# QUALIFYING EXAMINATION

Harvard University

Department of Mathematics

Tuesday 30 September 2003 (Day 1)

*There are six problems. Each question is worth 10 points, and parts of questions are of equal weight unless otherwise specified.*

1a. Let $S$ be an embedded closed surface in $\mathbf{R}^3$ with the position vector $\vec{X}(p)$ and the unit outward normal vector $\vec{N}(p)$ for $p \in S$. For a fixed (small) $t$, define a surface $S_t$ to be the set

$$S_t = \left\{ \vec{X}(p) + t\vec{N}(p) \in \mathbf{R}^3 \mid p \in S \right\}.$$

Let $\kappa_1$, $\kappa_2$ be the principal curvatures of $S$ at the point $p$ with respect to the outward normal vector. Let $H_t$ be the mean curvature of $S_t$ at the point $\vec{X}(p) + t\vec{N}(p)$ with respect to the outward normal vector (mean curvature is defined to be the sum of the two principal curvatures). Show that

$$H_t = \frac{\kappa_1}{1 - t\kappa_1} + \frac{\kappa_2}{1 - t\kappa_2}.$$

2a. Let $D = \{z \in \mathbf{C} : |z| < 1\}$ be the open unit disk in $\mathbf{C}$ and $\bar{D} = \{z \in \mathbf{C} : |z| \leq 1\}$ be the closed unit disk. Suppose $f : \bar{D} \to \bar{D}$ is analytic, one-to-one in $D$ and continuous in $\bar{D}$ . Also suppose $g : \bar{D} \to \bar{D}$ is analytic in $D$ and continuous in $\bar{D}$, with $g(0) = f(0)$ and $g(D) \subset f(D)$. Prove $|g'(0)| \leq |f'(0)|$.

3a. Use the Riemann-Hurwitz (or any other) method to compute the genus of the Fermat curve, which is given in $\mathbf{CP}^2$ with homogeneous coordinates $(x : y : z)$ by the equation $x^n + y^n = z^n$ (assume that the base field is $\mathbf{C}$).

4a. Let $k$ be a finite field with $q$ elements and let $\Gamma = \mathrm{GL}(2, k)$ denote the group of invertible $2 \times 2$ matrices over $k$.

(i) How many elements are there in $\Gamma$?

(ii) How many complex irreducible representations does $\Gamma$ have?

(iii) Consider the representation of $\Gamma$ on the space of complex-valued functions on $\mathbf{P}^1$ over $k$ (induced by the natural action of $\Gamma$ on $\mathbf{P}^1$). Let $V$ be the quotient of this space by the subspace of constant functions. Prove that $V$ is an irreducible representation of $\Gamma$.

5a. Let $V$ be a Hilbert space, and $W$ a vector subspace of $V$. Show that

$$V = \overline{W} \oplus W^{\perp}.$$

6a. What is $\pi_1(\mathbf{RP}^3)$? Show that any continuous map $f : \mathbf{RP}^3 \to S^1$ is null-homotopic.

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Wednesday 1 October 2003 (Day 2)

*There are six problems. Each question is worth 10 points, and parts of questions are of equal weight unless otherwise specified.*

1b. Let $(\mathbf{H}^2, g)$ be the two-dimensional hyperbolic space, where

$$\mathbf{H}^2 = \{(x, y) \in \mathbf{R}^2 : y > 0\}$$

is the upper half plane of $\mathbf{R}^2 = \mathbf{C}$ and the metric $g$ is given by

$$g = \frac{dx^2 + dy^2}{y^2}.$$

(i) Suppose $a$, $b$, $c$ and $d$ are real numbers such that $ad - bc = 1$. Define

$$\varphi(z) = \frac{az + b}{cz + d}$$

for any $z = x + \sqrt{-1}y$. Prove that $\varphi$ is an isometry for $(\mathbf{H}^2, g)$.

(ii) Prove that $(\mathbf{H}^2, g)$ has constant Gaussian curvature.

2b. Prove the open mapping theorem for analytic functions of one complex variable: "if $U$ is a connected open subset of $\mathbf{C}$ and $f : U \to \mathbf{C}$ is holomorphic and non-constant, then $f(U)$ is open." You may assume that a holomorphic function that is constant on an open subset of $U$ is constant on $U$.

3b. Prove that if $k$ is a field of characteristic $p$ and $f(x) \in k[x]$ is a polynomial, then the map from the curve $y^p + y = f(x)$ to $\mathbf{A}^1_k$ sending $(x, y)$ to $x$ is everywhere unramified.

