
IV

Universal Fourier expansions of modular forms

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Introduction

Let $\mathcal{X}$ be the set of matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z})$ such that $a > b \geq 0$ and $d > c \geq 0$. Let us consider the following series in $\mathbb{C}[M_2(\mathbb{Z})][[q]]$:

$$\sum_{M\in\mathcal{X}} M\ q^{\det M}$$

$$\begin{aligned}
&= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} q \\
&+ (\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} + \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix})\, q^2 \\
&+ (\begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} + \begin{pmatrix} 3 & 1 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 1 & 3 \end{pmatrix} + \begin{pmatrix} 3 & 2 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 2 & 3 \end{pmatrix})\, q^3 \\
&+ \ ...
\end{aligned}$$

The aim of this paper is to establish that this series produces (in a sense that will be made precise in a moment) Fourier expansions at infinity of modular forms of integral weight ≥ 2 for congruence subgroups of $SL_2(\mathbb{Z})$. This justifies the terminology "universal Fourier expansions of modular forms". Here and in what follows "Fourier expansion" will always mean "Fourier expansion at infinity".

Let k be an integer ≥ 2. Let N be an integer > 0. Let $\mathbb{C}_{k-2}[X,Y]$ be the complex vector space of homogeneous polynomials in two variables and degree $k-2$. Let $\Gamma_0(N)$ (resp. $\Gamma_1(N)$) be the group of matrixes $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ such that $N|c$ (resp. $N|c$, $N|(a-1)$). Let χ be a Dirichlet character modulo N. We $S_k(N)$ (resp. $S_k(N,\chi)$) the complex vector space of cusp forms of weight k for $\Gamma_1(N)$ (resp. for $\Gamma_0(N)$ with multiplicative character χ, see section 2.5).

Let $\mathbb{C}_{k-2}[X,Y][(\mathbb{Z}/N\mathbb{Z})^2]$ be the vector space of linear combinations of elements of $(\mathbb{Z}/N\mathbb{Z})^2$ with coefficients in $\mathbb{C}_{k-2}[X,Y]$. If ϕ is a linear map $\mathbb{C}_{k-2}[X,Y][(\mathbb{Z}/N\mathbb{Z})^2] \to \mathbb{C}$

and $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z})$, we denote by $\phi_{|g}$ the linear map $\mathbb{C}_{k-2}[X,Y][(\mathbb{Z}/N\mathbb{Z})^2] \to \mathbb{C}$ defined by the formula

$$\phi_{|g}(P(X,Y)[u,v]) = \phi(P(aX+bY, cX+dY)[au+cv, bu+dv]),$$

$P \in \mathbb{C}_{k-2}[X,Y]$, $(u,v) \in (\mathbb{Z}/N\mathbb{Z})^2$. We denote by E_N the set of elements (u,v) of $(\mathbb{Z}/N\mathbb{Z})^2$ satisfying the relation $\mathbb{Z}u + \mathbb{Z}v = \mathbb{Z}/N\mathbb{Z}$.

Let us denote by $P_N = \cup_{d|N}(\mathbb{Z}/d\mathbb{Z})^*$. By convention when $d=1$, $(\mathbb{Z}/d\mathbb{Z})^*$ has one element. Let $\mathbb{C}[P_N]_k$ be the quotient vector space of $\mathbb{C}[P_N]$ modulo the vector space generated by the elements of the form $[a] - (-1)^k[-a] = 0$, $a \in (\mathbb{Z}/d\mathbb{Z})^*$, $d|N$. If $a \in \mathbb{Z}/N\mathbb{Z}$, $d|N$ and a invertible modulo d, we denote by $[a]_d$ the image of $[a \pmod d]$ in $\mathbb{C}[P_N]_k$.

Let $b : \mathbb{C}_{k-2}[X,Y][E_N] \to \mathbb{C}[P_N]_k$ be the $\mathbb{C}$-bilinear map which associates to $P(X,Y)[u,v]$ the element $P(1,0)[v^{-1}]_{(u,N)} - P(0,1)[u^{-1}]_{(v,N)}$, where by abuse of notations v^{-1} is the inverse modulo (u,N) of v and (u,N) is the greater common divisor of u and N, i.e. the order of the sugroup of $\mathbb{Z}/N\mathbb{Z}$ generated by u.

Theorem 1 *Let ϕ be a linear map $\mathbb{C}_{k-2}[X,Y][(\mathbb{Z}/N\mathbb{Z})^2] \to \mathbb{C}$ verifying the following equalities*

$$\phi + \phi_{|\sigma} = \phi + \phi_{|\tau} + \phi_{|\tau^2} = \phi - \phi_{|J} = 0,$$

where $\sigma = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $\tau = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$ and $J = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$, and $\phi(P[u,v]) = 0$ if $(u,v) \notin E_N$, $P \in \mathbb{C}_{k-2}[X,Y]$. Let $x \in \mathbb{C}_{k-2}[X,Y][E_N]$ such that $b(x) = 0$. Then

$$\sum_{M \in \mathcal{X}} \phi_{|M}(x) q^{\det M}$$

is the Fourier expansion of an element f of $S_k(N)$. Furthermore all modular forms of such type can be produced by this method.

Let χ be a Dirichlet character $\mathbb{Z}/N\mathbb{Z} \to \mathbb{C}$. Let us suppose that ϕ satisfies the additional condition

$$\phi(P[\lambda u, \lambda v]) = \chi(\lambda)\phi(P[u,v])$$

($(\lambda, u, v) \in (\mathbb{Z}/N\mathbb{Z})^3$, $P \in \mathbb{C}_{k-2}[X,Y]$). Then f belongs to $S_k(N,\chi)$.

If x does not satisfy the relation $b(x) = 0$, the series obtained should be, except for the constant term, the Fourier expansion of a holomorphic modular form of weight k for $\Gamma_1(N)$ (see the remark in the section 3.2). This theorem can be refined to obtain only newforms (see sections 2.6 and 3.2).

We outline now the plan of the paper as well as the plan of the proof of the theorem 1. We recall in the first part the theory introduced by Manin and developed by Shokurov of modular symbols of arbitrary weight for subgroups of finite index of $SL_2(\mathbb{Z})$ ([4], [12], [11], [14], [13]). This theory can be related to the Eichler-Shimura theory connected with the cohomology of subgroups of finite index of $SL_2(\mathbb{Z})$. The Eichler-Shimura theory imbeds modular forms in a space of modular symbols; The

Manin-Shokurov theory constructs pairings between modular symbols and modular forms. We construct a complex vector space (of modular symbols) $\mathcal{M}_k(N)$, which is a quotient of $\mathbb{C}_{k-2}[X,Y][E_N]$ (see section 1.3). We denote by $[P,x]$ the image of $P[x] \in \mathbb{C}_{k-2}[X,Y][E_N]$ in $\mathcal{M}_k(N)$. There is a bilinear pairing of complex vector spaces between $\mathcal{M}_k(N)$ and $S_k(N) \oplus \overline{S_k(N)}$ (where $\overline{S_k(N)}$ is the space of antiholomorphic cusp forms), see section 1.5. This pairing is given as follows:

$$(f_1 + f_2, [P,x]) \mapsto \int_0^\infty f_{|g}(z) P(z,1)\, dz + \int_0^\infty f_{|g}(z) P(\bar{z},1)\, d\bar{z},$$

where $f_1 \in S_k(N)$, $f_2 \in \overline{S_k(N)}$, g is any element in $SL_2(\mathbb{Z})$ such that $\pi(g) = x$ and

$$f_{|g}(z) = (cz+d)^{-k} f(gz) \quad \text{and} \quad f_{|\bar{g}}(z) = (c\bar{z}+d)^{-k} f(gz).$$

We denote by $\mathcal{S}_k(N)$ the image in $\mathcal{M}_k(N)$ of the kernel of b. By a theorem of Shokurov, when restricted to $\mathcal{S}_k(N)$, the pairing defined above is nondegenerate (see section 1.5). We complete the theory of Shokurov by making explicit the relation between the modular symbols and the cohomology of $SL_2(\mathbb{Z})$. Moreover the action of the complex conjugation on modular curves defines an involution ι^* on $\mathcal{M}_k(N)$. This involution is studied in the section 1.6. In fact all the results in the first part of the paper are valid if $\Gamma_1(N)$ is replaced by any subgroup of finite index of $SL_2(\mathbb{Z})$.

The second part is devoted to the study of Hecke theory on $\mathcal{M}_k(N)$ (Hecke operators, Atkin-Lehner operators, old and new parts, degeneracy maps). We describe only here our main result. The Hecke operators operate by duality on $\mathcal{S}_k(N)$. In fact this action extends naturally to $\mathcal{M}_k(N)$. Let n be an integer > 0. We denote by $M_2(\mathbb{Z})_n$ the set of matrices of $M_2(\mathbb{Z})$ of determinant n. We say that an element $\sum_M u_M M \in \mathbb{C}[M_2(\mathbb{Z})_n]$ satisfies the condition (C_n) if for all $K \in M_2(\mathbb{Z})_n/SL_2(\mathbb{Z})$, we have in $\mathbb{C}[\mathbb{P}^1(\mathbb{Q})]$

$$\sum_{M \in K} u_M([M\infty] - [M0]) = [\infty] - [0].$$

We denote by T_n the Hecke operator on $\mathcal{M}_k(N)$. Our main result about Hecke operators is as follows.

Theorem 2 *Let $P[u,v] \in \mathbb{C}_{k-2}[X,Y][E_N]$. Let $\sum_M u_M M \in \mathbb{C}[M_2(\mathbb{Z})_n]$ satisfying the condition (C_n). Then we have (the sum is taken with respect to the matrixes $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$)*

$$T_n([P,(u,v)]) = \sum_M u_M[P(aX+bY, cX+dY), (au+cv, bu+dv)],$$

where the sum is restricted to the matrices M such that $(au+cv, bu+dv) \in E_N$ (if n and N are coprimes this restriction is unnecessary).

We remark that the condition (C_n) depends neither on the level N nor on the weight k. This theorem can be understood as a formula on linear forms on modular forms without knowing anything about modular symbols (see the pairing defined above). We give an analogous formula for the action of Atkin-Lehner operators on $\mathcal{M}_k(N)$ (see section 2.4). Let $\mathcal{X}_n$ be the set of matrices of $\mathcal{X}$ of determinant n. Then

$$\sum_{M\in\mathcal{X}_n} M$$

satisfies the condition (C_n) (see the section 3.1). The theorem 2 extends some results obtained in my thesis for the weight 2 (see [8], [7], [9]). It extends also a result in [16] concerning only the modular forms for the full modular group $SL_2(\mathbb{Z})$. We shall add that other elements of $\mathbb{C}[M_2(\mathbb{Z})_n]$ satisfy the condition (C_n) (see part 3).

For $x \in \mathbb{C}_{k-2}[X,Y][E_N]$ satisfying $b(x) = 0$ we can consider its image $m(x)$ in $\mathcal{S}_k(N)$. A complex linear form ϕ on $\mathbb{C}_{k-2}[X,Y][E_N]$ satisfying the conditions of the theorem 1 factorizes through a linear form on $\mathcal{M}_k(N)$, which induces a linear form ϕ_m on $\mathcal{S}_k(N)$. We obtain a linear form $\alpha(\phi,x)$ on the Hecke algebra (i.e. the $\mathbb{C}$-algebra generated by the Hecke operators T_n and $T_{n,n}$ of $\mathrm{End}_{\mathbb{C}}(S_k(N))$ for $n \geq 1$) which associates to T the complex number $\phi_m(Tm(x))$. We then use the following fact: if α is a linear map from the Hecke algebra to $\mathbb{C}$, then $\sum_{n=1}^{\infty}\alpha(T_n)q^n$ is the Fourier expansion of an element of $S_k(N)$. We apply this principle for $\alpha = \alpha(\phi,x)$ to obtain the theorem 1.

As an application of the theorem 1, we give a conditional method to construct bases of $S_k(N)$ (see section 3.3).

In the course of the completion of this work, I was generously invited several times to the Institut für experimentelle Mathematik. I would like to thank this institution for its nice hospitality.

1 The theory of Manin-Shokurov

1.1 The algebraic description of modular symbols

Let Γ be a subgroup of finite index of $SL_2(\mathbb{Z})$ and k an integer ≥ 2. Ik k is odd, we impose the following condition: $J = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \notin \Gamma$ (otherwise the following theory is empty).

First we consider the torsion free abelian group $\mathcal{M}$ generated by the expressions $\{\alpha,\beta\}$ ($(\alpha,\beta) \in \mathbb{P}^1(\mathbb{Q})^2$) with the following relations

$$\{\alpha,\beta\} + \{\beta,\gamma\} + \{\gamma,\alpha\} = 0 \qquad ((\alpha,\beta,\gamma) \in \mathbb{P}^1(\mathbb{Q})^3).$$

So we have $\{\alpha,\beta\} = -\{\beta,\alpha\}$ and $\{\alpha,\alpha\} = 0$. We consider now the complex vector space

$$\mathcal{M}_k = \mathbb{C}_{k-2}[X,Y] \otimes \mathcal{M},$$

where $\mathbb{C}_{k-2}[X,Y]$ is the space of complex homogeneous polynomials in two variables of degree $k-2$. We define a linear action of $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{Q})$ on $P \in \mathbb{C}_{k-2}[X,Y]$

and $P \otimes \{\alpha, \beta\} \in \mathcal{M}_k$ by the rules

$$P_{|g}(X, Y) = P(dX - bY, -cX + aY)$$

and

$$(P \otimes \{\alpha, \beta\})_{|g} = P_{|g} \otimes \{g\alpha, g\beta\}.$$

For $(g, h) \in GL_2(\mathbb{Q})^2$, we have $(P_{|g})_{|h} = P_{|hg}$. Let $\mathcal{M}_k(\Gamma)$ be the quotient vector space of $\mathcal{M}_k$ obtained by imposing the relations $x_{|\gamma} = x$ ($x \in \mathcal{M}_k$, $\gamma \in \Gamma$). We will denote by $P\{\alpha, \beta\}$ the image of $P \otimes \{\alpha, \beta\} \in \mathcal{M}_k$ in $\mathcal{M}_k(\Gamma)$. The elements of $\mathcal{M}_k(\Gamma)$ will be called *modular symbols of weight k for Γ*.

Remark .- We can replace the field $\mathbb{C}$ in this construction by any ring. In particular, the space $\mathcal{M}_k(\Gamma)$ carries a natural *integral structure*. The subgroup $\mathbb{Z}_{k-2}[X, Y] \otimes \mathcal{M}$ of $\mathcal{M}_k$ is stable by the action of $GL_2(\mathbb{Q}) \cap M_2(\mathbb{Z})$. It makes sense to consider its image $\mathcal{M}_k(\Gamma, \mathbb{Z})$ in $\mathcal{M}_k(\Gamma)$. The complex vector space $\mathcal{M}_k(\Gamma)$ is canonically isomorphic to $\mathcal{M}_k(\Gamma, \mathbb{Z}) \otimes \mathbb{C}$.

1.2 The Manin symbols

Let $g \in SL_2(\mathbb{Z})$ and $P \in \mathbb{C}_{k-2}[X, Y]$. We introduce the *Manin symbol* $[P, g] \in \mathcal{M}_k(\Gamma)$ by the formula

$$[P, g] = P_{|g}\{g0, g\infty\}.$$

We recall the notations of the introduction: $\sigma = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $\tau = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$ and $J = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$. These matrices verify $\sigma^2 = J$, $J^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\tau^3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Proposition 1 *Let $g \in SL_2(\mathbb{Z})$ and $P \in \mathbb{C}_{k-2}[X, Y]$. The Manin symbol $[P, g]$ depends only on the class Γg and on P. When g runs through $SL_2(\mathbb{Z})$ and when P runs through $\mathbb{C}_{k-2}[X, Y]$, the Manin symbols generate $\mathcal{M}_k(\Gamma)$. Furthermore they verify the following equalities:*

$$[P, g] + [P_{|\sigma^{-1}}, g\sigma] = 0,$$

$$[P, g] + [P_{|\tau^{-1}}, g\tau] + [P_{|\tau^{-2}}, g\tau^2] = 0$$

and

$$[P, g] = [P_{|J}, -g].$$

Proof .- The first assertion is a consequence of the construction of $\mathcal{M}_k(\Gamma)$. To prove the second assertion, we make use of a theorem of Manin which asserts that the elements $\{g0, g\infty\}$, for $g \in SL_2(\mathbb{Z})$ generate $\mathcal{M}$ (This is known as "Manin's trick" [4], proposition 1.6. The proof of that assertion relies on continued fractions expansion). It follows quickly that the Manin symbols generate $\mathcal{M}_k(\Gamma)$. To prove the first two equalities we remark the following relations in $\mathbb{P}^1(\mathbb{Q})$: $\infty = \sigma 0 = \sigma^2 \infty$ and $\infty = \tau 1 = \tau^2 0$. We have

$$\begin{aligned} [P, g] + [P_{|\sigma^{-1}}, g\sigma] &= P_{|g}\{g0, g\infty\} + (P_{|\sigma^{-1}})_{|g\sigma}\{g\sigma 0, g\sigma\infty\} \\ &= P_{|g}\{g0, g\infty\} + P_{|g\sigma\sigma^{-1}}\{g\infty, g0\} \\ &= 0 \end{aligned}$$

and

$$\begin{aligned}
&[P,g]+[P_{|\tau^{-1}},g\tau]+[P_{|\tau^{-2}},g\tau^2]\\
&= P_{|g}\{g0,g\infty\}+(P_{|\tau^{-1}})_{|g\tau}\{g\tau 0,g\tau\infty\}+(P_{|\tau^{-2}})_{|g\tau^2}\{g\tau^2 0,g\tau^2\infty\}\\
&= P_{|g}(\{g0,g\infty\}+\{g1,g0\}+\{g\infty,g1\}\\
&= 0.
\end{aligned}$$

The third equality follows from the fact that the matrix J operates trivially on $\mathbb{P}^1(\mathbb{Q})$.

