Stein ① 11/17/03

Bott — The Symbol of an Operator; a Reminiscience

(Basic Notions)

1949: IAS, I didn't know abstract math

$\bar{A}$ — closure? I didn't know.

Hermann Weyl: Hodge Theory — make proof rigorous!

Kodaira, deRham

$$M \quad \Omega^0(M) \xrightarrow[\text{"differentiation"}]{d} \Omega^1(M) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n(M)$$

$$H_{DR}(M) = \ker(d)/\operatorname{im}(d)$$

$$\underline{\underline{H^q(M)}}$$

Very beautiful

work well with pullback, etc.

Bad for physicits since they don't like "quotient spaces"

Hodge theory "breaks the symmetry".

First nontrivial, and surprising remark:

M compact $\Longrightarrow H^q(M)$ finite dimensional,

(a complicated proof is due to Leray — he gave it at IAS when I was there)

Riemannian metric:

$$ds^2 = g_{ij}\, dx^i dx^j$$

$$\langle \omega, \eta \rangle = \int_{[M]} \omega \wedge \eta \qquad \text{inner product on Riemannian manifold.}$$

induces maps $d^* : \Omega^i(M) \to \Omega^{i-1}(M)$

Let $\Box = dd^* + d^*d$: preserves each $\Omega^q(M)$.

Thm of Hodge: $\operatorname{Ker}\Box|^q = \mathcal{H}^q(M) \cong H^q(M)$

so each cohomology class has a ~~[illegible]~~ representative in $\operatorname{Ker}\Box|^q$, uniquely!

Thm

Example: $S^1 =: M$

$\Omega^0 = \mathcal{F}$ = functions, $\Omega^1 = \mathcal{F}$

= periodic C^∞ functions,

$f \longmapsto f'$ d = derivative $= \frac{d}{dx}$

$d^* = -d/dx$

so $\Box|^0 = -\frac{d^2}{dx^2}$, $\Box|^1 = -d^2/dx^2$

Fourier series: $f = a_0 + a_1 \cos x + b_1 \sin x + \cdots + a_n \cos nx + b_n \sin nx$

$\Box(f) = -\left(0 - a_1 \cos x - b_1 \sin x + \cdots - n^2 a_n \cos nx - n^2 b_n \sin nx\right)$

so $\ker(\Box)$ = constant functions.

More Modern Version of Hodge Theory —

"What is dx, Kodaira?" — • "A small piece of surface!" - Kodaira

• Slap! — Weil → "vector bundle" later terminology

• Chevalley — precise language

diff. = cross section of exterior power of tangent bundle

$$\Omega^0 \rightleftarrows \Omega^1 \cdots \rightleftarrows \Omega^n$$

$\Box$ $\Box$ $\Box$

Decompose into eigenspace of Laplacian:

$$\Omega^q(M) = \bigoplus_{\lambda \to \infty} \Omega^q_\lambda(M)$$

finite dimensional.

eigenvalues of $\Box$

over a discrete set

Beautiful generalization of Fourier series to any curved manifolds.

$d\Box = \Box d$ since $d^2 = 0$.

$dd^* + d^*d$

Direct sum of finite-dimensional problems:

$$d_\lambda : \Omega_\lambda^0(M) \longrightarrow \Omega_\lambda^1(M) \to \cdots \xrightarrow{d_\lambda} \Omega_\lambda^n(M)$$

Natural thing: "Cohomology of whole thing is direct sum of pieces."

yes but ~~[illegible]~~.

$$d_\lambda d_\lambda^* + d_\lambda^* d = \lambda \text{ on } \Omega_\lambda^q(M)$$

if $\lambda \neq 0$, get $\frac{d_\lambda d_\lambda^*}{\lambda} + \frac{d_\lambda^* d_\lambda}{\lambda} = 1$

$$d_\lambda\left(\frac{d_\lambda^*(\varphi)}{\lambda}\right) + \frac{d_\lambda^*}{\lambda} d_\lambda(\varphi) = \varphi \implies \text{No cohomology for } \lambda \neq 0!$$

To make precise, would have to characterize coeff. of C^∞ functions.

$\square$ We mainly thought in terms of "second order operators" diff. equations people

Ask more generally when do such theorems like Hodge's hold?

$\square$ $a_{11}\frac{\partial}{\partial x_1^2} + 2a_{12}\frac{\partial}{\partial x_1 \partial x_2} + a_{22}\frac{\partial}{\partial x_2^2} + \cdots$

$a_{ij}(x_1,x_2) =$

$\sigma_\square(x,\xi) = a_{11}\xi_1^2 + 2a_{12}\xi_1\xi_2 + a_{22}\xi_2^2 =$ symbol of the operator.

$\square$ Elliptic $\Leftrightarrow$ $a_\square(x,\xi) \neq 0$ for ξ real and not $\neq 0$.

Hyperbollic $\Leftrightarrow$ one eigenvalue negative

Parabollic $\Leftrightarrow$ semidefinite.

