = \sum_i k_i$ are periods of framed mixed Tate motives M such that

$$S(M) := \{i \mid \mathrm{gr}_i^W M \neq 0\}$$

is supported in $[0, n]$ and if $i < j$ are in $S(M)$, then $j - i$ is odd and ≥ 3. Using the structure of $\mathcal{L}_{\mathbb{Z}}^{\mathrm{mot}}$, one shows that the dimension of such motives (modulo framed equivalence) is exactly d_n. Thus their space of periods has dimension $\leq d_n$.

Conjecture 3 (Zagier) $\dim_{\mathbb{Q}} L_n = d_n$.

Thank you