Remark .- The Manin symbols can be written as complex linear combinations of Manin symbols of type $[P,g]$, where P is a monomial of the form X^qY^{k-2-q} ($q\in\{0,1,...,k-2\}$). Since these monomials are in finite number and since $\Gamma\backslash SL_2(\mathbb{Z})$ is finite, $\mathcal{M}_k(\Gamma)$ is generated by a finite number of elements. So it is a finite dimensional complex vector space.

1.3 Comparison with the theory of Shokurov

We recall the theory of Shokurov (see [12], [11], [14], [13], especially the lemma-definition 1.2 in the last paper). We use here the notations of Shokurov. Let $\mathfrak{H}$ be the Poincaré half-plane. Let Δ_Γ be the compactification of the Riemann surface $\Gamma\backslash\mathfrak{H}$. Let Π be the set of cusps of Δ_Γ. Shokurov considers a certain locally constant sheaf, which he denotes $(R_1\phi_*\mathbb{Q})^w$, on the modular curve Δ_Γ (where $w=k-2$, see [13]). This sheaf is the usual sheaf on the modular curve Δ_Γ constructed from the space of modular forms of weight k for Γ; But there is no need to describe it here.

Let $(\alpha,\beta)\in(\mathbb{P}^1(\mathbb{Q}))^2$ and $(n,m)\in(\mathbb{Z}^{k-2})^2$. The modular symbol of Shokurov $\{\alpha,\beta,n,m\}_\Gamma$ (which we will call provisionnally Shokurov symbol) lies in the homology group $\mathrm{H}_1(\Delta_\Gamma,\Pi;(R_1\phi_*\mathbb{Q})^w)$, which is a $\mathbb{Q}$-vector space. The Shokurov symbols enjoy the following properties (see lemma-definition 1.2 of [13]):

- The Shokurov symbol $\{\alpha,\beta,n,m\}_\Gamma$ depends only on the polynomial $\prod_{i=1}^{k-2}(n_iX+m_iY)\in\mathbb{C}_{k-2}[X,Y]$, where $n=(n_1,n_2,...,n_{k-2})$ and $m=(m_1,m_2,...,m_{k-2})$.
- They generate the $\mathbb{Q}$-vector space $\mathrm{H}_1(\Delta_\Gamma,\Pi;(R_1\phi_*\mathbb{Q})^w)$.
- They satisfy the following relations

  $$\{\alpha,\beta,n,m\}_\Gamma+\{\beta,\gamma,n,m\}_\Gamma+\{\gamma,\alpha,n,m\}_\Gamma=0,$$

  $(\alpha,\beta,\gamma)\in\mathbb{P}^1(\mathbb{Q})^3$, $(n,m)\in(\mathbb{Z}^{k-2})^2$.
- For $\gamma=\begin{pmatrix}a&b\\c&d\end{pmatrix}\in\Gamma$, we have

  $$\{\gamma\alpha,\gamma\beta,dn-cm,-bn+am\}_\Gamma=\{\alpha,\beta,n,m\}_\Gamma,$$

  $(\alpha,\beta)\in\mathbb{P}^1(\mathbb{Q})^2$, $(n,m)\in(\mathbb{Z}^{k-2})^2$.

Our present goal is to prove that the Shokurov symbols can be identified with our modular symbols. Because of the first property satisfied by the Shokurov symbols, we can denote by $\{\alpha,\beta,P\}$ the symbol $\{\alpha,\beta,n,m\}_\Gamma$, where $P=\prod_{i=1}^n(n_iX+m_iY)\in\mathbb{Z}_{k-2}[X,Y]$. The last two properties can be rewritten

$$\{\alpha,\beta,P\}+\{\beta,\gamma,P\}+\{\gamma,\alpha,P\}=0$$

and

$$\{\gamma\alpha,\gamma\beta,P_{|\gamma}\}=\{\alpha,\beta,P\}.$$

We deduce from these properties that there exists a surjective homomorphism of complex vector spaces:

$$s_1:\mathcal{M}_k(\Gamma)\to \mathrm{H}_1(\Delta_\Gamma,\Pi;(R_1\phi_*\mathbb{Q})^w)\otimes_{\mathbb{Q}}\mathbb{C}$$

which associates $P\{\alpha,\beta\}$ to $\{\alpha,\beta,P\}\otimes 1$, for P of the form $\prod_{i=1}^{k-2}(n_iX+m_iY)\in\mathbb{Z}_{k-2}[X,Y]$, where $n=(n_1,n_2,...,n_{k-2})\in\mathbb{Z}^{k-2}$ and $m=(m_1,m_2,...,m_{k-2})\in\mathbb{Z}^{k-2}$.

Proposition 2 *The linear map s_1 is an isomorphism of complex vector spaces.*

Proof .- For $j\in\Gamma\backslash SL_2(\mathbb{Z})$, and $q\in\{0,1,...,k-2\}$, Shokurov denotes by $\xi(j,q)$ (and calls *marked class*) the element $s_1([X^qY^{k-2-q}])$ of $\mathrm{H}_1(\Delta_\Gamma,\Pi;(R_1\phi_*\mathbb{Q})^w)\otimes_{\mathbb{Q}}\mathbb{C}$, where g is any element of the class j.

After extending the scalars from $\mathbb{Q}$ to $\mathbb{C}$, the theorem 2.3 of [13] asserts that the kernel of the linear map

$$\mathbb{C}_{k-2}[X,Y][\Gamma\backslash SL_2(\mathbb{Z})]\to \mathrm{H}_1(\Delta_\Gamma,\Pi;(R_1\phi_*\mathbb{Q})^w)\otimes_{\mathbb{Q}}\mathbb{C}$$

which associates to $P[\Gamma g]$ the element $s_1([P,g])$ is generated by the elements $s_1([P,g])+s_1([P_{|\sigma^{-1}},g\sigma])$, $s_1([P,g])+s_1([P_{|\tau^{-1}},g\tau])+s_1([P_{|\tau^{-2}},g\tau^2])$ and $s_1([P,g])-s_1([P_{|J},-g])$, where P is a monomial, i.e. $s_1([P,g])$ is a marked class. It is a consequence of the proposition 1 that this linear map factorizes through

$$\mathbb{C}_{k-2}[X,Y][\Gamma\backslash SL_2(\mathbb{Z})]\to\mathcal{M}_k(\Gamma)\xrightarrow{s_1}\mathrm{H}_1(\Delta_\Gamma,\Pi;(R_1\phi_*\mathbb{Q})^w)\otimes_{\mathbb{Q}}\mathbb{C},$$

where the first linear map associates to $P[\Gamma g]$ the Manin symbol $[P,g]$. By considering the kernels of these linear maps, we obtain that s_1 is injective.

Let $\mathbb{C}_{k-2}[X,Y][\Gamma\backslash SL_2(\mathbb{Z})]_k$ be the quotient vector space of $\mathbb{C}_{k-2}[X,Y][\Gamma\backslash SL_2(\mathbb{Z})]$ obtained by imposing the relations $P[j]=P_{|J}[j(J)]$, $j\in\Gamma\backslash SL_2(\mathbb{Z})$, $P\in\mathbb{C}_{k-2}[X,Y]$. We define a linear action on the right of $SL_2(\mathbb{Z})$ on $\mathbb{C}_{k-2}[X,Y][\Gamma\backslash SL_2(\mathbb{Z})]_k$ by the rule

$$(P[\Gamma g])\gamma=P_{|\gamma^{-1}}[\Gamma g\gamma],$$

$P\in\mathbb{C}_{k-2}[X,Y]$, $(g,\gamma)\in SL_2(\mathbb{Z})^2$.

We denote by $\mathbb{C}_{k-2}[X,Y][\Gamma\backslash SL_2(\mathbb{Z})]_k^\sigma$ (resp. $\mathbb{C}_{k-2}[X,Y][\Gamma\backslash SL_2(\mathbb{Z})]_k^\tau$) the set of elements of $\mathbb{C}_{k-2}[X,Y][\Gamma\backslash SL_2(\mathbb{Z})]_k$ invariant under the action of σ (resp. τ).

Proposition 3 *We have the following exact sequences of complex vector spaces*

$$0 \to \mathbb{C}_{k-2}[X,Y][\Gamma\backslash SL_2(\mathbb{Z})]_k^\sigma \times \mathbb{C}_{k-2}[X,Y][\Gamma\backslash SL_2(\mathbb{Z})]_k^\tau \xrightarrow{i} \\ \mathbb{C}_{k-2}[X,Y][\Gamma\backslash SL_2(\mathbb{Z})]_k \xrightarrow{m_k} \mathcal{M}_k(\Gamma) \to 0,$$

if $k > 2$, and

$$0 \to \mathbb{C} \xrightarrow{\epsilon} \mathbb{C}_{k-2}[X,Y][\Gamma\backslash SL_2(\mathbb{Z})]_k^\sigma \times \mathbb{C}_{k-2}[X,Y][\Gamma\backslash SL_2(\mathbb{Z})]_k^\tau \xrightarrow{i} \\ \mathbb{C}_{k-2}[X,Y][\Gamma\backslash SL_2(\mathbb{Z})]_k \xrightarrow{m_k} \mathcal{M}_k(\Gamma) \to 0,$$

if $k = 2$, where m_k is the map which associates to $P[\Gamma g]$ the Manin symbol $[P,g]$, i associates to (a,b) the element $a+b$, in the last exact sequence the linear map ϵ associates to λ the element $(\sum_{x\in E_N} \lambda[x], \sum_{x\in E_N} \lambda[x])$.

Proof .- By a theorem of Shokurov (see theorem 2.3 of [13], this result has already been used in the proof of the proposition 2), the kernel of m_k is equal to the sum of the complex vector spaces $\mathbb{C}_{k-2}[X,Y][\Gamma\backslash SL_2(\mathbb{Z})]_k^\sigma$ and $\mathbb{C}_{k-2}[X,Y][\Gamma\backslash SL_2(\mathbb{Z})]_k^\tau$. The surjectivity of m_k is proved by the proposition 1. It remains to prove that the intersection of $\mathbb{C}_{k-2}[X,Y][\Gamma\backslash SL_2(\mathbb{Z})]_k^\sigma$ and $\mathbb{C}_{k-2}[X,Y][\Gamma\backslash SL_2(\mathbb{Z})]_k^\tau$ is equal to $\{0\}$ if $k>2$ and is equal to the image of ϵ if $k=2$. Let $\sum_{j\in\Gamma\backslash SL_2(\mathbb{Z})} P_j[j] \in \mathbb{C}_{k-2}[X,Y][\Gamma\backslash SL_2(\mathbb{Z})]_k^\sigma \cap \mathbb{C}_{k-2}[X,Y][\Gamma\backslash SL_2(\mathbb{Z})]_k^\tau$. We have $P_{j|\sigma} = P_{(j\sigma)}$ and $P_{j|\tau} = P_{(j\tau)}$ for all $j \in \Gamma\backslash SL_2(\mathbb{Z})$. Since σ and τ generate $SL_2(\mathbb{Z})$ (see [10]), we have $P_{(jg)} = P_j = P_{j|g}$ for all $j \in \Gamma\backslash SL_2(\mathbb{Z})$ and all $g \in SL_2(\mathbb{Z})$. This implies that P_j is a constant polynomial, i.e. $P=0$ or $k=2$. If $k=2$, then P_j is a complex number independent of j ($SL_2(\mathbb{Z})$ operates transitively on $\Gamma\backslash SL_2(\mathbb{Z})$). So $\sum_{j\in\Gamma\backslash SL_2(\mathbb{Z})} P_j[j]$ is in the image of ϵ. This proves the proposition.

1.4 The boundary modular symbols

Let $\mathcal{B}$ be the abelian group generated by the elements $\{\alpha\}$ ($\alpha \in \mathbb{P}^1(\mathbb{Q})$). Let $\mathcal{B}_k$ be the complex vector space $\mathbb{C}_{k-2}[X,Y]\otimes\mathcal{B}$. We define a linear action of $g \in GL_2(\mathbb{Q})$ on $P\otimes\{\alpha,\beta\} \in \mathcal{B}_k$ by the formula

$$(P\otimes\{\alpha\})_{|g} = P_{|g}\otimes\{g\alpha\}.$$

We define $\mathcal{B}_k(\Gamma)$ as the complex vector space quotient of $\mathcal{B}_k$ obtained by imposing the relations

$$(P\otimes\{\alpha\})_{|\gamma} = P\otimes\{\alpha\},$$

for $\gamma\in\Gamma$.

For $\alpha \in \mathbb{P}^1(\mathbb{Q})$ and $(n,m) \in (\mathbb{Z}^{k-2})^2$, Shokurov introduces the boundary symbol $\{\alpha,n,m\}_\Gamma$ which is an element of the $\mathbb{Q}$-vector space $\mathrm{H}_0(\Pi;(R_1\phi_*\mathbb{Q})^w)$. This symbol depends only on α and on $\prod_{i=1}^{k-2}(n_iX+m_iY) \in \mathbb{C}_{k-2}[X,Y]$. For $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$, we have

$$\{\gamma\alpha, dn-cm, -bn+am\}_\Gamma = \{\alpha,n,m\}_\Gamma.$$

These symbols generate the $\mathbb{Q}$-vector space $\mathrm{H}_0(\Pi;(R_1\phi_*\mathbb{Q})^w)$. See the lemma-definition 1.2 of [13] for all these properties. We will denote by $\{\alpha, \prod_{i=1}^{k-2}(n_iX+m_iY)\}$ the boundary symbol $\{\alpha,n,m\}_\Gamma$.

Let R_k be the equivalence relation in $\mathbb{C}[\Gamma\backslash\mathbb{Q}^2]$ which identifies the element $[\Gamma\begin{pmatrix}\lambda u\\ \lambda v\end{pmatrix}]$ with $(\frac{\lambda}{|\lambda|})^k[\Gamma\begin{pmatrix}u\\ v\end{pmatrix}]$ $(\lambda\in\mathbb{Q}-\{0\})$. We denote by $\mathbb{C}[\Gamma\backslash\mathbb{Q}^2]_k$ the finite dimensional complex vector space $\mathbb{C}[\Gamma\backslash\mathbb{Q}^2]/R_k$. If k is even, this vector space is canonically isomorphic to $\mathbb{C}[\Gamma\backslash\mathbb{P}^1(\mathbb{Q})]$. In any event its dimension is equal to $|\Gamma\backslash\mathbb{P}^1(\mathbb{Q})|$, i.e. the number of cusps of the modular curve Δ_Γ.

Because of the properties satisfied by the boundary symbols of Shokurov there is an unique surjective linear map:

$$s_2\ :\ \mathcal{B}_k(\Gamma)\to\mathrm{H}_0(\Pi;(R_1\phi_*\mathbb{Q})^w)\otimes_\mathbb{Q}\mathbb{C}$$

which associates to $P\{\alpha\}$ the boundary symbol $\{\alpha,P\}\otimes 1$ for P a polynomial of the form $\prod_{i=1}^{k-2}(n_iX+m_iY)\in\mathbb{C}_{k-2}[X,Y]$, where $n=(n_1,n_2,...,n_{k-2})\in\mathbb{Z}^{k-2}$ and $m=(m_1,m_2,...,m_{k-2})\in\mathbb{Z}^{k-2}$.

We consider now the complex linear map

$$\mu\ :\ \mathcal{B}_k(\Gamma)\to\mathbb{C}[\Gamma\backslash\mathbb{Q}^2]_k,$$

which associates to $P\{\frac{u}{v}\}$ the element $P(u,v)[\Gamma\begin{pmatrix}u\\ v\end{pmatrix}]$ of $\mathbb{C}[\Gamma\backslash\mathbb{Q}^2]_k$. The fact that μ is well defined is a part of the following proposition.

