Lecture III: How Jacobians and Theta Functions Arise

I would like to begin by introducing Jacobians in the way that they actually were discovered historically. Unfortunately, my knowledge of 19th-century literature is very scant so this should not be taken too literally. You know the story began with Abel and Jacobi investigating general algebraic integrals

$$I = \int f(x)dx$$

where f was a multi-valued algebraic function of x, i.e., the solution to

$$g(x, f(x)) \equiv 0, \quad g \text{ polynomial in } 2 \text{ variables.}$$

So we can write I as

$$I = \int_{\gamma} y\, dx$$

where γ is a path in plane curve $g(x,y) = 0$; or we may reformulate this as the study of integrals

$$I(a) = \int_{a_0}^{a} \overbrace{\frac{P(x,y)}{Q(x,y)}dx}^{\omega}, \qquad \begin{array}{l} P,Q \text{ polynomials} \\ a,a_0 \in \text{plane curve } C: g(x,y) = 0 \end{array}$$

of <u>rational</u> differentials ω on plane curves C. The main result is that such integrals always admit an addition theorem: i.e., there is an integer g such that if a_0 is a base point, and $a_1,\ldots,a_{g+1}$ are any points of C, then one can determine up to permutation $b_1,\cdots,b_g \in C$ rationally in terms of the a's* such that

$$\int_{a_0}^{a_1} \omega + \cdots + \int_{a_0}^{a_{g+1}} \omega \equiv \int_{a_0}^{b_1} \omega + \cdots + \int_{a_0}^{b_g} \omega, \text{ mod periods of } \int\omega.$$

For instance, if $C = \mathbb{P}^1$, $\omega = dx/x$, then $g = 1$ and:

$$\int_1^{a_1} \frac{dx}{x} + \int_1^{a_2} \frac{dx}{x} = \int_1^{a_1 a_2} \frac{dx}{x} .$$

Iterating, this implies that for all $a_1,\cdots,a_g,b_1,\cdots,b_g \in C$, there are $c_1,\cdots,c_g \in C$ depending up to permutation rationally on the a's and b's such that

$$\sum_{i=1}^{g} \int_{a_0}^{a_i} \omega + \sum_{i=1}^{g} \int_{a_0}^{b_i} \omega \equiv \sum_{i=1}^{g} \int_{a_0}^{c_i} \omega \quad \text{(mod periods)} .$$

Now this looks like a group law! Only a very slight strengthening will lead us to a reformulation in which this most classical of all theorems will suddenly sound very modern. We introduce the concept of an algebraic group G: succinctly, this is a "group object in the category of varieties," i.e., it is simultaneously a variety and a group where the group law $m: G\times G \longrightarrow G$ and the inverse $i: G \longrightarrow G$ are morphisms of varieties. Such a G is, of course, automatically a complex analytic Lie group too, hence it has a Lie algebra Lie(G), and an exponential map exp: Lie(G) $\longrightarrow$ G. Now I wish to rephrase

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* E.g., one can find polynomials $g_i(x, y; a)$ in x, y and the coordinates of the a's such that the b_i's are the set of all $b \in C$ such that $g_i(b; a) = 0$.

Abel's theorem as asserting that if C is a curve, and ω is any rational differential on C, then the multi-valued function

$$a \longmapsto \int_{a_o}^{a} \omega$$

can be factored into a composition of 3 functions:

$$C\text{-(poles of } \omega) \xrightarrow{\ \phi\ } J \twoheadleftarrow^{\exp} \text{Lie } J \xrightarrow{\ \ell\ } \mathbb{C}$$

where:

i) J is a commutative algebraic group,

ii) ℓ is a linear map from Lie J to $\mathbb{C}$

iii) ϕ is a morphism of varieties; and, in fact, if $g = \dim J$, then if we use addition on J to extend ϕ to

$$\phi^{(g)}: [(C\text{-poles } \omega)\times\cdots\times(C\text{-poles } \omega)/\text{permutations}] \longrightarrow J$$
$$S_g$$

then $\phi^{(g)}$ is birational, i.e., is bijective on a Zariski-open set.

