11

Newforms and Euler Products: Math 252, Stein, 11/10/2003

In this chapter we discuss the work of Atkin, Lehner, and W. Li on newforms and their associated L-series and Euler products.

Bibliography: [36] refers to W-C. Li, *Newforms and functional equations*, Math. Ann. **212** (1975), 285–315.

11.1 Atkin, Lehner, Li Theory

The results of [36] about newforms are proved using many linear transformations that do not necessarily preserve $S_k(\Gamma_1(N),\varepsilon)$. Thus we introduce more general spaces of cusp forms, which these transformations preserve. These spaces are also useful because they make precise how the space of cusp forms for the full congruence subgroup $\Gamma(N)$ can be understood in terms of spaces $S_k(\Gamma_1(M),\varepsilon)$ for various M and ε, which justifies our usual focus on these latter spaces. This section follows [36] closely.

Let M and N be positive integers and define

$$\Gamma_0(M,N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbf{Z}) \,:\, M \mid c, N \mid b \right\},$$

and

$$\Gamma(M,N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(M,N) \,:\, a \equiv d \equiv 1 \pmod{MN} \right\}.$$

Note that $\Gamma_0(M,1) = \Gamma_0(M)$ and $\Gamma(M,1) = \Gamma_1(M)$. Let $S_k(M,N)$ denote the space of cusp forms for $\Gamma(M,N)$.

If ε is a Dirichlet character modulo MN such that $\varepsilon(-1) = (-1)^k$, let $S_k(M,N,\varepsilon)$ denote the space of all cups forms for $\Gamma(M,N)$ of weight k and character ε. This is the space of holomorphic functions $f : \mathfrak{h} \to \mathbf{C}$ that satisfy the usual vanishing conditions at the cusps and such that for all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(M, N)$,

$$f|\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \varepsilon(d) f.$$

We have

$$S_k(M, N) = \oplus_\varepsilon S_k(M, N, \varepsilon).$$

We now introduce operators between various $S_k(M, N)$. Note that, except when otherwise noted, the notation we use for these operators below is as in [36], which conflicts with notation in various other books. When in doubt, check the definitions.

Let

$$f|\begin{pmatrix} a & b \\ c & d \end{pmatrix}(\tau) = (ad - bc)^{k/2}(c\tau + d)^{-k} f\left(\frac{a\tau + b}{c\tau + d}\right).$$

This is like before, but we omit the weight k from the bar notation, since k will be fixed for the whole discussion.

For any d and $f \in S_k(M, N, \varepsilon)$, define

$$f|U_d^N = d^{k/2-1} f\Big|\left(\sum_{u \bmod d} \begin{pmatrix} 1 & uN \\ 0 & d \end{pmatrix}\right),$$

where the sum is over *any* set u of representatives for the integers modulo d. Note that the N in the notation is a superscript, not a power of N. Also, let

$$f|B_d = d^{-k/2} f|\begin{pmatrix} d & 0 \\ 0 & 1 \end{pmatrix},$$

and

$$f|C_d = d^{k/2} f|\begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix}.$$

In [36], C_d is denoted W_d, which would be confusing, since in the literature W_d is usually used to denote a completely different operator (the Atkin-Lehner operator, which is denoted V_d^M in [36]).

Since $\left(\begin{smallmatrix} 1 & N \\ 0 & 1 \end{smallmatrix}\right) \in \Gamma(M, N)$, any $f \in S_k(M, N, \varepsilon)$ has a Fourier expansion in terms of powers of $q_N = q^{1/N}$. We have

$$\left(\sum a_n q_N^n\right)|U_d^N = \sum_{n \geq 1} a_{nd} q_N^n,$$

$$\left(\sum a_n q_N^n\right)|B_d = \sum_{n \geq 1} a_n q_N^{nd},$$

and

$$\left(\sum a_n q_N^n\right)|C_d = \sum_{n \geq 1} a_n q_{Nd}^n.$$

The second two equalities are easy to see; for the first, write everything out and use that for $n \geq 1$, the sum $\sum_u e^{2\pi i u n/d}$ is 0 or d if $d \nmid n$, $d \mid n$, respectively.

The maps B_d and C_d define injective maps between various spaces $S_k(M, N, \varepsilon)$. To understand B_d, use the matrix relation

$$\begin{pmatrix} d & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{pmatrix} x & dy \\ z/d & w \end{pmatrix}\begin{pmatrix} d & 0 \\ 0 & 1 \end{pmatrix},$$

and a similar one for C_d. If $d \mid N$ then $B_d : S_k(M,N,\varepsilon) \to S_k(dM, N/d, \varepsilon)$ is an isomorphism, and if $d \mid M$, then $C_d : S_k(M,N) \to S_k(M/d, Nd, \varepsilon)$ is also an isomorphism. In particular, taking $d = N$, we obtain an isomorphism

$$B_N : S_k(M,N,\varepsilon) \to S_k(MN,1,\varepsilon) = S_k(\Gamma_1(MN),\varepsilon). \tag{11.1.1}$$

Putting these maps together allows us to completely understand the cusp forms $S_k(\Gamma(N))$ in terms of spaces $S_k(\Gamma_1(N^2),\varepsilon)$, for all Dirichlet characters ε that arise from characters modulo N. (Recall that $\Gamma(N)$ is the principal congruence subgroup $\Gamma(N) = \ker(\mathrm{SL}_2(\mathbf{Z}) \to \mathrm{SL}_2(\mathbf{Z}/N\mathbf{Z}))$. This is because $S_k(\Gamma(N))$ is isomorphic to the direct sum of $S_k(N,N,\varepsilon)$, as ε various over all Dirichlet characters modulo N.

For any prime p, the pth *Hecke operator* on $S_k(M,N,\varepsilon)$ is defined by

$$T_p = U_p^N + \varepsilon(p)p^{k-1}B_p.$$

Note that $T_p = U_p^N$ when $p \mid N$, since then $\varepsilon(p) = 0$. In terms of Fourier expansions, we have

$$\left(\sum a_n q_N^n\right)|T_p = \sum_{n\geq 1}\left(a_{np} + \varepsilon(p)p^{k-1}a_{n/p}\right)q_N^n,$$

where $a_{n/p} = 0$ if $p \nmid n$.

The operators we have just defined satisfy several commutativity relations. Suppose p and q are prime. Then $T_pB_q = B_qT_p$, $T_pC_q = C_qT_p$, and $T_pU_q^N = U_q^NT_p$ if $(p, qMN) = 1$. Moreover $U_d^NB_{d'} = B_{d'}U_d^N$ if $(d,d') = 1$.

Remark 11.1.1. Because of these relations, (11.1.1) describe $S_k(\Gamma(N))$ as a module over the ring generated by T_p for $p \nmid N$.

Definition 11.1.2 (Old Subspace). The *old subspace* $S_k(M,N,\varepsilon)_{\mathrm{old}}$ is the subspace of $S_k(M,N,\varepsilon)$ generated by all $f|B_d$ and $g|C_e$ where $f \in S_k(M',N)$, $g \in S_k(M,N')$, and M', N' are proper factors of M, N, respectively, and $d \mid M/M'$, $e \mid N/N'$.

Since T_p commutes with B_d and C_e, the Hecke operators T_p all preserve $S_k(M,N,\varepsilon)_{\mathrm{old}}$, for $p \nmid MN$. Also, B_N defines an isomorphism

$$S_k(M,N,\varepsilon)_{\mathrm{old}} \cong S_k(MN,1,\varepsilon)_{\mathrm{old}}.$$

Definition 11.1.3 (Petersson Inner Product). If $f, g \in S_k(\Gamma(N))$, the *Petersson inner product* of f and g is

$$\langle f, g\rangle = \frac{1}{[\mathrm{SL}_2(\mathbf{Z}) : \Gamma(N)]}\int_D f(z)\overline{g(z)}y^{k-2}\,dx\,dy,$$

where D is a fundamental domain for $\Gamma(N)$ and $z = x + iy$.

This Petersson pairing is normalized so that if we consider f and g as elements of $\Gamma(N')$ for some multiple N' of N, then the resulting pairing is the same (since the volume of the fundamental domain shrinks by the index).

Proposition 11.1.4 (Petersson). *If $p \nmid N$, then $\langle f|T_p, g\rangle = \langle f, g|T_p\rangle$.*

Remark 11.1.5. The proposition implies that the T_p, for $p \nmid N$, are diagonalizable. Be careful, because the T_p, with $p \mid N$, need not be diagonalizable.

Definition 11.1.6 (New Subspace). The *new subspace* $S_k(M,N,\varepsilon)_{\text{new}}$ is the orthogonal complement of $S_k(M,N,\varepsilon)_{old}$ in $S_k(M,N,\varepsilon)$ with respect to the Petersson inner product.

Both the old and new subspaces of $S_k(M,N,\varepsilon)$ are preserved by the Hecke operators T_p with $(p,NM)=1$.

Remark 11.1.7. Li [36] also gives a purely algebraic definition of the new subspace as the intersection of the kernels of various trace maps from $S_k(M,N,\varepsilon)$, which are obtained by averaging over coset representatives.

Definition 11.1.8 (Newform). A *newform* $f=\sum a_n q_N^n \in S_k(M,N,\varepsilon)$ is an element of $S_k(M,N,\varepsilon)_{\text{new}}$ that is an eigenform for all T_p, for $p\nmid NM$, and is normalized so that $a_1=1$.

Li introduces the crucial "Atkin-Lehner operator" W_q^M (denoted V_q^M in [36]), which plays a key roll in all the proofs, and is defined as follows. For a positive integer M and prime q, let $\alpha=\text{ord}_q(M)$ and find integers x,y,z such that $q^{2\alpha}x-yMz=q^\alpha$. Then W_q^M is the operator defined by slashing with the matrix $\begin{pmatrix} q^\alpha x & y \\ Mz & q^\alpha\end{pmatrix}$. Li shows that if $f\in S_k(M,1,\varepsilon)$, then $f|W_q^M|W_q^M=\varepsilon(q^\alpha)f$, so W_q^M is an automorphism. Care must be taken, because the operator W_q^M need not commute with $T_p=U_p^N$, when $p\mid M$.

After proving many technical but elementary lemmas about the operators B_d, C_d, U_p^N, T_p, and W_q^M, Li uses the lemmas to deduce the following theorems. The proofs are all elementary, but there is little I can say about them, except that you just have to read them.

Theorem 11.1.9. *Suppose $f=\sum a_n q_N^n\in S_k(M,N,\varepsilon)$ and $a_n=0$ for all n with $(n,K)=1$, where K is a fixed positive integer. Then $f\in S_k(M,N,\varepsilon)_{\text{old}}$.*

From the theorem we see that if f and g are newforms in $S_k(M,N,\varepsilon)$, and if for all but finitely many primes p, the T_p eigenvalues of f and g are the same, then $f-g$ is an old form, so $f-g=0$, hence $f=g$. Thus the eigenspaces corresponding to the systems of Hecke eigenvalues associated to the T_p, with $p\nmid MN$, each have dimension 1. This is known as "multiplicity one".

Theorem 11.1.10. Let $f=\sum a_n q_N^n$ be a newform in $S_k(M,N,\varepsilon)$, p a prime with $(p,MN)=1$, and $q\mid MN$ a prime. Then

1. $f|T_p=a_pf$, $f|U_q^N=a_qf$, and for all $n\geq 1$,
$$a_pa_n = a_{np}+\varepsilon(p)p^{k-1}a_{n/p},$$
$$a_qa_n = a_{nq}.$$
If $L(f,s)=\sum_{n\geq 1}a_nn^{-s}$ is the Dirichlet series associated to f, then $L(f,s)$ has an Euler product
$$L(f,s)=\prod_{q\mid MN}(1-a_qq^{-s})^{-1}\prod_{p\nmid MN}(1-a_pp^{-s}+\varepsilon(p)p^{k-1}p^{-2s})^{-1}.$$
2. (a) If ε is not a character mod MN/q, then $|a_q|=q^{(k-1)/2}$.
   (b) If ε is a character mod MN/q, then $a_q=0$ if $q^2\mid MN$, and $a_q^2=\varepsilon(q)q^{k-2}$ if $q^2\nmid MN$.