Proposition 4 *The linear maps s_2 and μ are isomorphisms of complex vector spaces.*

Proof .- Because of the way we constructed $\mathcal{B}_k(\Gamma)$, we can decompose it as a direct sum

$$\mathcal{B}_k(\Gamma)=\oplus_\alpha\mathcal{B}_k(\Gamma)_\alpha,$$

where α runs through a set of representatives of $\Gamma\backslash\mathbb{P}^1(\mathbb{Q})$ and $\mathcal{B}_k(\Gamma)_\alpha$ is the subspace of $\mathcal{B}_k(\Gamma)$ generated by the elements $P\{\alpha\}$ $(P\in\mathbb{C}_{k-2}[X,Y])$. For $\alpha\in\mathbb{P}^1(\mathbb{Q})$, we have a surjective linear map

$$\begin{array}{rcl}\psi_\alpha\ :\ \mathbb{C}_{k-2}[X,Y] & \to & \mathcal{B}_k(\Gamma)_\alpha\\ P & \mapsto & P\{\alpha\}.\end{array}$$

Its kernel is generated by the polynomials of the form $P-P_{|g_\alpha}$, where $g_\alpha\in\Gamma$ verifies the equality $g_\alpha\alpha=\alpha$. If we write $\alpha=\frac{u}{v}$ $((u,v)\in\mathbb{Z}^2)$, we find $(P-P_{|g_\alpha})(u,v)=0$. This proves that μ is well defined. Since the kernel of ψ_α contains the kernel of a linear form, its image is at most of dimension 1. The dimension of $\mathcal{B}_k(\Gamma)$ is at most $|\Gamma\backslash\mathbb{P}^1(\mathbb{Q})|$. Since the dimension of $\mathrm{H}_0(\Pi;(R_1\phi_*\mathbb{Q})^w)\otimes_\mathbb{Q}\mathbb{C}$ is exactly $|\Gamma\backslash\mathbb{P}^1(\mathbb{Q})|$ (see [13], theorem 3.4), the map s_2 must be bijective, and the dimension of $\mathcal{B}_k(\Gamma)$ must be $|\Gamma\backslash\mathbb{P}^1(\mathbb{Q})|$. We deduce that the image of each map ψ_α is of dimension 1. So the linear map μ must be bijective.

Shokurov considers the canonical boundary map:

$$\partial \ : \ \mathrm{H}_1(\Delta_\Gamma, \Pi; (R_1\phi_*\mathbb{Q})^w) \otimes_{\mathbb{Q}} \mathbb{C} \to \mathrm{H}_0(\Pi; (R_1\phi_*\mathbb{Q})^w) \otimes_{\mathbb{Q}} \mathbb{C}.$$

We have ([13], lemma-definition 1.2)

$$\partial(\{\alpha, \beta, n, m\} \otimes 1) = \{\beta, n, m\} \otimes 1 - \{\alpha, n, m\} \otimes 1.$$

The kernel of ∂ is canonically isomorphic to $\mathrm{H}_1(\Delta_\Gamma; (R_1\phi_*\mathbb{Q})^w) \otimes \mathbb{C}$. By abuse of notations, we still denote by ∂ the linear map $\mathcal{M}_k(\Gamma) \to \mathcal{B}_k(\Gamma)$ obtained from ∂ by the identifications made. We denote by $\mathcal{S}_k(\Gamma)$ the kernel of ∂ in $\mathcal{M}_k(\Gamma)$. We have

$$\partial(P\{\alpha, \beta\}) = P\{\beta\} - P\{\alpha\}.$$

Proposition 5 *We have the exact sequences*

$$0 \to \mathcal{S}_k(\Gamma) \to \mathcal{M}_k(\Gamma) \xrightarrow{\partial} \mathcal{B}_k(\Gamma) \to 0,$$

if $k > 2$, *and*

$$0 \to \mathcal{S}_k(\Gamma) \to \mathcal{M}_k(\Gamma) \xrightarrow{\partial} \mathcal{B}_k(\Gamma) \xrightarrow{\theta} \mathbb{C} \to 0,$$

if $k = 2$, *where the linear form* θ *associates to* $\lambda\{\alpha\}$ *the complex number* λ.

Proof .- Let us prove that ∂ is surjective if $k > 2$. Let $\beta = \frac{u}{v} \in \mathbb{P}^1(\mathbb{Q})$. Let $\lambda \in \mathbb{C}$. Let $\alpha = \frac{u'}{v'} \in \mathbb{P}^1(\mathbb{Q})$ such that $\alpha \neq \beta$. Let $P \in \mathbb{C}_{k-2}[X, Y]$ such that $P(u, v) = \lambda$ and $P(u', v') = 0$ (since $k > 2$ such a polynomial exists). We have in $\mathbb{C}[\Gamma\backslash\mathbb{Q}^2]_k$

$$\mu \circ \partial(P\{\alpha, \beta\}) = P(u, v)[\Gamma\beta] - P(u', v')[\Gamma\alpha] = \lambda[\Gamma\beta].$$

So ∂ is surjective.

If $k = 2$, the map θ is well defined and the image of ∂ is contained in the kernel of θ. The image of ∂ and the kernel of θ are both generated by elements of the type $\{\beta\} - \{\alpha\}$. Since θ is surjective we deduce the validity of the second exact sequence.

We give now a formula for the boundary of Manin symbols.

Proposition 6 *Let* $P \in \mathbb{C}_{k-2}[X, Y]$ *and* $g \in SL_2(\mathbb{Z})$. *We have in* $\mathbb{C}[\Gamma\backslash\mathbb{Q}^2]_k$

$$\mu \circ \partial([P, g]) = P(1, 0)[\Gamma g \begin{pmatrix} 1 \\ 0 \end{pmatrix}] - P(0, 1)[\Gamma g \begin{pmatrix} 0 \\ 1 \end{pmatrix}].$$

Proof .- We set $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. We have

$$\begin{aligned} \partial([P, g]) &= \partial(P_{|g}\{\frac{b}{d}, \frac{a}{c}\}) \\ &= P_{|g}\{\frac{a}{c}\} - P_{|g}\{\frac{b}{d}\} \end{aligned}$$

and

$$\begin{aligned}\mu \circ \partial([P, g]) &= P_{|g}(a, c)[\Gamma \begin{pmatrix} a \\ c \end{pmatrix}] - P_{|g}(b, d)[\Gamma \begin{pmatrix} b \\ d \end{pmatrix}] \\ &= P(1, 0)[\Gamma g \begin{pmatrix} 1 \\ 0 \end{pmatrix}] - P(0, 1)[\Gamma g \begin{pmatrix} 0 \\ 1 \end{pmatrix}].\end{aligned}$$

1.5 Pairings with modular forms

Let f be a map from the Poincaré half plane to $\mathbb{C}$. For $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{Q})$ and $z \in \mathfrak{H}$, we recall the usual notations

$$f_{|g}(z) = (cz + d)^{-k} f(gz)(\det g)^{\frac{k}{2}}$$

and

$$f_{|\bar{g}}(z) = (c\bar{z} + d)^{-k} f(gz)(\det g)^{\frac{k}{2}},$$

where the horizontal bar above complex numbers denotes the complex conjugation. We denote by $S_k(\Gamma)$ (resp. $\overline{S_k(\Gamma)}$) the complex vector space of holomorphic (resp. antiholomorphic) cusp forms of weight k for the group Γ. There is a canonical isomorphism of real vector spaces between $S_k(\Gamma)$ and $\overline{S_k(\Gamma)}$ which associates to f the antiholomorphic modular form $z \mapsto \overline{f(z)}$; we denote by $\bar{f}$ this antiholomorphic modular form.

Shokurov defines a pairing ([13], lemma-definition 1.2)

$$(S_k(\Gamma) \oplus \overline{S_k(\Gamma)}) \times \mathrm{H}_1(\Delta_\Gamma, \Pi; (R_1\phi_*\mathbb{Q})^w) \to \mathbb{C}$$

given by the formula

$$(f_1 + f_2, \{\alpha, \beta, n, m\}_\Gamma) \mapsto \int_\alpha^\beta f_1(z) \prod_{i=1}^{k-2} (n_i z + m_i)\, dz + \int_\alpha^\beta f_2(z) \prod_{i=1}^{k-2} (n_i \bar{z} + m_i)\, d\bar{z}$$

($f_1 \in S_k(\Gamma)$, $f_2 \in \overline{S_k(\Gamma)}$), where the path is, except for the extremities, contained in the upper half-plane. Using the identifications of section 1.3, we deduce from this a pairing of complex vector spaces

$$(S_k(\Gamma) \oplus \overline{S_k(\Gamma)}) \times \mathcal{M}_k(\Gamma) \to \mathbb{C}$$

given by the rule

$$(f_1 + f_2, P\{\alpha, \beta\}) \mapsto < f_1 + f_2, P\{\alpha, \beta\} > = \int_\alpha^\beta f_1(z) P(z, 1)\, dz + \int_\alpha^\beta f_2(z) P(\bar{z}, 1)\, d\bar{z}$$

$f_1 \in S_k(\Gamma)$, $f_2 \in \overline{S_k(\Gamma)}$, $(\alpha, \beta) \in \mathbb{P}^1(\mathbb{Q})^2$, $P \in \mathbb{C}_{k-2}[X, Y]$. In general, the pairing $< .,. >$ is degenerate. But we have the following theorem, which is essentially a restatement of the theorem 0.2 of [14].

Theorem 3 *The bilinear pairing $< .,. >$ is nondegenerate if restricted to a pairing*

$$(S_k(\Gamma) \oplus \overline{S_k(\Gamma)}) \times \mathcal{S}_k(\Gamma) \to \mathbb{C}.$$

Proof .- Shokurov proves([14], theorem 0.2) that the bilinear pairing between $(S_k(\Gamma) \oplus \overline{S_k(\Gamma)})$ and $\mathrm{H}_1(\Delta_\Gamma; (R_1\phi_*\mathbb{Q})^w) \otimes_{\mathbb{Q}} \mathbb{C}$ is nondegenerate. The theorem 3 is deduced from the identification between $\mathrm{H}_1(\Delta_\Gamma; (R_1\phi_*\mathbb{Q})^w) \otimes_{\mathbb{Q}} \mathbb{C}$ and $\mathcal{S}_k(\Gamma)$.

Remark .- 1) Let $f_1 \in S_k(\Gamma)$ and $f_2 \in \overline{S_k(\Gamma)}$. Let $g \in SL_2(\mathbb{Z})$ and $P \in \mathbb{C}_{k-2}[X,Y]$. We have

$$< f_1 + f_2, [P,g] >= \int_0^\infty f_{1|g}(z)P(z,1)\,dz + \int_0^\infty f_{2|\bar{g}}(z)P(\bar{z},1)\,d\bar{z}.$$

So we find the formula given in the introduction.

2) The pairing $< .,. >$ is degenerate in general. The space of elements $m \in \mathcal{M}_k(\Gamma)$ such that $< f, m >= 0$ for all $f \in (S_k(\Gamma) \oplus \overline{S_k(\Gamma)})$ is the space of *Eisenstein elements*. By a theorem of Shokurov, it admits a basis in $\mathcal{M}_k(\Gamma,\mathbb{Z})$ ([13], theorem 4.3 and corollary 4.4, see the remark in the section 1.1 for the meaning of $\mathcal{M}_k(\Gamma,\mathbb{Z})$). It would be interesting to find expressions in terms of Manin symbols of elements of such a basis.

3) The dimension of $\mathcal{M}_k(\Gamma)$ is equal to twice the dimension of $S_k(\Gamma)$ plus the dimension of the space of Eisenstein series of weight k for Γ. Using the exact sequences appearing in the proposition 3, one can find the dimension of these spaces of modular forms.

1.6 The action of the complex conjugation

In this section we suppose that the matrix $\eta = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ normalizes the group Γ.

Proposition 7 *The map ι which associates to $f \in S_k(\Gamma) \oplus \overline{S_k(\Gamma)}$ the function $z \mapsto f(-\bar{z}) = f(\eta\bar{z}) = f_{|\eta}(\bar{z}) = f_{|\bar{\eta}}(\bar{z})$ is a complex linear involution of $S_k(\Gamma) \oplus \overline{S_k(\Gamma)}$ which exchanges $S_k(\Gamma)$ and $\overline{S_k(\Gamma)}$.*

The involution ι^* on $\mathcal{M}_k(\Gamma)$ given by the rule

$$\iota^*(P\{\alpha,\beta\}) = -P_{|\tilde{\eta}}\{\eta\alpha,\eta\beta\}$$

($P \in \mathbb{C}_{k-2}[X,Y]$, $(\alpha,\beta) \in \mathbb{P}^1(\mathbb{Q})^2$) is adjoint to ι with respect to the pairing $< .,. >$ of section 1.5. Moreover ι^* acts as follows on Manin symbols

$$\iota^*([P,g]) = -[P_{|\tilde{\eta}}, \eta g \eta^{-1}].$$

Proof .- Let $f \in S_k(\Gamma)$. Let $\gamma \in \Gamma$. We have, since $\eta^{-1}\gamma\eta \in \Gamma$,

$$\iota(f)_{|\bar{\gamma}} = f_{|\overline{\gamma\eta}} = f_{|\overline{\eta\eta^{-1}\gamma\eta}} = f_{|\bar{\eta}} = \iota(f).$$

So $\iota(f)$ is an antiholomorphic modular from. The analogous result with respect to $f \in \overline{S_k(\Gamma)}$ can be proved similarly.

Since η normalizes Γ, the definition of ι^* is compatible with the construction of $\mathcal{M}_k(\Gamma)$. Let $f \in S_k(\Gamma)$, $P \in \mathbb{C}_{k-2}[X,Y]$ and $(\alpha,\beta) \in \mathbb{P}^1(\mathbb{Q})^2$. We have

$$\begin{aligned}
< \iota(f), P\{\alpha,\beta\} > &= \int_\alpha^\beta f(-\bar{z})P(\bar{z},1)\, d\bar{z} \\
&= -\int_{-\alpha}^{-\beta} f(z)P(-z,1)\, dz \\
&= < f, -P_{|\tilde{\eta}}\{\eta\alpha,\eta\beta\} > \\
&= < f, \iota^*(P\{\alpha,\beta\}) > .
\end{aligned}$$

The same equalities hold with respect to $f \in \overline{S_k(\Gamma)}$.

Finally we prove the formula with respect to the Manin symbols. We have

$$\begin{aligned}
\iota^*([P,g]) &= \iota^*(P_{|g}\{g0, g\infty\}) \\
&= -(P_{|g})_{|\tilde{\eta}}\{\eta g0, \eta g\infty\} \\
&= -P_{|\tilde{\eta}\tilde{\eta}^{-1}g\tilde{\eta}}\{\eta g\eta^{-1}\eta 0, \eta g\eta^{-1}\eta\infty\} \\
&= -(P_{|\tilde{\eta}})_{|\tilde{\eta}^{-1}g\tilde{\eta}}\{\eta g\eta^{-1}0, \eta g\eta^{-1}\infty\} \\
&= -(P_{|\tilde{\eta}})_{|\eta^{-1}g\eta}\{\eta g\eta^{-1}0, \eta g\eta^{-1}\infty\} \\
&= -[P_{|\tilde{\eta}}, \eta g\eta^{-1}].
\end{aligned}$$

Let $\mathcal{S}_k(\Gamma)^+$ (resp. $\mathcal{S}_k(\Gamma)^-$) be the subspace of $\mathcal{S}_k(\Gamma)$ constituted by the elements of $\mathcal{S}_k(\Gamma)$ invariant (resp. antiinvariant) under the action of ι^*. We have a direct sum

$$\mathcal{S}_k(\Gamma) = \mathcal{S}_k(\Gamma)^+ \oplus \mathcal{S}_k(\Gamma)^-.$$

Proposition 8 *The bilinerar pairings induced by the pairing $< .,. >$ on*

$$S_k(\Gamma) \times \mathcal{S}_k(\Gamma)^+ \to \mathbb{C} \quad \textit{and} \quad S_k(\Gamma) \times \mathcal{S}_k(\Gamma)^- \to \mathbb{C}$$

respectively are nondegenerate.

Proof .- Let $(S_k(\Gamma) \oplus \overline{S_k(\Gamma)})^+$ be the subspace of $S_k(\Gamma) \oplus \overline{S_k(\Gamma)}$ invariant under the action of ι. The pairing $< .,. >$ restricted to $(S_k(\Gamma) \oplus \overline{S_k(\Gamma)})^+ \times \mathcal{S}_k(\Gamma)^+ \to \mathbb{C}$ is nondegenerate. Since the map

$$\begin{aligned}
S_k(\Gamma) &\to (S_k(\Gamma) \oplus \overline{S_k(\Gamma)})^+ \\
f &\mapsto f + \iota(f)
\end{aligned}$$

is an isomorphism, we deduce the proposition from the equality

$$< f + \iota(f), x > = < f, x > + < f, \iota^*(x) > = 2 < f, x >$$

($f \in S_k(\Gamma)$, $x \in \mathcal{S}_k(\Gamma)^+$). The assertion concerning $\mathcal{S}_k(\Gamma)^-$ can be proved in a similar way.

1.7 Connection with the cohomology of $SL_2(\mathbb{Z})$

The complex vector space $\mathbb{C}_{k-2}[X,Y][\Gamma\backslash SL_2(\mathbb{Z})]_k$ is endowed with an action on the right of $SL_2(\mathbb{Z})$ given by the formula

$$(P[\Gamma g])\gamma = P_{|\tilde{\gamma}}[\Gamma g\gamma].$$

Proposition 9 *We have an isomorphism of complex vector spaces*

$$H^1(SL_2(\mathbb{Z}), \mathbb{C}_{k-2}[X,Y][\Gamma\backslash SL_2(\mathbb{Z})]) \simeq \mathcal{M}_k(\Gamma).$$

Proof .- We note $M = \mathbb{C}_{k-2}[X,Y][\Gamma\backslash SL_2(\mathbb{Z})]$. Let M^σ (resp. M^τ) be the subspace of M constituted by elements of M fixed under the action of σ (resp. τ). In view of the exact sequences of the proposition 5, we only have to prove that there is an isomorphism of complex vector spaces

$$\mathrm{H}^1(SL_2(\mathbb{Z}), \mathbb{C}_{k-2}[X,Y][\Gamma\backslash SL_2(\mathbb{Z})]) \simeq M/(M^\sigma + M^\tau).$$

We recall that $\mathrm{Z}^1(SL_2(\mathbb{Z}), M)$ (resp. $\mathrm{B}^1(SL_2(\mathbb{Z}), M)$) is the complex vector space of functions $\phi : SL_2(\mathbb{Z}) \to M$ such that $\phi(gh) = \phi(g)h + \phi(h)$, $(g,h) \in SL_2(\mathbb{Z})^2$ (resp. such that there exists $m \in M$ with $\phi(g) = m - mg$, $g \in SL_2(\mathbb{Z})$). We have $\mathrm{H}^1(SL_2(\mathbb{Z}), M) = \mathrm{Z}^1(SL_2(\mathbb{Z}), M)/\mathrm{B}^1(SL_2(\mathbb{Z}), M)$.

Since $SL_2(\mathbb{Z})$ is the free product of the groups generated respectively by σ and by τ, the map

$$\begin{array}{rcl} \mathrm{Z}^1(SL_2(\mathbb{Z}), M) & \to & \ker(1+\sigma+\sigma^2+\sigma^3) \times \ker(1+\tau+\tau^2) \\ \phi & \mapsto & (\phi(\sigma), \phi(\tau)) \end{array}$$

is an isomorphism of complex vector spaces ($\ker(1+\sigma+\sigma^2+\sigma^3)$ and $\ker(1+\tau+\tau^2)$ denote the kernels in M of the multiplications on the right by $(1+\sigma+\sigma^2+\sigma^3)$ and $(1+\tau+\tau^2)$ respectively). Since M is a complex vector space and since $\sigma^4 = \tau^3 = 1$, we have $\ker(1+\sigma+\sigma^2+\sigma^3) = M(1-\sigma)$ and $\ker(1+\tau+\tau^2) = M(1-\tau)$. We have an obvious isomorphism between M/M^σ and $M(1-\sigma)$ (resp. between M/M^τ and $M(1-\tau)$). So there is an isomorphism of complex vector spaces $\Lambda : \mathrm{Z}^1(SL_2(\mathbb{Z}), M) \to M/M^\sigma \times M/M^\tau$. Since $SL_2(\mathbb{Z})$ is generated by σ and τ, $\mathrm{B}^1(SL_2(\mathbb{Z}), M)$ is the set of functions $\phi : SL_2(\mathbb{Z}) \to M$ such that there exists $m \in M$ with $\phi(\sigma) = m(1-\sigma)$ and $\phi(\tau) = m(1-\tau)$. The image by Λ of $\mathrm{B}^1(SL_2(\mathbb{Z}), M) \subset \mathrm{Z}^1(SL_2(\mathbb{Z}), M)$ is the diagonal image of M in $M/M^\sigma \times M/M^\tau$. The map

$$\begin{array}{rcl} M/M^\sigma \times M/M^\tau & \to & M/(M^\sigma + M^\tau) \\ (m_1 + M^\sigma, m_2 + M^\tau) & \mapsto & m_1 - m_2 + M^\sigma + M^\tau \end{array}$$

is surjective. We only have to prove that its kernel is precisely $\Lambda(\mathrm{B}^1(SL_2(\mathbb{Z}), M))$ in order to prove the proposition. Let $(a,b) \in M^2$ such that $a - b \in M^\sigma + M^\tau$. There exists $(a', b') \in M^\sigma \times M^\tau$ such that $a - b = a' - b'$. So we have $a - a' = b - b' = c$. We have $a + M^\sigma = c + M^\sigma$ and $b + M^\tau = c + M^\tau$. So $(a + M^\sigma, b + M^\tau)$ belongs to the diagonal image of M in $M/M^\sigma \times M/M^\tau$ and the proposition has been proved.

The space $S_k(\Gamma) \oplus \overline{S_k(\Gamma)}$ is *imbedded*, by the theory of Eichler-Shimura combined with the Shapiro lemma, in $\mathrm{H}^1(SL_2(\mathbb{Z}), \mathbb{C}_{k-2}[X,Y][\Gamma\backslash SL_2(\mathbb{Z})])$. This theory can be extended to include, not only cups forms, but also Eisenstein series. This point of view has been adopted by Wang ([15]).

1.8 Some comments on the the whole space of modular forms

This paper is concerned mainly with cusp forms and does not fully take care of the Eisenstein series. The vector space $\mathcal{M}_k(\Gamma)$ does not correspond to the full space of holomorphic modular forms of weight k for Γ just as $\mathcal{S}_k(\Gamma)$ corresponds $S_k(\Gamma)$. To summarize this one might say that the space of cusp forms is reflected in $\mathcal{M}_k(\Gamma)$ with multiplicity two and the space of Eisenstein series with multiplicity one. We explain informally in this section how to construct a more complete theory.

Let us consider the torsion free abelian group $\tilde{\mathcal{M}}$ generated by the expressions $\widetilde{\{\alpha,\beta\}}$ $((\alpha,\beta) \in (\mathbb{P}^1(\mathbb{Q}))^2)$ with the relations

$$\widetilde{\{\alpha,\beta\}} - \widetilde{\{\gamma,\beta\}} + \widetilde{\{\gamma,\delta\}} - \widetilde{\{\alpha,\delta\}} = 0,$$

(α, β, γ, δ all in $\mathbb{P}^1(\mathbb{Q})$). We set $\tilde{\mathcal{M}}^0 = \{\sum \lambda_{\alpha,\beta}\widetilde{\{\alpha,\beta\}} \in \tilde{\mathcal{M}} / \sum \lambda_{\alpha,\beta} = 0\}$; It is a subgroup of $\tilde{\mathcal{M}}$ of $\mathbb{Z}$-corank 1. Let us notice that in $\tilde{\mathcal{M}}$ we do not have the relation $\widetilde{\{\alpha,\alpha\}} = 0$. There is an obvious exact sequence

$$0 \to \mathbb{Z}[\mathbb{P}^1(\mathbb{Q})] \to \tilde{\mathcal{M}} \to \mathcal{M} \to 0,$$

where the injection associates to $[\alpha]$ the element $\widetilde{\{\alpha,\alpha\}}$ and the surjection associates to the element $\widetilde{\{\alpha,\beta\}}$ of $\tilde{\mathcal{M}}$ the element $\{\alpha,\beta\}$ of $\mathcal{M}$.

We continue the construction just as in section 1.1. We use the notations $\tilde{\mathcal{M}}_k = \mathbb{C}_{k-2}[X,Y] \otimes \tilde{\mathcal{M}}$ and $\tilde{\mathcal{M}}_k^0 = \mathbb{C}_{k-2}[X,Y] \otimes \tilde{\mathcal{M}}^0$. There is a linear action of $GL_2(\mathbb{Q})$ on $\tilde{\mathcal{M}}_k$ and $\tilde{\mathcal{M}}_k^0$ given by the formula $(P \otimes \widetilde{\{\alpha,\beta\}})_{|g} = P_{|g} \otimes \widetilde{\{g\alpha, g\beta\}}$. So we may consider the quotient groups $\tilde{\mathcal{M}}_k(\Gamma)$ and $\tilde{\mathcal{M}}_k^0(\Gamma)$ obtained by quotienting $\tilde{\mathcal{M}}_k$ and $\tilde{\mathcal{M}}_k^0$ by the action of Γ. We denote by $P\widetilde{\{\alpha,\beta\}}$ the image in $\mathcal{M}_k(\Gamma)$ of the element $P \otimes \widetilde{\{\alpha,\beta\}}$ of $\mathcal{M}_k$. When $k > 2$, there is actually a canonical isomorphism between $\tilde{\mathcal{M}}_k(\Gamma)$ and $\tilde{\mathcal{M}}_k^0(\Gamma)$; When $k = 2$, $\tilde{\mathcal{M}}_k^0(\Gamma)$ can be identified with a subspace of $\tilde{\mathcal{M}}_k(\Gamma)$ of codimension one. Moreover one has the exact sequence

$$0 \to \mathcal{B}_k(\Gamma) \to \tilde{\mathcal{M}}_k(\Gamma) \to \mathcal{M}_k(\Gamma) \to 0$$

deduced from the previous exact sequence. Notice that the dimension of the complex vector space $\mathcal{M}_k^0(\Gamma)$ is twice the dimension of the full space of holomorphic modular forms of weight k for Γ.

We introduce the Manin symbol $\widetilde{[P,g]} \in \tilde{\mathcal{M}}_k(\Gamma)$ as the image of $P \otimes \widetilde{\{g0, g\infty\}}$ in $\tilde{\mathcal{M}}_k(\Gamma)$, where $P \in \mathbb{C}_{k-2}[X,Y]$ and $g \in SL_2(\mathbb{Z})$. For $\Lambda = \sum_h u_h[h] \in \mathbb{Z}[SL_2(\mathbb{Z})]$, we denote by $\widetilde{[P,g]} \bullet \Lambda$ the linear combination of Manin symbols $\sum_h u_h \widetilde{[P_{|h^{-1}}, gh]}$ (see the notations of section 1.2). We use the notation

$$\theta = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \in \mathbb{Z}[SL_2(\mathbb{Z})].$$

We have following relations for all Manin symbols in $\mathcal{M}_k(\Gamma)$:

$$\widetilde{[P,g]} \bullet \theta = 0$$

and

$$\widetilde{[P,g]} - \widetilde{[P,g]} \bullet (J) = 0.$$

Let us consider again the vector space $\mathbb{C}_{k-2}[X,Y][\Gamma\backslash SL_2(\mathbb{Z})]_k$ of the section 1.3. Let us denote by $\mathbb{C}_{k-2}[X,Y][\Gamma\backslash SL_2(\mathbb{Z})]_k^\theta$ the image of $\mathbb{C}_{k-2}[X,Y][\Gamma\backslash SL_2(\mathbb{Z})]_k$ under the action of the right of θ (there is a unique action of $\mathbb{Z}[SL_2(\mathbb{Z})]$ on $\mathbb{C}_{k-2}[X,Y][\Gamma\backslash SL_2(\mathbb{Z})]_k$ which extends by $\mathbb{Z}$-linearity the action of $SL_2(\mathbb{Z})$ defined in the section 1.3). We expect then to have an exact sequence of complex vector spaces:

$$0 \to \mathbb{C}_{k-2}[X,Y][\Gamma\backslash SL_2(\mathbb{Z})]_k^\theta \to \mathbb{C}_{k-2}[X,Y][\Gamma\backslash SL_2(\mathbb{Z})]_k \to \tilde{\mathcal{M}}_k(\Gamma) \to 0,$$

where the injection is the inclusion and the surjection associates to the class of $P[g]$ the Manin symbol $\widetilde{[P,g]}$.

We consider now the generalisation of the theorem 2 to $\tilde{\mathcal{M}}_k(\Gamma)$, when $\Gamma = \Gamma_1(N)$. In that case the Manin symbols can be written under the form $\widetilde{[P,x]}$, where $P \in \mathbb{C}_{k-2}[X,Y]$ and $x \in E_N$ (see the section 2.2). Let n be an integer > 0. There is an obvious action of Hecke operators T_n on $\tilde{\mathcal{M}}_k(N)$ which extends the action on $\mathcal{M}_k(\Gamma)$. This action is given by the formula

$$T_n(P\widetilde{\{\alpha,\beta\}}) = \sum_{\delta \in R} P_{|\delta}\widetilde{\{\delta\alpha, \delta\beta\}},$$

where R is a set of representatives of $\Gamma_1(N)\backslash\Delta_n$; Δ_n is the set of matrixes $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z})_n$ such that $N|c$ and $N|(a-1)$. Let $\sum_M u_M[M] \in \mathbb{Z}[M_2(\mathbb{Z})_n]$ satisfying the following condition $(\tilde{C}_n)$: For all class $K \in M_2(\mathbb{Z})_n/SL_2(\mathbb{Z})$, we have the relations $\sum_{M\in K} u_M[M\infty] = [\infty]$ and $\sum_{M\in K} u_M[M0] = [0]$ in $\mathbb{Z}[\mathbb{P}^1(\mathbb{Q})]$. This condition is stronger than the condition (C_n) given in the introduction.

We have then the following formula for the action of Hecke operators on Manin symbols

$$T_n\widetilde{[P,(u,v)]} = \sum_M u_M\widetilde{[P(aX+bY, cX+dY), (au+cv, bu+dv)]},$$

where the sum is taken with respect to the matrixes $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and restricted to the terms for which $(au+cv, bu+dv)$ belongs to E_N, $P \in \mathbb{C}_{k-2}[X,Y]$ and $(u,v) \in E_N$.

This formula can be proved easily by extending the techniques used in the sections 2.1 and 2.3. We indicate a family of elements of $\mathbb{Z}[M_2(\mathbb{Z})_n]$ satisfying the condition $(\tilde{C}_n)$ in the section 3.4.

Moreover we expect the existence of a nondegenerate pairing between $\tilde{\mathcal{M}}_k(\Gamma)^0$ and $M_k(\Gamma) \oplus \overline{M_k(\Gamma)}$, where $M_k(\Gamma)$ (resp. $\overline{M_k(\Gamma)}$) is the space of holomorphic (resp. antiholomorphic) modular forms of weight k for Γ and the existence of an alternate bilinear pairing $\tilde{\mathcal{M}}_k^0(\Gamma) \times \tilde{\mathcal{M}}_k^0(\Gamma) \to \mathbb{C}$. The first of these pairings should extend the pairing considered in the section 1.6. To make the picture complete one expects a nondegenerate self-pairing extending the Petersson scalar product on $M_k(\Gamma)$. When $\Gamma = SL_2(\mathbb{Z})$, such a pairing has been constructed by Zagier ([17]).

2 Hecke theory on modular symbols

2.1 The general principle for the action of linear operators

We keep in this section the notations of the first part. Let $\Delta \subset GL_2(\mathbb{Q})$ such that $\Gamma\Delta = \Delta\Gamma$ and such that $\Gamma\backslash\Delta$ is finite.

Let R be a set of representatives of $\Gamma\backslash\Delta$. There is a well defined linear map

$$T_\Delta \;:\; \mathcal{M}_k(\Gamma) \to \mathcal{M}_k(\Gamma)$$

which associates to $P\{\alpha,\beta\}$ the element $\sum_{\delta\in R} P_{|\delta}\{\delta\alpha,\delta\beta\}$. This map does not depend on the choice of R.

For $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Q})$, we note $\tilde{g} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = g^{-1}\mathrm{det}g$ and $\tilde{\Delta} = \{g \in GL_2(\mathbb{Q}) / \tilde{g} \in \Delta\}$.

Let ϕ be a map

$$\tilde{\Delta}SL_2(\mathbb{Z}) \to SL_2(\mathbb{Z})$$

satisfying the following three conditions (more properly the conditions are satisfied by the pair (Δ,ϕ))

1. For all $\gamma \in \tilde{\Delta}SL_2(\mathbb{Z})$, and $g \in SL_2(\mathbb{Z})$ we have $\Gamma\phi(\gamma g) = \Gamma\phi(\gamma)g$.
2. For all $\gamma \in \tilde{\Delta}SL_2(\mathbb{Z})$, we have $\gamma\phi(\gamma)^{-1} \in \tilde{\Delta}$ (or equivalently $\phi(\gamma)\tilde{\gamma} \in \Delta$).
3. The map $\Gamma\backslash\Delta \to \tilde{\Delta}SL_2(\mathbb{Z})/SL_2(\mathbb{Z})$ which associates to $\Gamma\delta$ the element $\tilde{\delta}SL_2(\mathbb{Z})$ is injective (it is necessarily surjective).

Let $\sum u_M M \in \mathbb{C}[M_2(\mathbb{Z})]$. We will say that $\sum u_M M$ satisfies the condition (C_Δ) if and only if, for all $K \in \tilde{\Delta}SL_2(\mathbb{Z})/SL_2(\mathbb{Z})$ we have the following equality in $\mathbb{C}[\mathbb{P}^1(\mathbb{Q})]$:

$$\sum_{M\in K} u_M([M\infty] - [M0]) = [\infty] - [0].$$

Theorem 4 *Let $P \in \mathbb{C}_{k-2}[X,Y]$ and $g \in SL_2(\mathbb{Z})$. Let $\sum u_M M \in \mathbb{C}[M_2(\mathbb{Z})]$ satisfying the condition (C_Δ). We have in $\mathcal{M}_k(\Gamma)$*

$$T_\Delta([P,g]) = \sum_{M,gM\in\tilde{\Delta}SL_2(\mathbb{Z})} u_M[P_{|\tilde{M}}, \phi(gM)].$$

Proof .- We start from the right side of the equality. Let S be a set of representatives of $g^{-1}\tilde{\Delta}SL_2(\mathbb{Z})/SL_2(\mathbb{Z})$. We have

$$\begin{aligned}
&\sum_{M,gM\in\tilde{\Delta}SL_2(\mathbb{Z})} u_M[P_{|\tilde{M}},\phi(gM)] \\
&= \sum_{s\in S}\sum_{M\in sSL_2(\mathbb{Z})} u_M P_{|\phi(gM)\tilde{M}}\{\phi(gM)0,\phi(gM)\infty\} \\
&= \sum_{s\in S}\sum_{M\in sSL_2(\mathbb{Z})} u_M P_{|\phi(gss^{-1}M)\tilde{M}}\{\phi(gss^{-1}M)0,\phi(gss^{-1}M)\infty\} \\
&= \sum_{s\in S}\sum_{M\in sSL_2(\mathbb{Z})} u_M P_{|\phi(gs)s^{-1}M\tilde{M}}\{\phi(gs)s^{-1}M0,\phi(gs)s^{-1}M\infty\}.
\end{aligned}$$

The last equality is a consequence of the property 1 satisfied by ϕ. Since s and M have the same determinant we have $s^{-1}M\tilde{M}=\tilde{s}$. We make use now of the condition (C_Δ) and of the properties of modular symbols to obtain the equalities:

$$\begin{aligned}
\sum_{M,gM\in\tilde{\Delta}SL_2(\mathbb{Z})} u_M[P_{|\tilde{M}},\phi(gM)] &= \sum_{s\in S} P_{|\phi(gs)\tilde{s}}\{\phi(gs)s^{-1}0,\phi(gs)s^{-1}\infty\} \\
&= \sum_{s\in S} P_{|\phi(gs)\tilde{s}\tilde{g}g}\{\phi(gs)\tilde{s}\tilde{g}g0,\phi(gs)\tilde{s}\tilde{g}g\infty\}.
\end{aligned}$$

Because of the property 2 satisfied by ϕ, $\phi(gs)\tilde{s}\tilde{g}$ belongs to Δ. Because of the property 3 satisfied by ϕ, for $s\neq s'$ we have $\Gamma\phi(gs)\tilde{s}\tilde{g}\neq\Gamma\phi(gs')\tilde{s'}\tilde{g}$. We deduce that $\phi(gs)\tilde{s}\tilde{g}$ runs through a set of representatives of $\Gamma\backslash\Delta$ when s runs through S. This concludes the proof of the theorem.

We denote by T^*_Δ the linear operator on $S_k(\Gamma)$ (resp. $\overline{S_k(\Gamma)}$) defined by the rule

$$f\mapsto\sum_{\delta\in R}(\det\delta)^{\frac{k}{2}-1}f_{|\delta}$$

(resp.

$$f\mapsto\sum_{\delta\in R}(\det\delta)^{\frac{k}{2}-1}f_{|\bar{\delta}}).$$

This operator is independant of the choice of the set of representatives R of $\Gamma\backslash\Delta$ made at the beginning of the section.

Proposition 10 *The operators T^*_Δ and T_Δ are adjoint with respect to the bilinear pairing $<.,.>$ defined in the section 1.5.*

Proof .- Let $(\alpha,\beta)\in\mathbb{P}^1(\mathbb{Q})^2$ and $P\in\mathbb{C}_{k-2}[X,Y]$. Let $f\in S_k(\Gamma)$. For $g=\begin{pmatrix} a & b \\ c & d\end{pmatrix}\in GL_2(\mathbb{Q})$, we use the otation $j(g,z)=(cz+d)$. We have

$$<T^*_\Delta(f),P\{\alpha,\beta\}> \;=\; \int_\alpha^\beta T^*_\Delta(f)P(z,1)\,dz$$

$$= \sum_{\delta \in R} \int_{\alpha}^{\beta} (\det\delta)^{k-1} f(\delta z) j(\delta, z)^{-k} P(z,1)\, dz$$

We make now the change of variables $u = \delta z$. We obtain

$$\begin{aligned}
< T_\Delta^*(f), P\{\alpha,\beta\} > &= \sum_{\delta \in R} \int_{\delta\alpha}^{\delta\beta} (\det\delta)^{k-1} f(u) j(\delta, \delta^{-1}u)^{-k} P(\delta^{-1}u, 1)\, d\delta^{-1}u \\
&= \sum_{\delta \in R} \int_{\delta\alpha}^{\delta\beta} (\det\delta)^{k-1} f(u) j(\tilde{\delta}, u)^{k} (\det\delta)^{-k} P(\tilde{\delta}u, 1) \frac{\det\delta\, du}{(j(\tilde{\delta}, u))^2} \\
&= \sum_{\delta \in R} \int_{\delta\alpha}^{\delta\beta} f(u) P_{|\delta}(u,1)\, du \\
&= < f, T_\Delta(P\{\alpha,\beta\}) > .
\end{aligned}$$

The similar equality holds for $f \in \overline{S_k(\Gamma)}$. This proves the proposition.

Proposition 11 *The operator T_Δ maps $\mathcal{S}_k(\Gamma)$ into itself.*

Proof .- Let T_Δ^∂ be the endomorphism of $\mathcal{B}_k(\Gamma)$ which associates to $P\{\alpha\}$ the element

$$\sum_{\delta \in R} P_{|\delta}\{\delta\alpha\}.$$

It does not depend on the choice of R. The proposition follows from the equality

$$\partial \circ T_\Delta = T_\Delta^\partial \circ \partial.$$

Remark .- 1) If $\Delta \subset M_2(\mathbb{Z})$, then T_Δ maps $\mathcal{M}_k(\Gamma, \mathbb{Z})$ into itself. As we will see in the section 2.3, the Hecke operators preserve the integral structure of $\mathcal{M}_k(\Gamma)$.

2) If $\eta = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ normalizes Γ and Δ (see section 1.6) then we have

$$T_\Delta \circ \iota = \iota \circ T_\Delta.$$

In particular this applies to Hecke operators.

2.2 Manin symbols for $\Gamma_1(N)$

Let N be an integer > 0. From now, we apply the results previously obtained to $\Gamma = \Gamma_1(N)$. We use thereafter the notations $\mathcal{M}_k(N) = \mathcal{M}_k(\Gamma_1(N))$, $\mathcal{S}_k(N) = \mathcal{S}_k(\Gamma_1(N))$, $\mathcal{B}_k(N) = \mathcal{B}_k(\Gamma_1(N))$ ... We remark that the matrix $\eta = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ normalizes $\Gamma_1(N)$ (see section 1.6). The surjection

$$\begin{aligned}
\pi \;:\; SL_2(\mathbb{Z}) &\to E_N \subset (\mathbb{Z}/N\mathbb{Z})^2 \\
\begin{pmatrix} a & b \\ c & d \end{pmatrix} &\mapsto (c,d)
\end{aligned}$$

defines a bijection between $\Gamma_1(N)\backslash SL_2(\mathbb{Z})$ and E_N. Let λ be a section of π. Let $x \in E_N$. The Manin symbol $[P, \lambda(x)]$ depends only on $\Gamma_1(N)\lambda(x)$ and P (see proposition 1). So it depends only on P and x. By abuse of notations we denote it by $[P, x]$. We define an action of $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z})$ on $x = (u, v) \in (\mathbb{Z}/N\mathbb{Z})^2$ by the formula

$$xM = (au + cv, bu + dv).$$

Proposition 12 *Let $x = (u, v) \in E_N$ and $P \in \mathbb{C}_{k-2}[X, Y]$. We have*

$$\iota^*([P, x]) = -[P_{|\tilde{\eta}}, x\eta] = -[P(-X, Y), (-u, v)].$$

Proof .- Let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ such that $\pi(g) = x$. By application of the proposition 7, we have

$$\iota^*([P, x]) = \iota^*([P, g]) = -[P_{|\tilde{\eta}}, \eta g \eta^{-1}].$$

We have

$$\pi(\eta g \eta^{-1}) = \pi(\begin{pmatrix} a & -b \\ -c & d \end{pmatrix}) = (-u, v).$$

The proposition follows.

There is a bijection between $\Gamma\backslash(\mathbb{Q}^2 - \{0\})/\mathbb{Q}_+^*$ and P_N (see the introduction for the definition of P_N) which associates to $\Gamma_1(N)\begin{pmatrix} u \\ v \end{pmatrix}\mathbb{Q}_+^*$ the element $u \pmod{()} v, N) \in (\mathbb{Z}/(v, N)\mathbb{Z})^* \subset P_N$. This bijection defines an isomorphism of complex vector spaces between $\mathbb{C}[\Gamma_1(N)\backslash\mathbb{Q}^2]_k$ and $\mathbb{C}[P_N]_k$. The image of $\partial([P, x])$ ($P \in \mathbb{C}_{k-2}[X, Y]$, $x \in E_N$) by this isomorphism is equal to $b([P, x])$ (see the introduction).

2.3 Hecke operators

We use the notations already introduced, especially those of section 2.2. Let n be an integer ≥ 1. Let Δ_n be the set of matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z})$, of determinant n, such that $N|c$ and $N|(a-1)$. We have $\Gamma_1(N)\Delta_n = \Delta_n\Gamma_1(N) = \Delta_n$. The set $\Gamma_1(N)\backslash\Delta_n$ is finite. We denote by T_n the *Hecke operator* T_{Δ_n} on $\mathcal{M}_k(N)$ (see section 1.1). We remark that $\eta\Delta_n = \Delta_n\eta$. So the Hecke operators commute with the involution ι^* defined in the section 1.6.

We denote by $M_2(\mathbb{Z})_n$ the set of matrices of $M_2(\mathbb{Z})$ of determinant n. We recall (see the introduction) that $\sum_M u_M M \in \mathbb{C}[M_2(\mathbb{Z})_n]$ satisfies the condition (C_n) if for all classes $K \in M_2(\mathbb{Z})_n/SL_2(\mathbb{Z})$ we have the following equality in $\mathbb{C}[\mathbb{P}^1(\mathbb{Q})]$:

$$\sum_{M \in K} u_M([M\infty] - [M0]) = [\infty] - [0].$$

We prove now the theorem 2.

Proof .- We will prove this theorem as a special case of the theorem 4. The set $\tilde{\Delta}_n SL_2(\mathbb{Z})$ is the set of matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of $M_2(\mathbb{Z})$ of determinant n such that (c,d,N) are globally coprime (i.e. $c\mathbb{Z}/N\mathbb{Z} + d\mathbb{Z}/N\mathbb{Z} = \mathbb{Z}/N\mathbb{Z}$). Let ϕ_n be a map

$$\tilde{\Delta}_n SL_2(\mathbb{Z}) \to SL_2(\mathbb{Z})$$

such that $\pi(\phi_n(\begin{pmatrix} a & b \\ c & d \end{pmatrix})) = (c,d) \in E_N$. In particular we have $\pi(\phi_n(M)) = (0,1)M$.

Lemma 1 *The map ϕ_n and the set Δ_n satisfy the conditions 1., 2. and 3. of the section 2.1.*

Proof .- Let $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \tilde{\Delta}_n$ and $g \in SL_2(\mathbb{Z})$, we have

$$\pi(\phi_n(\begin{pmatrix} a & b \\ c & d \end{pmatrix} g)) = (c,d)g = \pi(\phi_n(\begin{pmatrix} a & b \\ c & d \end{pmatrix})g).$$

This equality proves that the pair (Δ_n, ϕ_n) satisfies the property 1.

An element $g \in \tilde{\Delta}_n SL_2(\mathbb{Z})$ belongs to $\tilde{\Delta}_n$ if and only if $(0,1)g = (0,1)$ in E_N. Let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \tilde{\Delta}_n SL_2(\mathbb{Z})$. We have $(0,1)g = (0,1)\phi_n(g)$ and so $(0,1)g\phi_n(g)^{-1} = (0,1)$. We deduce that $g\phi_n(g)^{-1}$ belongs to $\tilde{\Delta}_n$. So we have the property 2.

Let $(\delta, \delta') = (\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}) \in \Delta_n^2$ such that

$$\delta'\delta^{-1} = \begin{pmatrix} \frac{da'-b'c}{n} & \frac{-ba'+ab'}{n} \\ \frac{dc'-d'c}{n} & \frac{-bc'+ad'}{n} \end{pmatrix} \in SL_2(\mathbb{Z}).$$

Because of the definition of Δ_n we have $n \equiv d \equiv d' \pmod{N}$. So we have $N | \frac{dc'-d'c}{n}$. The matrices δ, δ' and $\delta'\delta^{-1}$ are all trigonal matrices modulo N. Since the upper left entry of δ and δ' are both 1 modulo N, the upper left entry of $\delta'\delta^{-1}$ must be 1 modulo N. So $\delta'\delta^{-1}$ belongs to $\Gamma_1(N)$. We deduce that the set Δ_n satisfies the property 3.

Let $g \in SL_2(\mathbb{Z})$ such that $\pi(g) = x$. We have $gM \in \tilde{\Delta}_n SL_2(\mathbb{Z})$ if and only if xM belongs to E_N.

Since $\sum_M u_M M$ satisfies the condition (C_n) it satisfies the condition (C_{Δ_n}) of the section 2.1. By application of the theorem 4, we obtain the theorem 2.

2.4 Atkin-Lehner operators

Let N' be a positive integer dividing N such that N and N/N' are coprimes. Let $\Delta'_{N'}$ be the set of matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z})$, of determinant N', such that $N|c$, $N'|a$, $N'|d$, $N'|(b-1)$, $(N/N')|(a-1)$. We have $\Gamma_1(N)\Delta'_{N'} = \Delta'_{N'}\Gamma_1(N) = \Delta'_{N'}$. We have $\iota\Delta'_{N'} = \Delta'_{N'}\iota$.

The operator $T_{\Delta'_{N'}}$ (see section 2.1) is an *Atkin-Lehner operator*. It will be denoted by $W_{N'}$.

Theorem 5 *Let $x \in E_N$ and $P \in \mathbb{C}_{k-2}[X,Y]$. Let $g \in SL_2(\mathbb{Z})$ such that $\pi(g) = x$. Let $\sum_M u_M M \in \mathbb{C}[M_2(\mathbb{Z})_{N'}]$ satisfying the condition $(C_{N'})$. We have*

$$W_{N'}([P,x]) = \sum_{M, xM=(0,0) \pmod{N'}} u_M [P_{|\tilde{M}}, \epsilon_{N'}(gM)],$$

where $\epsilon_{N'}(gM)$ is the unique element of E_N congruent to $(1,0)gM$ modulo N' and congruent to $(0,1)gM = xM$ modulo N/N'.

Proof .- We will apply the theorem 4 for $\Delta = \Delta'_{N'}$. The set $\tilde{\Delta}'_{N'} SL_2(\mathbb{Z})$ is the set of matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z})$ of determinant N' such that $N'|c$ and $N'|d$ (we have necessarily $a\mathbb{Z}/N\mathbb{Z} + b\mathbb{Z}/N\mathbb{Z} = \mathbb{Z}/N\mathbb{Z}$).

Let $\delta_0 \in \Delta'_{N'}$. We consider the map

$$\begin{array}{rcl} \phi'_{N'} \ : \ \tilde{\Delta}'_{N'} SL_2(\mathbb{Z}) & \to & SL_2(\mathbb{Z}) \\ \delta_0 g & \mapsto & g. \end{array}$$

Since $\tilde{\Delta}'_{N'} = \delta_0 \Gamma_1(N)$, the pair $(\Delta'_{N'}, \phi'_{N'})$ satisfies the conditions 1 and 2. Let $\delta = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \tilde{\Delta}'_{N'}$ and $g = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \in SL_2(\mathbb{Z})$, such that $\delta g = \begin{pmatrix} aa'+bc' & ab'+bd' \\ ca'+dc' & cb'+dd' \end{pmatrix} \in \tilde{\Delta}'_{N'}$. We have the congruences

$$\begin{array}{rcl} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} & \equiv & \delta g \equiv \begin{pmatrix} c' & d' \\ 0 & 0 \end{pmatrix} \pmod{N'} \\ \begin{pmatrix} 1 & * \\ 0 & * \end{pmatrix} & \equiv & \delta g \equiv \begin{pmatrix} a'+bc' & * \\ dc' & d'd \end{pmatrix} \pmod{N/N'}. \end{array}$$

We obtain the congruences

$$\begin{array}{rcl} dd' & \equiv & d \pmod{N/N'} \\ c'd & \equiv & 0 \pmod{N/N'} \\ c' & \equiv & 0 \pmod{N'} \\ d' & \equiv & 1 \pmod{N'}. \end{array}$$

Since d is prime to N/N', we have $g \in \Gamma_1(N)$. This implies that the pair $(\Delta'_{N'}, \phi'_{N'})$ satisfies the condition 3 of the section 2.1.