In our example

$$C = \mathbb{P}^1, \qquad \omega = dx/x,$$

then $J = \mathbb{P}^1-(0,\infty)$ which is an algebraic group where the group law is multiplication, and ϕ is the identity. The point is that J is the object that realizes the rule by which 2 g-tuples $(a_1,\cdots,a_g),(b_1,\cdots,b_g)$ are "added" to form a third $(c_1,\cdots,c_g)$, and so that the integral

$\sum_{i=1}^{g} I(x_i)$ becomes a homomorphism from J to $\mathbb{C}$. A slightly less fancy way to put it is that there is a ϕ: C-(poles ω) $\longrightarrow$ J and a <u>translation-invariant</u> differential η on J such that

$$\phi^* \eta = \omega,$$

hence

$$\int_{\phi(a_o)}^{\phi(a)} \eta \equiv \int_{a_o}^{a_o} \omega \quad \text{(mod periods).}$$

Among the ω's, the most important are those of 1st kind, i.e., without poles, and if we integrate all of them at once, we are led to the most important J of all: the <u>Jacobian</u>, which we call Jac. From property (iii), we find that Jac must be a <u>compact</u> commutative algebraic group, i.e., a complex torus, and we want that

$$\phi\colon \; C \longrightarrow \text{Jac},$$

should set up a bijection:

iv) $$\phi^*\colon \begin{bmatrix} \text{translation-} \\ \text{invariant 1-forms} \\ \eta \text{ on Jac} \end{bmatrix} \longrightarrow \begin{bmatrix} \text{rational differentials} \\ \omega \text{ on C w/o poles} \end{bmatrix} =$$

Thus

$$\dim \text{Jac} = \dim R_1(C)$$
$$= \text{genus } g \text{ of C.}$$

To construct Jac explicitly, there are 2 simple ways:

v) Analytically: write Jac = V/L, V complex vector space, L a lattice. Define:

$$V = \text{dual of } R_1(C)$$

$$L = \left\{ \begin{array}{l} \text{set of } \ell \in V \text{ obtained as periods, i.e.,} \\ \ell(\omega) = \int_\gamma \omega \quad \text{for some 1-cycle } \gamma \text{ on } C. \end{array} \right.$$

Fixing a base point $a_o \in C$, define for all $a \in C$

$$\phi(a) = \left\{ \begin{array}{l} \text{image in } V/L \text{ of any } \ell \in V \text{ defined by} \\ \qquad \ell(\omega) = \int_{a_o}^{a} \omega, \\ \text{where we fix a path from } a_o \text{ to } a. \end{array} \right.$$

Note that since Jac is a group,

$$V^* \cong \left(\begin{array}{c} \text{translation-invariant} \\ \text{1-forms on Jac} \end{array} \right) \cong \left(\begin{array}{c} \text{cotangent sp. to Jac at } \alpha \\ \text{any } \alpha \in \text{Jac} \end{array} \right) \cong R_1(C).$$

vi) Algebraically: following Weil's original idea, introduce $S^gC = C \times \cdots \times C/S_g$ and construct by the Riemann-Roch theorem, a "group-chunk" structure on S^gC, i.e., a partial group law:

$$m: \; U_1 \times U_2 \longrightarrow U_3$$

$$U_i \subset S^gC \quad \text{Zariski-open.}$$

He then showed that any such algebraic group-chunk prolonged automatically into an algebraic group J with

$S^gC \supset U_4 \subset J$ (some Zariski-open U_4).

An important point is that ϕ is an integrated form of the canonical map $\Phi: C \longrightarrow \mathbb{P}^{g-1}$ discussed at length above –

vii) Φ is the Gauss map of ϕ, i.e., for all $x \in C$, $d\phi(T_{x,C})$ is a 1-dimensional subspace of $T_{\phi(x),Jac}$, and by translation this is isomorphic to Lie(Jac). If $\mathbb{P}^{g-1}$ = [space of 1-dim$^{\underline{1}}$ subsp. of Lie(Jac)], then $d\phi: C \longrightarrow \mathbb{P}^{g-1}$ is just Φ.

(Proof: this is really just a rephrasing of (iv).)

The Jacobian has always been the corner-stone in the analysis of algebraic curves and compact Riemann surfaces. Its power lies in the fact that it <u>abelianizes</u> the curve and is a <u>reification</u> of H_1, e.g.,

viii) Via $\phi: C \longrightarrow Jac$, every abelian covering $\pi: C_1 \longrightarrow C$ is the "pull-back" of a unique covering $p: G_1 \longrightarrow Jac$ (i.e., $C_1 \cong C \times_{Jac} G_1$).

Weil's construction in vi) above was the basis of his epoch-making proof of the Riemann Hypothesis for curves over finite fields, which really put characteristic p algebraic geometry on its feet.

There are very close connections between the geometry of the curve C (e.g., whether or not C is hyperelliptic) and Jac. We want to describe these next in order to tie in Jac with the special cases studied in Lecture I, and in order to "see" Jac very concretely in low genus. The main tool we want to use is: