Grigor, in W. Stein's Class

references
- Cycl thy : Iwasawa's book, Washington
- Mazur–Tate–Teitelbaum, Invent Math 84
- Greenberg, "Iwa. Thy. Past & Present"

plan/
- §0. Philosophy
- §1. Cycl. Thy
- §2. Modularity
- §3. Modular Symbols
- §4. $\hookrightarrow$ measure

tomorrow/ p-adic BSD
Iwasawa Thy

§0 X – varieties $\rightsquigarrow L(X,s)$ $\quad L(X,n)$ $n \in \mathbb{Z}$ – special vals

(e.g. E/Q ell. curve) $\longrightarrow L(E,s)$ $\quad$ have certain rationality properties

p-adic analogue?

§1 χ char conductor f $\rightsquigarrow L(\chi,s) = \sum \frac{\chi(n)}{n^s}$

$$\frac{t e^{xt}}{e^t - 1} = \sum B_n(x)\frac{t^n}{n!}, \quad B_n = B_n(0).$$

Thm: $n \geq 1 \Rightarrow L(\chi, 1-n) = -\frac{B_{\chi,n}}{n}$, where $f^{n-1}\sum_{a=1}^{f} \chi(a) B_n\left(\frac{a}{f}\right) \in \mathbb{Q}(\chi)$.

Thm: $\exists$ a p-adic mero. f'n $L_p(\chi,s)$ on $\{s \in \mathbb{C}_p \mid |s| < p^{1-\frac{1}{p-1}}\}$ s.t. $L_p(\chi, 1-n) = \left(1 - \chi\omega^{-n}(p)\, p^{n-1}\right)\frac{B_{\chi\omega^{-n},n}}{n}$

with ω = Teichmüller.

"p-adic L-f'ns for Ell. Curves/Q"

$E/\mathbb{Q} \rightsquigarrow L(E,s) = \prod_p L^{(p)}_{\text{local}}(E,s)$, $\quad L^{(p)}_{\text{local}}(E,s) = \begin{cases} \frac{1}{1 - a_p p^{-s} + p^{1-2s}} & p \nmid N_E \\ \frac{1}{1-p^{-s}} & \text{split mult.} \\ \frac{1}{1+p^{-s}} & \text{nonsplit mult.} \\ 1 & \text{additive} \end{cases}$

$= \sum a_n n^{-s}$

$L(E,\chi,s) = \sum \chi(n) a_n n^{-s}$.

§2. Modularity: $E/\mathbb{Q}$ modular $\Longleftrightarrow \exists f \in S_2(\Gamma_0(N_E))$ s.t. $L(E,s) = L(f,s) = \sum \frac{a(n)}{n^s}$
norm. eigenform, $a(n) \in \mathbb{Q}$

$$L(f,s) = \frac{(2\pi)^s}{\Gamma(s)}\int_0^\infty f(iy)\, y^s \frac{dy}{y} \quad \text{Mellin transform}$$

$f \in S_2(N,\varepsilon)$ ε mod N, $r \in \mathbb{P}^1(\mathbb{Q}) = \mathbb{Q} \cup \infty$ def $\Phi(f,r) = 2\pi i \int_{i\infty}^{r} f(z)\,dz = \begin{cases} 2\pi \int_0^\infty f(r+it)\,dt & r \in \mathbb{Q}, \\ 0 & r = \infty. \end{cases}$

so $\boxed{\Phi(f,0) = L(f,1)}$. $\quad \chi$ mod $m \rightsquigarrow \tau(n,\chi) = \sum_{a \bmod m} \chi(a) e^{2\pi i a n/m}$, $\quad \tau(n,\chi) = \bar{\chi}(n)\underset{\tau(1,\chi)}{\tau(\chi)}$.

if $f_\chi(z) = \sum \chi(n) a_n q^n$ then $f_{\bar{\chi}}(z) = \frac{1}{\tau(\chi)} \sum_{a \bmod m} \chi(a) f(z + a/m)$

$\rightsquigarrow \Phi(f_{\bar{\chi}}, r) = \frac{1}{\tau(\chi)} \sum_a \chi(a)\, \Phi(f, r + a/m)$

Thm: $L(f_{\bar{\chi}}, 1) = \frac{\tau(\bar{\chi})}{m} \sum_a \chi(a)\, \Phi\left(f, \frac{a}{m}\right)$

§3. $\lambda(f; a, m) \stackrel{\text{def}}{=} \Phi(f, -a/m)$ $(m, a \in \mathbb{Q}, m > 0)$. think of $\{i\infty, a/m\}$ as "modular" symbols"

§4. Construction of the p-adic L-function

χ Dirichlet char. cond. m.

I. $L(f_{\bar\chi},1) = \frac{\tau(\bar\chi)}{m}\sum_{a \bmod m} \chi(a)\,\lambda(f;a,m)$

II. "remove the Euler factor at p"

<u>Assume</u>: p good ordinary for E, so $a_p \neq 0$.

then $X^2 - a_p X + p = (X-\alpha_p)(X-\beta_p)$, $\alpha \in \mathbb{Z}_p^\times$, $\beta \in p\mathbb{Z}_p$

⇝ $\tilde\lambda(f;a,m) = \lambda(f;a,m) - \alpha_p^{-1}\lambda(f;pa,m)$.

III. $\tilde\lambda$ is a distribution: $\sum_{\substack{b \bmod mp \\ b\equiv a(m)}} \tilde\lambda(f;b,mp) = \alpha_p\,\tilde\lambda(f;a,m)$

(comes from formula for action of T_p on modular symbols)

<u>Mazur's Measure:</u> $\tilde\lambda_{f,\mathbb{Q}}(B_{p^n}(a)) = \alpha_p^{-n}\,\tilde\lambda(f;a,p^n)$ gives $\tilde\lambda_{f,\mathbb{Q}}$ on $\mathbb{Z}_p^\times$

<u>Prop</u>: $\chi:\mathbb{Z}_p^\times \to \mathbb{C}_p^\times$ cont. fin. order char conductor p^n $\Rightarrow$ $\int_{\mathbb{Z}_p^\times}\chi(x)\,d\tilde\lambda_{f,\mathbb{Q}}(x) = \begin{cases} p^n L(f,\bar\chi,1)/\tau(\bar\chi) & \chi \neq 1 \\ (1-\alpha_p^{-1})L(E,1) & \chi = 1 \end{cases}$

<u>remark</u>: the $\mathbb{Z}$-module gen by the $\tilde\lambda(f;a,m)$ is a lattice in $\mathbb{C}$, for f fixed.

$\lambda(f;a,m) = (a,m)^+\Omega_E^+ + (a,m)^-\Omega_E^-$, where $(a,m)^\pm$ rational w/ bounded denominators.

$\mu = \tilde\lambda/\Omega_E^+$ ⇝ $L_p(E,s) = \int_{\mathbb{Z}_p^\times}\langle x\rangle^{s-1}\,d\mu(x)$.

<u>Conjecture</u> (p-adic BSD for p good ordinary):

1) $\mathrm{ord}_{s=1}L_p(E,s) = r = \mathrm{rank}_{\mathbb{Z}}E(\mathbb{Q})$

2) $\frac{L_p^{(r)}(E,1)}{r!} = \#\text{Ш}(E/\mathbb{Q})\cdot \mathrm{Reg}_p(E/\mathbb{Q})\cdot\prod_v c_v$.

<u>Thm</u> (Kato): $\mathrm{ord}_{s=1}L_p(E,s) \geq r$

Iwasawa Thy: $\mathbb{Q}_\infty/\mathbb{Q}$ cycl. $\mathbb{Z}_p$-ext'n, $\mathrm{Gal}(\mathbb{Q}_\infty/\mathbb{Q}) = \Gamma$, $X_n = Cl(\mathbb{Q}_n)[p^\infty]$, $X = \varprojlim X_n \in \overset{\Lambda \,=\, \mathbb{Z}_p[[X]] \,\cong}{\mathbb{Z}_p[[\Gamma]]}$-mod.

Λ = measures on Γ w/ vals in $\mathbb{Z}_p$.

<u>Thm</u>: $e_n = v_p(\#X_n)$ $\Rightarrow$ $\exists\, \mu,\lambda,\nu$ s.t. for $n \gg 0$, $e_n = \mu p^n + \lambda n + \nu$.

<u>Main Conjecture</u>: Structure of X as a Λ-mod (including μ,λ,ν) can be deduced from the $L_p(\chi,s)$

$E(\mathbb{Q}_n)$? rank $E(\mathbb{Q}_n)$? $\#\text{Ш}(E/\mathbb{Q}_n)$? ⇝ <u>Mazur proved</u>: If $E(\mathbb{Q})$ and $\text{Ш}(E/\mathbb{Q})[p^\infty]$ are finite then rank $E(\mathbb{Q}_\infty) < \infty$.

p ordinary ⇝ <u>Kato</u>: don't need hypotheses.

Conj: $|\text{Ш}(E/\mathbb{Q}_n)|_p = p^{\mu p^n + \lambda n + \nu}$ $n \gg 0$.