We remark that $\pi(\phi'_{N'}(gM)) = \epsilon_{N'}(gM)$. By application of the theorem 4, the theorem 5 follows.

2.5 Modular forms with characters

Let n be an integer prime to N. Let $D_n = \begin{pmatrix} n & 0 \\ 0 & n \end{pmatrix}$. Let $\Delta_{n,n}$ be the set of matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in D_n SL_2(\mathbb{Z})$ such that $N|c$ and $N|(a-1)$. We have $\Gamma_1(N)\Delta_{n,n} = \Delta_{n,n}\Gamma_1(N)$. By definition the operator $T_{n,n}$ is the operator $T_{\Delta_{n,n}}$ in the sense of the section 2.1. The group $\Gamma_1(N)$ operates transitively by multiplication on the right on $\Delta_{n,n}$.

Proposition 13 *Let $x=(u,v)\in E_N$ and $P\in\mathbb{C}_{k-2}[X,Y]$. We have*

$$T_{n,n}([P,x])=[P_{|D_n},xD_n]=n^{k-2}[P,(nu,nv)].$$

Proof .- Let $g\in SL_2(\mathbb{Z})$ such that $\pi(g)=x$. Let $\gamma\in\Delta_{n,n}$. We can write $\gamma=D_n\gamma_0$ with $\gamma_0=\begin{pmatrix}a&b\\c&d\end{pmatrix}\in\Gamma_0(N)$. We have $d\equiv n\pmod N$. So we have $\pi(\gamma_0 g)=(du,dv)=xD_n$. The proposition is proved by the following equalities

$$\begin{aligned}T_{n,n}([P,x]) &= P_{|D_n\gamma_0 g}\{D_n\gamma_0 g0,D_n\gamma_0 g\infty\}\\ &= P_{|D_n\gamma_0 g}\{\gamma_0 g0,\gamma_0 g\infty\}\\ &= [P_{|D_n},xD_n]\end{aligned}$$

Let χ be a Dirichlet character $\mathbb{Z}/N\mathbb{Z}\to\mathbb{C}$. We say that $f\in S_k(N)$ (resp. $f\in\overline{S_k(N)}$) belongs to $S_k(N,\chi)$ (resp. $\overline{S_k(N,\chi)}$) if we have $f_{|\gamma}=\chi(d)f$ (resp. $f_{|\bar\gamma}=\chi(d)f$) for all $\gamma=\begin{pmatrix}a&b\\c&d\end{pmatrix}\in\Gamma_0(N)$. This condition is equivalent to $T^*_{n,n}(f)=n^{k-2}\chi(n)f$ for all $n>0$ inversible modulo N, where $T^*_{n,n}$ is the operator dual to $T_{n,n}$ in the sense of the section 2.1.

We define $\mathcal{M}_k(N,\chi)$ as the quotient vector space of $\mathcal{M}_k(N)$ by the equivalence relation which identify $\chi(n)x$ and $n^{2-k}T_{n,n}(x)$ for all $n>0$ inversible modulo N and all $x\in\mathcal{M}_k(N)$. Equivalently $\mathcal{M}_k(N,\chi)$ is the quotient vector space of $\mathcal{M}_k(N)$ by the equivalence relation which identify the Manin symbol $[P,(\lambda u,\lambda v)]$ ($(\lambda,u,v)\in(\mathbb{Z}/N\mathbb{Z})^3$, $P\in\mathbb{C}_{k-2}[X,Y]$) in $\mathcal{M}_k(N,\chi)$ to $\chi(\lambda)[P,(u,v)]$.

We denote by $\mathcal{B}_k(N,\chi)$ the image of ∂ in $\mathcal{B}_k(N)$ and $\mathcal{S}_k(N,\chi)$ the kernel of ∂ in $\mathcal{M}_k(N,\chi)$. We can state the following refinement of the theorem 3.

Proposition 14 *The pairing*

$$(S_k(N,\chi)\oplus\overline{S_k(N,\chi)})\times\mathcal{S}_k(N,\chi)\to\mathbb{C}$$

deduced from the pairing $<.,.>$ of the section 1.5 is nondegenerate.

2.6 The new and old parts of $\mathcal{M}_k(N)$

We do not give the detailed proofs of some statements in this section, since these statements parallel the classical theory of old and newforms (see for instance [3]). Let N' be a positive divisor of N. Let t be a divisor of N/N'. The matrix $T=\begin{pmatrix}1&0\\0&t\end{pmatrix}$ satisfies $T^{-1}\Gamma_1(N)\subset\Gamma_1(N')T^{-1}$. So there are well defined linear maps

$$\begin{aligned}\epsilon_t\ :\ \mathcal{M}_k(N) &\to \mathcal{M}_k(N')\\ P(X,Y)\{\alpha,\beta\} &\mapsto P(X,tY)\{t\alpha,t\beta\}=P_{|\tilde T}\{\tilde T\alpha,\tilde T\beta\}\end{aligned}$$

and

$$\begin{aligned}\epsilon'_t\ :\ \mathcal{M}_k(N') &\to \mathcal{M}_k(N)\\ P\{\alpha,\beta\} &\mapsto \sum_\gamma P_{|\gamma}\{\gamma\alpha,\gamma\beta\},\end{aligned}$$

where γ runs through a set of representatives of $\Gamma_1(N)\backslash\Gamma_1(N')T$. We have $\epsilon_t \circ \epsilon'_t =$ multiplication by a nonzero scalar. The linear maps ϵ_t and ϵ'_t are respectively surjective and injective.

The intersection of the kernels of ϵ_t when N' runs through all the positive divisors of N and t runs through all the divisors of N/N' is by definition the *new part* $\mathcal{M}_k(N)^n$ of $\mathcal{M}_k(N)$. The space generated by the images of ϵ'_t when N' runs through all the positive divisors of N and t runs through all the divisors of N/N' is by definition the *old part* $\mathcal{M}_k(N)^o$ of $\mathcal{M}_k(N)$. We have a direct sum

$$\mathcal{M}_k(N) = \mathcal{M}_k(N)^n \oplus \mathcal{M}_k(N)^o.$$

This decomposition respects the decomposition of $S_k(N)$ in old and new part, i.e. $\mathcal{M}_k(N)^n$ (resp. $\mathcal{M}_k(N)^o$) is orthogonal to the old (resp. new) part of $S_k(N)$, when one considers the bilinear pairing $< .,. >$.

The two following propositions are not, properly speaking, applications of the general principle of the section 2.1. Nevertheless we use the same techniques to prove them.

Proposition 15 *Let $x \in E_N$. Let $P \in \mathbb{C}_{k-2}[X,Y]$. Let $\sum_M u_M M \in \mathbb{C}[M_2(\mathbb{Z})_t]$ satisfying the condition (C_t). We have in $\mathcal{M}_k(N)$*

$$\epsilon_t([P,x]) = \sum u_M [P_{|\tilde{M}}, \frac{1}{t}xM],$$

where the sum is restricted to the matrices M such that $xM \in tE_N$, and where the multiplication by $\frac{1}{t}$ is well defined from E_N to $E_{N'}$.

Proof .- Let $g \in SL_2(\mathbb{Z})$ such that $\pi(g) = x$. We use the notation $T = \begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix}$. Let $M \in M_2(\mathbb{Z})_t$ such that $xM \in tE_N$. Then we have $\pi(T^{-1}gM) = \frac{1}{t}xM$. So two matrices M and M' of $M_2(\mathbb{Z})_t$ satisfying $xM \in tE_N$ and $xM' \in tE_N$ are in the same class modulo multiplication on the right by an element of $SL_2(\mathbb{Z})$. If $y \in tE_N$ and $h \in SL_2(\mathbb{Z})$ then we have $yh \in tE_N$. So the sum in the formula in the proposition is taken with respect to all elements belonging to only one class of $M_2(\mathbb{Z})_t/SL_2(\mathbb{Z})$. We prove now the validity of the formula. We begin with the right side of the equality:

$$\begin{aligned} \sum u_M [P_{|\tilde{M}}, \frac{1}{t}xM] &= \sum u_M P_{|T^{-1}gM\tilde{M}}\{T^{-1}gM0, T^{-1}gM\infty\} \\ &= \sum u_M P_{|\tilde{T}g}\{T^{-1}gM0, T^{-1}gM\infty\}. \end{aligned}$$

We use the condition (C_t) and the fact that the sum is taken over only one class of $M_2(\mathbb{Z})_t/SL_2(\mathbb{Z})$. We have

$$\begin{aligned} \sum u_M P_{|\tilde{T}g}\{T^{-1}gM0, T^{-1}gM\infty\} &= P_{|\tilde{T}g}\{T^{-1}g0, T^{-1}g\infty\} \\ &= P_{|\tilde{T}g}\{tg0, tg\infty\} \\ &= \epsilon_t([P,x]). \end{aligned}$$

The proposition is proved.

Proposition 16 *Let $g \in SL_2(\mathbb{Z})$. Let H be a system of representatives in $\Gamma_1(N')$ of $\Gamma_1(N)\backslash\Gamma_1(N')g$. Let $\sum_M u_M M \in \mathbb{Z}[M_2(\mathbb{Z})_t]$ satisfying the condition (C_t). We have*

$$\epsilon'_t([P,\pi(g)]) = \sum_{h\in H}\sum_M u_M[P_{|\tilde{M}}, \pi(\begin{pmatrix} \frac{1}{t} & 0 \\ 0 & 1 \end{pmatrix} hM)],$$

where the second sum is restricted to the matrices M such that $hM \in \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix} SL_2(\mathbb{Z})$.

Proof .- First we remark that, given $h \in H$, the set of matrices $M \in M_2(\mathbb{Z})_t$ such that $hM \in \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix} SL_2(\mathbb{Z})$ is exactly a class of $M_2(\mathbb{Z})_t/SL_2(\mathbb{Z})$. We use the notation $T = \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}$. We have

$$\begin{aligned}
\sum_{h\in H}\sum_M u_M[P_{|\tilde{M}}, \pi(\begin{pmatrix} \frac{1}{t} & 0 \\ 0 & 1 \end{pmatrix} hM)] &= \sum_{h\in H}\sum_M u_M P_{|\tilde{T}^{-1}hM\tilde{M}}\{\frac{1}{t}hM0, \frac{1}{t}hM\infty\} \\
&= \sum_{h\in H}\sum_M u_M P_{|\tilde{T}h}\{\frac{1}{t}hM0, \frac{1}{t}hM\infty\} \\
&= \sum_{h\in H} P_{|\tilde{T}h}\{\frac{1}{t}h0, \frac{1}{t}h\infty\} \\
&= \epsilon'_t([P,\pi(g)]).
\end{aligned}$$

The proposition is proved.

To determine the new part of $\mathcal{M}_k(N)$ it is useful to consider the following proposition (which is a counterpart of a classical statement in the theory of modular forms, see [3]).

Proposition 17 *Let $x \in \mathcal{M}_k(N)$. This element belongs to $\mathcal{M}_k(N)^n$ if and only if x and $W_N x$ belong to the kernel of $\epsilon_1 : \mathcal{M}_k(N) \to \mathcal{M}_k(N/t)$ for all divisors t of N.*

3 Families of elements satisfying the condition (C_n)

3.1 The Manin-Heilbronn family

Let $\mathcal{S}$ (resp. $\mathcal{S}'$) be the set of matrices $\begin{pmatrix} x & -y \\ y' & x' \end{pmatrix} \in M_2(\mathbb{Z})$ of determinant > 0 and satisfying one at least of the following three conditions

- $x > |y|$, $x' > |y'|$, $yy' > 0$
- $y = 0$, $|y'| < \frac{x'}{2}$
- $y' = 0$, $|y| < \frac{x}{2}$

(resp. one of the following two conditions

- $y = 0$, $|y'| = \frac{x'}{2}$
- $y' = 0$, $|y| = \frac{x}{2}$).

Let n be an integer > 0. Let $\mathcal{S}_n$ (resp. $\mathcal{S}'_n$) be the set of elements of $\mathcal{S}$ (resp. $\mathcal{S}'$) of determinant n.

Proposition 18 *The element*

$$\sum_{M \in \mathcal{S}_n} M + \frac{1}{2} \sum_{M \in \mathcal{S}'_n} M$$

of $\mathbb{C}[M_2(\mathbb{Z})_n]$ *satisfies the condition* (C_n)*.*

Proof .- We use several lemmas.

Lemma 2 *The set of elements* $\begin{pmatrix} d & b \\ 0 & \frac{n}{d} \end{pmatrix} \in M_2(\mathbb{Z})$ *(resp.* $\begin{pmatrix} \frac{n}{d} & 0 \\ b & d \end{pmatrix}$*), with* $d > 0$ *and* $-\frac{d}{2} < b \leq \frac{d}{2}$ *is a set of representatives of* $M_2(\mathbb{Z})_n/SL_2(\mathbb{Z})$*. The same assertion holds if we replace the condition* $-\frac{d}{2} < b \leq \frac{d}{2}$ *by* $-\frac{d}{2} \leq b < \frac{d}{2}$*. These elements are the only elements* $\begin{pmatrix} x & -y \\ y' & x' \end{pmatrix} \in \mathcal{S}_n \cup \mathcal{S}'_n$ *such that* $y' = 0$ *(resp.* $y = 0$*).*

Proof .- We prove the lemma with respect to the set of matrices $\begin{pmatrix} d & b \\ 0 & \frac{n}{d} \end{pmatrix} \in M_2(\mathbb{Z})$ with $d > 0$ and $-\frac{d}{2} < b \leq \frac{d}{2}$. The other situations can be dealt with similarly. Let $g \in M_2(\mathbb{Z})_n$. Since $SL_2(\mathbb{Z})$ operates transitively on $\mathbb{P}^1(\mathbb{Q})$, there exists $\gamma \in SL_2(\mathbb{Z})$ such that $g\gamma\infty = \infty$. So there exists an element $g' = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in gSL_2(\mathbb{Z})$ such that $g'\infty = \infty$. So we have $c = 0$. We can multiply by $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ to obtain $a > 0$. Two matrices of $M_2(\mathbb{Z})$ of the type $\begin{pmatrix} d & b \\ 0 & \frac{n}{d} \end{pmatrix}$ and $\begin{pmatrix} d' & b' \\ 0 & \frac{n}{d'} \end{pmatrix}$, with $d > 0$ and $d' > 0$ are congruent modulo $SL_2(\mathbb{Z})$ if and only if $d = d'$ and $b \equiv b' \pmod{d}$. This proves the first statement of the lemma.

The second statement follows immediately from the definition of the sets $\mathcal{S}_n$ and $\mathcal{S}'_n$.

Lemma 3 *Let* $g = \begin{pmatrix} x & -y \\ y' & x' \end{pmatrix} \in \mathcal{S}_n$ *such that* $y' \neq 0$ *(resp.* $y \neq 0$*). There exists an unique element* $g_0 = \begin{pmatrix} x_0 & -y_0 \\ y'_0 & x'_0 \end{pmatrix} \in \mathcal{S}_n \cap gSL_2(\mathbb{Z})$ *such that* $\epsilon(y')\begin{pmatrix} x \\ y' \end{pmatrix} = \begin{pmatrix} -y_0 \\ x'_0 \end{pmatrix}$ *(resp.* $\begin{pmatrix} -y \\ x' \end{pmatrix} = \epsilon(y'_0)\begin{pmatrix} x_0 \\ y'_0 \end{pmatrix}$*) where* $\epsilon(y')$ *(resp.* $\epsilon(y'_0)$*) is the sign of* y' *(resp.* y'_0*); We have* $g\infty = g_0 0$ *(resp.* $g0 = g_0\infty$*).*

Proof .- We prove the lemma with respect to the hypothesis $y' \neq 0$. The other assertion can be proved similarly. We consider two cases.

