2.4 The Conjecture for Non-Modular Abelian Varieties

Conjecture 2.3.1 can be extended to general abelian varieties over global fields. Here we discuss only the case of a general abelian variety A over $\mathbf{Q}$. We follow the discussion in [Lan91, 95-94] (Lang, Number Theory III), which describes Gross's formulation of the conjecture for abelian varieties over number fields, and to which we refer the reader for more details.

For each prime number ℓ, the ℓ-adic *Tate module* associated to A is

$$\mathrm{Ta}_\ell(A) = \varprojlim_n A(\overline{\mathbf{Q}})[\ell^n].$$

Since $A(\overline{\mathbf{Q}})[\ell^n] \cong (\mathbf{Z}/\ell^n\mathbf{Z})^{2\dim(A)}$, we see that $\mathrm{Ta}_\ell(A)$ is free of rank $2\dim(A)$ as a $\mathbf{Z}_\ell$-module. Also, since the group structure on A is defined over $\mathbf{Q}$, $\mathrm{Ta}_\ell(A)$ comes equipped with an action of $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$:

$$\rho_{A,\ell} : \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \to \mathrm{Aut}(\mathrm{Ta}_\ell(A)) \approx \mathrm{GL}_{2d}(\mathbf{Z}_\ell).$$

Suppose p is a prime and let $\ell \neq p$ be another prime. Fix any embedding $\overline{\mathbf{Q}} \hookrightarrow \overline{\mathbf{Q}}_p$, and notice that restriction defines a homorphism $r : \mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p) \to \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$. Let $G_p \subset \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ be the image of r. The inertia group $I_p \subset G_p$ is the kernel of the natural surjective reduction map, and we have an exact sequence

$$0 \to I_p \to \mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p) \to \mathrm{Gal}(\overline{\mathbf{F}}_p/\mathbf{F}_p) \to 0.$$

The Galois group $\mathrm{Gal}(\overline{\mathbf{F}}_p/\mathbf{F}_p)$ is isomorphic to $\widehat{\mathbf{Z}}$ with canonical generator $x \mapsto x^p$. Lifting this generator, we obtain an element $\mathrm{Frob}_p \in \mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$, which is well-defined up to an element of I_p. Viewed as an element of $G_p \subset \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$, the element Frob_p is well-defined up I_p and our choice of embedding $\overline{\mathbf{Q}} \hookrightarrow \overline{\mathbf{Q}}_p$. One can show that this implies that $\mathrm{Frob}_p \in \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ is well-defined up to I_p and conjugation by an element of $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$.

For a G_p-module M, let

$$M^{I_p} = \{x \in M : \sigma(x) = x \text{ all } \sigma \in I_p\}.$$

Because I_p acts trivially on M^{I_p}, the action of the element $\mathrm{Frob}_p \in \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ on M^{I_p} is well-defined up to conjugation (I_p acts trivially, so the "up to I_p" obstruction vanishes). Thus the characteristic polynomial of Frob_p on M^{I_p} is well-defined, which is why $L_p(A,s)$ is well-defined. The *local L-factor* of $L(A,s)$ at p is

$$L_p(A,s) = \frac{1}{\det\left(I - p^{-s}\,\mathrm{Frob}_p^{-1} \,|\, \mathrm{Hom}_{\mathbf{Z}_\ell}(\mathrm{Ta}_\ell(A), \mathbf{Z}_\ell)^{I_p}\right)}.$$

Definition 2.4.1. $L(A,s) = \displaystyle\prod_{\text{all } p} L_p(A,s)$

For all but finitely many primes $\mathrm{Ta}_\ell(A)^{I_p} = \mathrm{Ta}_\ell(A)$. For example, if $A = A_f$ is attached to a newform $f = \sum a_n q^n$ of level N and $p \nmid \ell{\cdot}N$, then $\mathrm{Ta}_\ell(A)^{I_p} = \mathrm{Ta}_\ell(A)$. In this case, the Eichler-Shimura relation implies that $L_p(A,s)$ equals $\prod L_p(f_i,s)$, where the $f_i = \sum a_{n,i}q^n$ are the Galois conjugates of f and $L_p(f_i,s) = (1 - a_{p,i}\cdot p^{-s} + p^{1-2s})^{-1}$. The point is that Eichler-Shimura can be used to show that the characteristic polynomial of Frob_p is $\prod_{i=1}^{\dim(A)}(X^2 - a_{p,i}X + p^{1-2s})$.

Theorem 2.4.2. $L(A_f,s) = \prod_{i=1}^{d} L(f_i,s)$.