If D is a differential operator on $\mathbb{R}^n$ (linear)

$$D = \sum_{|J|\le m} a_J \frac{\partial^J}{\partial x^J}$$

$\uparrow$ multi-indexes

$$\text{Symbol}(D) = \sigma(x,\xi) = \sum_{|J|=m} a_J \xi^J$$

Elliptic diff. equation: $\sigma(x,\xi) \neq 0$ for ξ real and $\neq 0$

Shrewdness of Hodge theory to use $\square$ cost us 10 years!

$$\Omega^{even} \xrightarrow{d+d^*} \Omega^{odd}$$

easy: Haramonic forms = kernel of this operator.

So Hodge theory tells you about a system of diff. operators.

I thought about this in 1960s ... I was drinking seriously at this cocktail party ... Atiyah learned:

Elliptic: ker, coker finite dimensional

difference of dim is invariant under deformation.

$$D = \sum_{|J|\le m} A_J \frac{\partial}{\partial x^J}$$

$\uparrow$ $n\times n$ matrices

$$\sigma_D(x,\xi) = \sum_{|J|=m} A_J \xi^J.$$

Call it elliptic if $\det(\sigma_D(x,\xi)) \neq 0$ for ξ real and $\neq 0$.

Mathematicians don't like matrices, so say

$\sigma_D(x,\xi)$ in $GL_K(\mathbb{R})$.

For all real $\xi \neq 0$,

$\boxed{|\xi| = 1. \quad \sigma_D(x)}$

Interpret the symbol as a map: $\sigma_D(x) : S^{n-1} \longrightarrow GL_k(\mathbb{R})$ $\rightsquigarrow$ homotopy class

$[\sigma_D(x)] \in \pi_{n-1}(GL_k(\mathbb{R})) \longrightarrow \pi_{n-1}(GL_\infty(\mathbb{R}))$ dilute to stable homotopy groups. (4)

Virulence

~ I don't know why I call it this
Maybe has to do with male phenomenon.

Example: $\frac{\partial}{\partial z}(u+iv) = \left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)(u+iv)$

Cauchy Riemann eqn: write them like you *should* but never do

$$\Downarrow$$

$$\begin{pmatrix} \partial_x & -\partial_y \\ \partial_y & \partial_x \end{pmatrix}$$

$$\downarrow$$

to find homotopy element

symbol: $\det\begin{pmatrix} \xi_1 & \xi_2 \\ -\xi_2 & \xi_1 \end{pmatrix} = \xi_1^2 + \xi_2^2 = 1$

$S^1 \longrightarrow\!\!\!\!\!\rightarrow$ orthogonal matrices $\subseteq GL_2(\mathbb{R})$

σ_D induces an isomorphism to homotopy

$[\sigma_D]$ generates $\pi_1(GL_2) \rightsquigarrow \pi_1(GL_\infty)$.

Late '50, early '60s — we thought in terms of symbols.

One evening $\underbrace{}_{}$ Dirac equation ~ I fiddled with them:

Symbol of Dirac equation in various dimensions.

$$\not{\partial}$$

$$\boxed{\xi_1 \cdot 1 + \xi_2 \cdot J}$$

$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $J^2 = -1$

independently, there is Clifford algebra:

$C_k = \{1, e_i, i = 1, \dots, k$ with $e_i^2 = -1$, $e_i e_j + e_j e_i = 0$ if $i \neq j\}$

Dirac operator:

$\not{\partial} \longleftrightarrow (\xi_0 - \sum \xi_i e_i)$

Why elliptic: $(\xi_0 - \sum \xi_i e_i)(\xi_0 + \sum \xi_i e_i) = \underbrace{\xi_0^2 + \xi_1^2 + \dots + \xi_n^2}$

Laplacian has been factored

That night I computed these Clifford algebras.

n	C_n Clifford algebra	M	dim(M)	structure of group of representations (Grothendieck group)
0	$\mathbb{R}$	$\mathbb{Z}$	1	$\mathbb{Z}/2\mathbb{Z}$
1	$\mathbb{C}$	$\mathbb{Z}$	2	$\mathbb{Z}/2\mathbb{Z}$
2	$\mathbb{H}$ (quaternions)	$\mathbb{Z}$	4	0
3	$\mathbb{H} \oplus \mathbb{H}$	$\mathbb{Z} \oplus \mathbb{Z}$	4	$\mathbb{Z}$
4	$M_2(\mathbb{H})$ = 2x2 matrices over $\mathbb{H}$	$\mathbb{Z}$	8	0
5	$M_4(\mathbb{C})$	$\mathbb{Z}$	8	0
6	$M_8(\mathbb{R})$	$\mathbb{Z}$	8	0
7	$M_8(\mathbb{R})$	$\mathbb{Z} \oplus \mathbb{Z}$	8	$\mathbb{Z}$
8	$M_{16}(\mathbb{R})$			

So 2 distinct reps.

Natural map

$M_s \to M_{s-}$

Then it repeats — just multiply by simple factor.

This reminds me of my periodicity theorem.

Final theorem, 1964, Atiyah, Shapiro:

Irreducible representations applied to ϕ generate $\pi_*(GL)$.

Tricky to do directly w/ matrices.

Compare C_k with algebra $C_k' = \{ e_i^2 = 1, \cdot e \ldots \}$

tensor $C_k \otimes C_k'$ and get recursive formula.