First we suppose that $\frac{x'}{y'}$ is not integral. Let q be the unique integer such that $|q + \frac{x'}{y'}| < 1$ and $(qy' + x') > 0$. Since $|y'| < x'$, we have $\epsilon(y')q < 0$. The matrix $\begin{pmatrix} -q\epsilon(y' & \epsilon(y') \\ -\epsilon(y') & 0 \end{pmatrix}$ belongs to $SL_2(\mathbb{Z})$ and we have

$$g_0 = g \begin{pmatrix} -q\epsilon(y') & \epsilon(y') \\ -\epsilon(y') & 0 \end{pmatrix} = \begin{pmatrix} -(qx - y)\epsilon(y') & \epsilon(y')x \\ -(qy' + x')\epsilon(y') & \epsilon(y')y' \end{pmatrix}.$$

The nondiagonal entries of this matrix are nonzero. Since $|y| < x$, we have $(qx - y)\epsilon(y') > |-\epsilon(y')x| = x$. Because of the construction of q we have $|(qy' + x')\epsilon(y')| < \epsilon(y')y'$. We have $(qy' + x')\epsilon(y')(-\epsilon(y')x) = -(qy' + x')x < 0$. We have proved that g_0 belongs to $\mathcal{S}_n$. This proves the existence of the matrix. We now prove the unicity. Let $g_0 \in M_2(\mathbb{Z})$ satisfying the conditions of the lemma. Let $\gamma \in SL_2(\mathbb{Z})$ such that $g\gamma = g_0$. We have $g\gamma 0 = g_0 0 = g\infty$. So we have $\gamma 0 = \infty$ and γ is of the form $\begin{pmatrix} -q\epsilon & \epsilon \\ -\epsilon & 0 \end{pmatrix}$, with $q \in \mathbb{Z}$ and $\epsilon \in \{1, -1\}$. We have

$$g_0 = g \begin{pmatrix} -q\epsilon & \epsilon \\ -\epsilon & 0 \end{pmatrix} = \begin{pmatrix} -(qx - y)\epsilon & \epsilon x \\ -(qy' + x')\epsilon & \epsilon y' \end{pmatrix}.$$

Since $\frac{x'}{y'}$ is not integral, we have $qy' + x' \neq 0$. So we have $\epsilon y' > 0$ and $\epsilon = \epsilon(y')$. We have also $|(qy' + x')\epsilon| < \epsilon y'$, that implies $|q + \frac{x'}{y'}| < 1$. Moreover the inequality $\epsilon x(qy' + x')\epsilon > 0$ implies $qy' + x' > 0$. So we have proved the unicity of q and consequently of g_0.

Now we consider the second case: $\frac{x'}{y'}$ is integral. We first prove the existence property. We find that the matrix $g \begin{pmatrix} -q\epsilon & \epsilon \\ -\epsilon & 0 \end{pmatrix}$, with $q = -\frac{x'}{y'}$ and $\epsilon = \epsilon(y')$, is equal to $\begin{pmatrix} \frac{n\epsilon}{y'} & \epsilon x \\ 0 & \epsilon y' \end{pmatrix}$. We only have to prove that $|x| < \frac{n\epsilon}{2y'}$, i.e. that $|xy'| < \frac{n}{2}$. Since $y'|x'$ and $|y'| < x'$, we have $|\frac{x'}{y'}| \geq 2$. Since $xx' + yy' = n$, we have $0 < xx' < n$, except perhaps if $y = 0$. If $y \neq 0$, we obtain the inequalities

$$|xy'| \leq \frac{1}{2}|xx'| < \frac{n}{2}.$$

If $y = 0$, we have $|y'| < \frac{1}{2}x'$ (because $g \in \mathcal{S}_n$) and the inequality

$$|xy'| < \frac{1}{2}|xx'| \leq \frac{n}{2}.$$

So the existence is proved. To prove the unicity we proceed as in the first case.

Lemma 4 *Let d be an even positive divisor of n. The elements of the set*

$$\{\begin{pmatrix} d & \frac{d}{2} \\ 0 & \frac{n}{d} \end{pmatrix}, \begin{pmatrix} d & -\frac{d}{2} \\ 0 & \frac{n}{d} \end{pmatrix}, \begin{pmatrix} \frac{d}{2} & 0 \\ \frac{n}{d} & \frac{2n}{d} \end{pmatrix}, \begin{pmatrix} \frac{d}{2} & 0 \\ -\frac{n}{d} & \frac{2n}{d} \end{pmatrix}\}$$

belongs to the same class modulo $SL_2(\mathbb{Z})$. Moreover these sets form a partition of $\mathcal{S}'_n$ which respects the classes modulo $SL_2(\mathbb{Z})$ when d runs through the positive even divisors of n.

Proof .- The first assertion is proved by straightforward calculations. The second assertion is obtained by comparing with the definition of $\mathcal{S}'_n$ and by using the lemma 2.

Lemma 5 *We have*

$$\mathcal{S}_n SL_2(\mathbb{Z}) \cap \mathcal{S}'_n SL_2(\mathbb{Z}) = \emptyset.$$

Proof .- Let $g = \begin{pmatrix} x & -y \\ y' & x' \end{pmatrix} \in \mathcal{S}_n \cap \mathcal{S}'_n SL_2(\mathbb{Z})$. We can not have $y' = 0$ because of the lemma 2. Since $g \in \mathcal{S}_n$ and $y' \neq 0$, we use the lemma 3 to prove the existence of an element $g_0 = \begin{pmatrix} x_0 & -y_0 \\ y'_0 & x'_0 \end{pmatrix} \in \mathcal{S}_n \cap gSL_2(\mathbb{Z}) = \mathcal{S}_n \cap \mathcal{S}'_n SL_2(\mathbb{Z})$ with $|y'_0| < y'$. By repeated use of the lemma 3 we obtain a sequence of matrices $g_m = \begin{pmatrix} x_m & -y_m \\ y'_m & x'_m \end{pmatrix} \in \mathcal{S}_n \cap \mathcal{S}'_n SL_2(\mathbb{Z})$ with decreasing $|y_m|$. For some m we must have $y_m = 0$. So we obtain a contradiction.

We turn now to the proof of the proposition. The equation $\det \begin{pmatrix} x & -y \\ y' & x' \end{pmatrix} = xx' + yy' = n$ has only a finite number of integral solutions with $xx' \geq 0$ and $yy' \geq 0$. So the set $\mathcal{S}_n$ is finite. Let $K \in M_2(\mathbb{Z})_n / SL_2(\mathbb{Z})$.

If $K \subset \mathcal{S}'_n SL_2(\mathbb{Z})$, then the lemma 4 tells us that there exists a unique positive divisor d of n such that

$$K \cap \mathcal{S}'_n = \{\begin{pmatrix} d & \frac{d}{2} \\ 0 & \frac{n}{d} \end{pmatrix}, \begin{pmatrix} d & -\frac{d}{2} \\ 0 & \frac{n}{d} \end{pmatrix}, \begin{pmatrix} \frac{d}{2} & 0 \\ \frac{n}{d} & \frac{2n}{d} \end{pmatrix}, \begin{pmatrix} \frac{d}{2} & 0 \\ -\frac{n}{d} & \frac{2n}{d} \end{pmatrix}\}.$$

We have in $\mathbb{C}[\mathbb{P}^1(\mathbb{Q})]$

$$\begin{aligned} \sum_{M \in K \cap \mathcal{S}'_n} ([M\infty] - [M0]) &= [\infty] - [\frac{d^2}{2n}] + [\infty] - [-\frac{d^2}{2n}] + [\frac{d^2}{2n}] - [0] + [-\frac{d^2}{2n}] - [0] \\ &= 2([\infty] - [0]). \end{aligned}$$

If $K \subset \mathcal{S}_n SL_2(\mathbb{Z})$, we have in $\mathbb{C}[\mathbb{P}^1(\mathbb{Q})]$ (the sums are taken with respect to the matrices $M = \begin{pmatrix} x & -y \\ y' & x' \end{pmatrix}$)

$$\sum_{M \in K \cap \mathcal{S}_n} [M\infty] - [M0] = \sum_{M \in K \cap \mathcal{S}_n} [\frac{x}{y'}] - \sum_{M \in K \cap \mathcal{S}_n} [-\frac{y}{x'}]$$

Because of the lemma 3, all the terms cancel each other except those corresponding to $y' = 0$ or $y = 0$. We obtain

$$\sum_{M \in K \cap \mathcal{S}_n} [M\infty] - [M0] = \sum_{M \in K \cap \mathcal{S}_n, y'=0} [\frac{x}{y'}] - \sum_{M \in K \cap \mathcal{S}_n, y=0} [-\frac{y}{x'}]$$

We now use the lemma 2 and we obtain that the last member is equal to $[\infty]-[0]$. The proposition is proved.

This result is very close to one obtained in [8], where the element appearing in the proposition 18 is called the Manin-Heilbronn element for n odd (if n is even it is called the Manin-Heilbronn element symmetrized by the complex conjugation). The phenomenon underlying the proposition 18 was discovered by Heilbronn in a work about continued fractions ([2]) and it was later used by Manin in the theory of modular symbols ([4], [5], [6]).

Let us remark that the lemmas 2 and 3 enable us to construct systematically the set $\mathcal{S}_n$. The set $\mathcal{S}'_n$ is given directly by the lemma 4.

In [8], using a result of Heilbronn we gave an estimate for the cardinality of $\mathcal{S}_n$: We have the following asymptotic formula when $n \to \infty$

$$|\mathcal{S}_n| \sim \frac{12\log 2}{\pi^2}\sigma_1(n)\log n,$$

where $\sigma_1(n)$ is the sum of the positive divisors of n.

We can now introduce another universal expansion of modular forms, which seems for practical purposes (i.e. explicit construction of bases of spaces of modular forms), to be more useful than the series considered in the introduction.

We call the following series in $\mathbb{C}[M_2(\mathbb{Z})][[q]]$ the *Manin-Heilbronn expansion* :

$$\begin{aligned}
& \sum_{M\in\mathcal{S}} Mq^{\det M} + \frac{1}{2}\sum_{M\in\mathcal{S}'} Mq^{\det M} \\
= \; & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} q \\
+ \; & (\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 2 & -1 \\ 0 & 1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 & 0 \\ -1 & 2 \end{pmatrix})q^2 \\
+ \; & (\begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} + \begin{pmatrix} 3 & 1 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 1 & 3 \end{pmatrix} + \begin{pmatrix} 3 & -1 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ -1 & 3 \end{pmatrix})q^3 + \ldots
\end{aligned}$$

3.2 The family related to $\Gamma(2)$

Let n be an odd integer > 0. Let

$$\begin{aligned} U_n = \{\begin{pmatrix} x & -y \\ y' & x' \end{pmatrix} \in M_2(\mathbb{Z})/ \quad & xx'+yy'=n,\ x\in(1+4\mathbb{Z}),\ x \text{ and } x' \text{ odd}, \\ & y \text{ and } y' \text{ even}, x>|y|, x'>|y'|\} \end{aligned}$$

and

$$\begin{aligned} V_n = \{\begin{pmatrix} x & -y \\ y' & x' \end{pmatrix} \in M_2(\mathbb{Z})/ \quad & xx'+yy'=n,\ x\in(1+4\mathbb{Z}),\ x \text{ and } x' \text{ odd}, \\ & y \text{ and } y' \text{ even}, y>|x|, y'>|x'|\}. \end{aligned}$$

Proposition 19 *The element*

$$\theta_n = \sum_{M \in U_n} M - \sum_{M \in V_n} M$$

satisfies the condition (C_n).

See [7] for the proof. These elements θ_n are somewhat more canonical than the Manin-Heilbronn elements. Let us mention that they satisfy the following relations in $\mathbb{Z}[M_2(\mathbb{Z})]$ ([7])

$$\theta_n \theta_{n'} = \theta_{nn'}$$

if n and n' are coprimes odd integers, and

$$\theta_{p^q} = \theta_p \theta_{p^{q-1}} - p \begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix} \theta_{p^{q-2}}$$

if p is an odd prime number and q an integer ≥ 2.

3.3 The set $\mathcal{X}$

Let n be an integer > 0.

Proposition 20 *The element*

$$\sum_{a>b\geq 0, d>c\geq 0, ad-bc=n} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{C}[M_2(\mathbb{Z})_n]$$

satisfies the condition (C_n).

Proof .- The proof is similar to the proof of the proposition 18. Let

$$\mathcal{X}_n = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z})_n / a > b \geq 0,\ d > c \geq 0,\ ad - bc = n\}.$$

Lemma 6 *The set of elements* $\begin{pmatrix} d & b \\ 0 & \frac{n}{d} \end{pmatrix} \in M_2(\mathbb{Z})$ *(resp.* $\begin{pmatrix} \frac{n}{d} & 0 \\ b & d \end{pmatrix}$*), with* $d > 0$ *and* $0 \leq b < d$ *is a set of representatives of* $M_2(\mathbb{Z})_n / SL_2(\mathbb{Z})$*. These elements are the only elements* $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathcal{X}_n$ *such that* $\gamma = 0$ *(resp.* $\beta = 0$*).*

Proof .- The proof of this lemma is similar to the proof of the lemma 2.

Lemma 7 *Let* $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathcal{X}_n$ *such that* $c \neq 0$ *(resp.* $b \neq 0$*). There exists an unique element* $g_0 = \begin{pmatrix} a_0 & b_0 \\ c_0 & d_0 \end{pmatrix} \in \mathcal{X}_n \cap gSL_2(\mathbb{Z})$ *such that* $\begin{pmatrix} a \\ c \end{pmatrix} = \begin{pmatrix} b_0 \\ d_0 \end{pmatrix}$ *(resp.* $\begin{pmatrix} b \\ d \end{pmatrix} = \begin{pmatrix} a_0 \\ c_0 \end{pmatrix}$*). We have* $g\infty = g_0 0$ *(resp.* $g0 = g_0\infty$*).*

Proof .- We prove the assertion corresponding to $c \neq 0$. We prove first the existence of g_0. Let m be the smallest integer $\geq d/c$. We have $m \geq 2$. We have $ma - b > a > 0$ and $c > mc - d \geq 0$ by construction of m. So the matrix

$$g\begin{pmatrix} m & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} ma-b & a \\ mc-d & c \end{pmatrix}$$

belongs to $\mathcal{X}_n$ and the existence is proved. Now we consider the problem of the unicity. We have

$$g^{-1}g_0 = \begin{pmatrix} \frac{a_0d-bc_0}{n} & \frac{b_0d-bd_0}{n} \\ \frac{ac_0-a_0c}{n} & \frac{ad_0-b_0c}{n} \end{pmatrix} = \begin{pmatrix} \frac{a_0d-bc_0}{n} & 1 \\ -1 & 0 \end{pmatrix} \in SL_2(\mathbb{Z}).$$

We use the notation $m = \frac{a_0d-bc_0}{n}$. We have

$$g_0 = g\begin{pmatrix} m & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} ma-b & a \\ mc-d & c \end{pmatrix}.$$

Since g_0 belongs to $\mathcal{X}_n$, we must have $c > mc - d \geq 0$. We deduce that m must be the smallest integer $\geq d/c$. The unicity is proved.

Lemma 8 *The set $\mathcal{X}_n$ is finite.*

Proof .- We have to prove that the set of quadruples $(a, b, c, d) \in \mathbb{Z}^4$ solutions of the equation $n = ad - bc$, with $a > b \geq 0$ and $d > c \geq 0$, is finite. We have the inequality $n = ad - bc \geq ad - (a-1)(d-1) = a + d - 1$. Therefore the four numbers a,,b, c and d are all positive and smaller than n.

We finish now the proof of the proposition 20. Let $K \in M_2(\mathbb{Z})_n/SL_2(\mathbb{Z})$. We have in $\mathbb{C}[\mathbb{P}^1(\mathbb{Q})]$ (the sums are taken with respect to the matrices $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$)

$$\sum_{M\in K\cap\mathcal{X}_n} [M\infty] - [M0] = \sum_{M\in K\cap\mathcal{X}_n} [\frac{a}{c}] - \sum_{M\in K\cap\mathcal{X}_n} [\frac{b}{d}].$$

Because of the lemma 7, all the terms destroy each other except those corresponding to $c = 0$ or $b = 0$. We obtain

$$\sum_{M\in K\cap\mathcal{X}_n} [M\infty] - [M0] = \sum_{M\in K\cap\mathcal{X}_n, c=0} [\frac{a}{c}] - \sum_{M\in K\cap\mathcal{X}_n, d=0} [\frac{b}{d}].$$

We now use the lemma 6 and we obtain that the last member is equal to $[\infty] - [0]$. The proposition is proved.

Remark .- The linear combination of matrices appearing in the proposition 20, satisfies a stronger condition than the condition (C_n): for all $K \in M_2(\mathbb{Z})_n/SL_2(\mathbb{Z})$ we have in $\mathbb{Z}[\mathbb{Q}_+^*\backslash\mathbb{Q}^2]$

$$\sum_{M\in K} u_M([M\begin{pmatrix} 1 \\ 0 \end{pmatrix}] - [M\begin{pmatrix} 0 \\ 1 \end{pmatrix}]) = [\begin{pmatrix} 1 \\ 0 \end{pmatrix}] - [\begin{pmatrix} 1 \\ 0 \end{pmatrix}].$$

This is an easy consequence of the lemmas used in the proof of the proposition 20.