4b. (i) Let $k$ be an algebraically closed field. Assume that $k$ is *uncountable.* Let now $V$ be a vector space over $k$ of at most countable dimension and $A : V \to V$ be a linear operator. Prove that there exists $\lambda \in k$ such that the operator $A - \lambda\, \mathrm{id}_V$ is not invertible. (Hint: show first that in the field $k(t)$ of rational functions over $k$ the elements $\frac{1}{t-\lambda}$ are linearly independent (for different values of $\lambda$) and then use this fact.)

(ii) Show that (i) is not necessarily true if $k$ is countable.

(iii) Use (i) to show that for $k$ uncountable every maximal ideal in the ring $k[x_1, \ldots, x_n]$ is generated by $(x - \lambda_1, \ldots, x - \lambda_n)$ for some $(\lambda_1, \ldots, \lambda_n) \in k^n$.

5b. Give an example or show that none exist.

(i) A function $f : \mathbf{R} \to \mathbf{R}$ whose set of discontinuities is precisely the set $\mathbf{Q}$ of rational numbers.

(ii) A function $f : \mathbf{R} \to \mathbf{R}$ whose set of discontinuities is precisely the set $\mathbf{R} \setminus \mathbf{Q}$ of irrational numbers.

6b. Let $X$ be the manifold-with-boundary $D^2 \times S^2$. Calculate $H_2(X; \mathbf{Z})$, $H^2(X; \mathbf{Z})$ and $H^2(X, \partial X; \mathbf{Z})$, using any techniques you choose. Calculate the map $j_* : H^2(X, \partial X; \mathbf{Z}) \to H^2(X; \mathbf{Z})$ that arises from the inclusion $j : (X, \emptyset) \to (X, \partial X)$.

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Thursday 2 October 2003 (Day 3)

*There are six problems. Each question is worth 10 points, and parts of questions are of equal weight unless otherwise specified.*

1c. Let $\Sigma$ be an embedded, compact surface without boundary in $\mathbf{R}^3$. Show that there exists at least one point $p$ in $\Sigma$ which has strictly positive Gaussian curvature.

2c. Determine for which $x \in \mathbf{Q}_p$ the exponential power series $\sum x^n/n!$ converges. Do the same for the logarithmic power series $\sum x^n/n$.

3c. Let $V$ be a variety over an algebraically closed field $k$, and suppose $V$ is also a group, i.e., there are morphisms $\varphi : V \times V \to V$ (multiplication or addition), and $\psi : V \to V$ (inverse) that satisfy the group axioms. Then $V$ is called an *algebraic group*.

(i) Suppose that $V$ is a nonsingular plane cubic. Describe a way to put a group structure on $V$. You do not have to prove that the maps you define are morphisms, but you do have to prove that they satisfy the axioms of a group.

(ii) Let $V$ be defined by $y^2z = x^3$ in $\mathbf{P}^2$. Prove that $V - \{(0,0,1)\}$ can be equipped with the structure of algebraic group.

(iii) Let $V$ be defined by $x^3 + y^3 = xyz$ in $\mathbf{P}^2$. Prove that $V - \{(0,0,1)\}$ can be equipped with the structure of algebraic group.

4c. Compute the following integral:

$$\int_0^\infty \frac{\log x}{(1+x^2)^2}\ \mathrm{d}x.$$

5c. (i) Let $a, b$ be nonnegative numbers, and $p, q$ such that $1 < p < \infty$ and $1/p + 1/q = 1$. Establish Young's inequality:

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}.$$

(ii) Using Young's inequality, prove the Hölder inequality: If $f \in L^p[0,1]$ and $g \in L^q[0,1]$, where $p$ and $q$ are as above, then $fg \in L^1[0,1]$, and

$$\int |fg| \leq \|f\|_p \cdot \|g\|_q.$$

(iii) For $1 < p < \infty$, and $g \in L^q$, consider the linear functional $F$ on $L^p$ given by

$$F(f) = \int fg.$$

Show that $\|F\| = \|g\|_q$. (Recall that $\|F\| = \sup\{|F(f)|/\|f\| : f \in L^p\}$.)

(iv) Establish similar results for $p = 1$ and $p = \infty$.

6c. (i) Prove that every continuous map $f : \mathbf{CP}^6 \to \mathbf{CP}^6$ has a fixed point.

(ii) Exhibit a continuous map $f : \mathbf{CP}^3 \to \mathbf{CP}^3$ without a fixed point. (Hint: Try the case of $\mathbf{CP}^1$ first and write your answer in terms of homogeneous coordinates.)