3.4 An additional family

Let n be an integer > 0. Let us mention an other family of elements of $\mathbb{Z}[M_2(\mathbb{Z})_n]$ satisfying the condition (C_n), and even the condition $(\tilde{C_n})$ of the section 1.8.

Proposition 21 *The element*

$$\sum_{a>b\geq 0, d>-c\geq 0, ad-bc=n} \begin{pmatrix} a & b \\ c & d \end{pmatrix} - \sum_{b\geq a>0, -c\geq d>0, ad-bc=n} \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

of $\mathbb{Z}[M_2(\mathbb{Z})_n]$ *satisfies the condition* (C_n)*.*

The proof of this proposition is based on the same kind of arguments than those used in the proofs of the propositions 18, 19 and 20. We leave this to the reader.

Remark .- The same proposition holds if we exchange simultaneously the signs of b and c in the two sums of the proposition 21.

If we set $\mathcal{A} = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z}) / a > b \geq 0, d > -c \geq 0\}$, and $\mathcal{B} = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z}) / b \geq a > 0, -c \geq d > 0\}$ then we obtain an other universal Fourier expansion:

$$\begin{aligned}
& \sum_{M\in\mathcal{A}} M q^{\det M} - \sum_{M\in\mathcal{B}} M q^{\det M} \\
= \; & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} q \\
+ \; & (\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} + \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ -1 & 2 \end{pmatrix})q^2 \\
+ \; & (\begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} + \begin{pmatrix} 3 & 1 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 1 \\ -2 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ -2 & 3 \end{pmatrix} \\
& + \begin{pmatrix} 3 & 2 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ -1 & 3 \end{pmatrix})q^3 + \ldots
\end{aligned}$$

It would be interesting to find other elements satisfying the condition (C_n). It seems that Zagier has recently found such elements closely related to the trace formula of Hecke operators.

4 Construction of modular forms

4.1 Modular forms as linear maps on the Hecke algebra

Let M be any of the following vector spaces: $S_k(N)$, $S_k(N,\chi)$, any of the previous ones but extending spaces of cusp forms to spaces of holomorphic modular forms (i.e. including Eisenstein series), any of the previous ones but restricting to spaces of newforms, any of the previous ones but considering antiholomorphic modular forms instead.

Let $\mathbb{T}$ be the corresponding *Hecke algebra*, i.e. the complex commutative subalgebra of $\mathrm{End}_{\mathbb{C}}(M)$ generated by the Hecke operators T_n^* and $T_{n,n}^*$ (see section 2.3 and 2.5, n is an integer > 0) which we will denote simply by T_n and $T_{n,n}$.

The statement and the proof of the following theorem is modeled on [1], page 306.

Theorem 6 *Let α be a linear map $\mathbb{T} \to \mathbb{C}$. Then*

$$\sum_{n=1}^{\infty} \alpha(T_n) q^n$$

is, except for the constant coefficient, the Fourier expansion of an element of M.

Proof .- We denote by a_n the linear form on M which associates to a modular form its n-th Fourier coefficient. We have $a_n = a_1 \circ T_n$.

Lemma 9 *The bilinear pairing of complex vector spaces*

$$\begin{array}{ccc} M \times \mathbb{T} & \to & \mathbb{C} \\ (f, T) & \mapsto & a_1(Tf) \end{array}$$

is nondegenerate.

Proof .- Let $T \in \mathbb{T}$. Suppose that for all $f \in M$ we have $a_1(Tf) = 0$. We have then for all $n \geq 1$

$$0 = a_1(TT_n f) = a_1(T_n T f) = a_n(Tf).$$

So we have $Tf = 0$ for all $f \in M$. We deduce the equality $T = 0$.

Conversely, let $f \in M$. Suppose that for all $T \in \mathbb{T}$, we have $a_1(Tf) = 0$. Then for all $n \geq 1$, we have

$$a_1(T_n f) = a_n(f) = 0.$$

So we have $f = 0$.

We turn now to the proof of the theorem. It follows from the lemma that there exists $f \in M$ such that $\alpha(T) = a_1(Tf)$ $(T \in \mathbb{T})$. Then we have the equalities

$$\sum_{n=1}^{\infty} \alpha(T_n) q^n = \sum_{n=1}^{\infty} a_1(T_n f) q^n = \sum_{n=1}^{\infty} a_n(f) q^n.$$

The later expression is the Fourier expansion of f. This concludes the proof of the theorem.

4.2 Proof of the main theorem

We prove now the theorem 1 by putting together all the parts of this paper. We write

$$x = \sum_{\lambda \in E_N} P_\lambda [\lambda].$$

We consider $m(x) = \sum_{\lambda \in E_N} [P_\lambda, \lambda] \in \mathcal{M}_k(N)$. The equality $b(x) = 0$ is equivalent to $m(x) \in \mathcal{S}_k(N)$ (proposition 6 combined with the fact that $\mathcal{S}_k(N)$ is the kernel of ∂). Because of the relations

$$\phi + \phi_{|\sigma} = \phi + \phi_{|\tau} + \phi_{|\tau^2} = \phi - \phi_{|J} = 0,$$

the linear map ϕ factorizes through a linear map on $\mathcal{M}_k(N)$ (see proposition 3), which induces a linear map ϕ_m on $\mathcal{S}_k(N)$. It is a consequence of the theorem 3 and the proposition 10 that the Hecke algebra on $S_k(N)$ is canonically isomorphic to the complex algebra generated by the operators T_n on $\mathcal{S}_k(N)$. The map

$$\begin{array}{ccc} \mathbb{T} & \to & \mathbb{C} \\ T & \mapsto & \phi_m(Tm(x)) \end{array}$$

is a linear map on the Hecke algebra. We use the theorem 2 and the proposition 20 to establish the following equalities

$$\begin{aligned} \phi_m(T_n m(x)) &= \sum_{\lambda \in E_N} \phi_m\Big(\sum_{M \in \mathcal{X}_n} [P_{\lambda|\tilde{M}}, \lambda M]\Big) \\ &= \sum_{\lambda \in E_N} \phi\Big(\sum_{M \in \mathcal{X}_n} P_{\lambda|\tilde{M}}[\lambda M]\Big) \\ &= \sum_{M \in \mathcal{X}_n} \phi_{|M}(x). \end{aligned}$$

Because of the property $\phi(P[x]) = 0$, if $x \notin E_N$, we do not have to make the restriction $\lambda M \in E_N$. By application of the theorem 6 we obtain that

$$\begin{aligned} \sum_{n=1}^{\infty} \phi_m(T_n m(x)) q^n &= \sum_{n=1}^{\infty} \Big(\sum_{M \in \mathcal{X}_n} \phi_{|M}(x)\Big) q^n \\ &= \sum_{M \in \mathcal{X}} \phi_{|M}(x) q^{\det M} \end{aligned}$$

is the Fourier expansion of an element of $S_k(N)$.

Let A be an element of $\mathcal{S}_k(N)$ such that for all $f \in S_k(N)$, we have $a_1(f) =< f, A >$ (We can take the element A_1 of section 3.3).

Let $f \in S_k(N)$. Let ϕ_f be the unique linear form $\mathcal{S}_k(N) \to \mathbb{C}$ such that $\phi_f(y) =< f, y >$ for all $y \in \mathcal{S}_k(N)$. By considering the linear form ϕ equal to ϕ_f composed with the canonical surjection from the kernel of b to $\mathcal{S}_k(N)$ and by choosing an element $x \in \mathbb{C}_{k-2}[X, Y][(\mathbb{Z}/N\mathbb{Z})^2]$ such that $b(x) = 0$ and of image A in $\mathcal{S}_k(N)$ as parameters, we obtain the following Fourier expansion of f

$$\sum_{n=1}^{\infty} \phi(T_n x) q^n = \sum_{n=1}^{\infty} \phi_f(T_n A) q^n = \sum_{n=1}^{\infty} a_1(T_n f) q^n = \sum_{n=1}^{\infty} a_n(f) q^n.$$

This proves that all modular forms can be obtained by this method.

If ϕ satisfies the additional condition

$$\phi(P[\lambda u, \lambda v] = \chi(\lambda)\phi(P[u, v])$$

$((\lambda, u, v) \in (\mathbb{Z}/N\mathbb{Z})^3$, $P \in \mathbb{C}_{k-2}[X, Y])$, then f belongs to $S_k(N, \chi)$. This follows from the proposition 14 and the theorem 6.

Remark .- 1) We will outline now the proof of the statement following the theorem 1 in the introduction. The Hecke operators operate on $\mathcal{B}_k(N)$ (see the operator T_Δ^∂ of the section 2.1). One can construct an isomorphism of complex vector spaces between the space of Eisenstein series of weight k for $\Gamma_1(N)$ and $\partial(\mathcal{M}_k(N)) \subset \mathcal{B}_k(N)$. Moreover this isomorphism is compatible with the action of Hecke operators. This can be seen through the Eichler-Simura isomorphism. Since we have the exact sequence

$$0 \to \mathcal{S}_k(N) \to \mathcal{M}_k(N) \to \mathcal{B}_k(N),$$

the Hecke algebra for $\mathcal{M}_k(N)$ is isomorphic to the Hecke algebra for the space $S_k(N)\oplus$ Eisenstein series. By application of the theorem 6 we should then prove the following result. If the element x in the theorem 1 does not verify $b(x) = 0$, then the series

$$\sum_{M \in \mathcal{X}} \phi_{|M}(x) q^{\det M}$$

is, except for the constant term, the Fourier expansion at infinity of a modular form of weight k for $\Gamma_1(N)$ which is not necessarily parabolic.

2) If x belongs to the new part of $\mathcal{M}_k(N)$, then the Fourier expansion is the Fourier expansion of a newform. This follows from an application of the theorem 6.

3) As noticed in the introduction, we can replace the series connected to $\mathcal{X}$ by the Manin-Heilbronn series: With the hypotheses of the theorem 1, the series

$$\sum_{M \in \mathcal{S}} \phi_{|M}(x) q^{\det M} + \frac{1}{2} \sum_{M \in \mathcal{S}'} \phi_{|M}(x) q^{\det M}$$

is the Fourier expansion of an element of $S_k(N)$. The possibility of producing modular forms using the sets $\mathcal{S}$ and $\mathcal{S}'$ was already noticed by Manin (see [4] and [6]).

4.3 Construction of bases of $S_k(N)$

Proposition 22 *Let $(\psi_i)_{i \in I}$ be a basis of the complex vector space $Hom_{\mathbb{C}}(\mathcal{S}_k(N)^+, \mathbb{C})$ (resp. $Hom_{\mathbb{C}}(\mathcal{S}_k(N)^-, \mathbb{C})$). Let $A \in \mathcal{S}_k(N)^+$ (resp. $\mathcal{S}_k(N)^-$) such that $\mathbb{T}A = \mathcal{S}_k(N)^+$ (resp. $\mathbb{T}A = \mathcal{S}_k(N)^-$). Then, for $i \in I$, the modular form f_i of Fourier expansion*

$$\sum_{n=1}^{\infty} \psi_i(T_n A) q^n$$

runs through a basis of $S_k(N)$ when i runs through I.

Proof .- The fact that f_i $(i \in I)$ is a modular form is a consequence of the theorem 6 and of the theorem 3. We suppose that the family of modular forms $(f_i)_{i\in I}$ is linearly dependant, i.e. there exists a family of complex numbers $(\lambda_i)_{i\in I}$ such that $\sum_{i\in I}\lambda_i f_i = 0$. We set $\psi = \sum_{i\in I}\lambda_i\psi_i$. We have $\psi(T_nA) = 0$ for all $n \geq 1$. Since $\mathbb{T}A = \mathcal{S}_k(N)^+$, the elements T_nA generate $\mathcal{S}_k(N)^+$ when n runs through the integers ≥ 1. So we have $\psi = 0$. Since $(\psi_i)_{i\in I}$ is a basis of $\mathrm{Hom}_{\mathbb{C}}(\mathcal{S}_k(N)^+, \mathbb{C})$, we have $\lambda_i = 0$ for all $i \in I$. Because of the equality of the dimensions of $\mathcal{S}_k(N)^+$ and $S_k(N)$ the proposition is proved with respect to $\mathcal{S}_k(N)^+$ (for $\mathcal{S}_k(N)^-$ the proof is of course similar).

Remark .- It is in fact difficult to find an element A in $\mathcal{S}_k(N)^+$ expressed as a linear combination of Manin symbols and satisfying $\mathbb{T}A = \mathcal{S}_k(N)^+$. But the set of such elements is $\mathcal{S}_k(N)^+$ minus the union of a finite number of proper subspaces.

We can restate the proposition 22 as follows.

Corollary 1 *Let A be as in the proposition 20. Let $x \in \mathbb{C}_{k-2}[X,Y][E_N]$ such that the image of x in $\mathcal{M}_k(N)$ is equal to A. Let $(\phi_i)_{i\in I}$ be a linearly free family in the space of linear forms $\phi \in Hom_{\mathbb{C}}(\mathbb{C}_{k-2}[X,Y][E_N], \mathbb{C})$ satisfying*

$$\phi + \phi_{|\sigma} = \phi + \phi_{|\tau} + \phi_{|\tau^2} = \phi - \phi_{|-I} = 0,$$

and

$$\phi = \phi_{|\eta}.$$

We suppose also that the space generated by the family $(\phi_i)_{i\in I}$ is in direct sum with the space of elements of $Hom_{\mathbb{C}}(\mathbb{C}_{k-2}[X,Y][E_N], \mathbb{C})$ which factorize through the linear map

$$b \ : \ \mathbb{C}_{k-2}[X,Y][E_N] \to \mathbb{C}[\Gamma_1(N)\backslash\mathbb{Q}^2]_k.$$

Moreover we suppose that the family $(\phi_i)_{i\in I}$ is maximal with respect to the above properties (i.e. the cardinality of I is equal to the dimension of $S_k(N)$). Then

$$\sum_{M\in\mathcal{X}} \phi_{i|M}(x) q^{\det M}$$

runs through the Fourier expansion of a basis of $S_k(N)$ when i runs through I.

Proof .- This is a direct reformulation of the proposition 22, obtained by considering the results obtained in the section 1.3, 1.4, 2.2 and 4.2. The list of equalities satisfied by ϕ_i $(i \in I)$ expresses that ϕ_i factorizes through $\mathcal{M}_k(N)/(1-\iota^*)\mathcal{M}_k(N)$. Because of the direct sum, the family of linear forms induced by $(\phi_i)_{i\in I}$ on $\mathcal{S}_k(N)/((1-\iota^*)\mathcal{M}_k(N) \cap \mathcal{S}_k(N)) \simeq \mathcal{S}_k(N)^+)$ is linearly free. So the family of linear forms on $\mathcal{S}_k(N)^+$ defined by $(\phi_i)_{i\in I}$ is linearly free. It is a basis because of the maximality condition. We are in position to apply the proposition 22, with $(\psi_i)_{i\in I}$ equal to the family of linear forms on $\mathcal{S}_k(N)^+$ defined by $(\phi_i)_{i\in I}$.

Let A_1 be the unique modular symbol in $\mathcal{S}_k(N)$ such that for all $f \in S_k(N)$, we have $a_1(f) =< f, A_1 >$ and $a_1(f) =< \iota(f), A_1 >$. Let $A_1^+ + A_1^-$ be the decomposition of A_1 with respect to the direct sum

$$\mathcal{S}_k(N) = \mathcal{S}_k(N)^+ \oplus \mathcal{S}_k(N)^-.$$

Proposition 23 *The modular symbols $A_1^+ \in \mathcal{S}_k(N)^+$ and $A_1^- \in \mathcal{S}_k(N)^-$ verify*

$$\mathbb{T}A_1^+ = \mathcal{S}_k(N)^+ \quad \textit{and} \quad \mathbb{T}A_1^- = \mathcal{S}_k(N)^-.$$

Proof .- Because of the proposition 7, we only have to prove that no modular form $f \in S_k(N)$ is orthogonal to $\mathbb{T}A_1^+$. We have for $f \in S_k(N)$ and n an integer ≥ 1

$$\begin{aligned} < f, T_n A_1^+ > &= < T_n f, \frac{1}{2}(A_1 + \iota^*(A_1)) > \\ &= \frac{1}{2} < T_n f + \iota(T_n f), A_1 > \\ &= \frac{1}{2}(a_1(T_n f) + a_1(\iota(\iota(T_n f)))) \\ &= a_1(T_n f) \\ &= a_n(f). \end{aligned}$$

If f is orthogonal to $\mathbb{T}A_1^+$, we have $a_n(f) = 0$ for all n. So $f = 0$. The proposition is proved.

Remark .- The modular symbol A_1 defined above is a canonical element of $\mathcal{M}_k(N)$. It would be interesting to obtain an expression as a linear combination of Manin symbols of A_1.

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