10. Jacobians of modular curves

In this section we will examine the Jacobians of modular curves, their reduction modulo primes, and the endomorphisms induced by Hecke operators.

10.1. Abelian varieties and Jacobians.

PRIMARY REFERENCES:
[**Mum1**], [**Rosen**], [**Mil2**], [**Mil3**] and [**BLRa**, Chapters 8,9].

We now review some generalities concerning abelian varieties and Jacobians of algebraic curves.

We first recall that an abelian variety A over an algebraically closed field k is a proper group variety over k. It is necessarily smooth, projective and commutative [**Mil2**, §1,2]. One can consider more generally abelian schemes, or families of abelian varieties, over an arbitrary base scheme S. An abelian scheme over S is a smooth proper group scheme over S whose geometric fibers are abelian varieties [**Mil2**, §20].

If $k = \mathbf{C}$ and A is a g-dimensional abelian variety, then the complex manifold $A(\mathbf{C})$ is isomorphic to a complex torus V/L where V is a g-dimensional vector space and L is a discrete subgroup of rank $2g$ [**Rosen**, §1]. An arbitrary complex torus V/L can be identified with the set of complex points of an abelian variety over $\mathbf{C}$ if and only if V/L possesses a non-degenerate Riemann form [**Rosen**, §3], i.e., a positive definite Hermitian form on V whose imaginary part is integer valued on L. In this case, the same is true for the complex torus V^*/L^* where $V^* \subset \mathrm{Hom}_{\mathbf{R}}(V,\mathbf{C})$ is the space of conjugate linear functions on V (i.e., additive functions ϕ satisfying $\phi(zv) = \bar{z}\phi(v)$ for all $z \in \mathbf{C}$, $v \in V$), and $L^* = \{\phi \in V^* | \phi(L) \subset \mathbf{R} + i\mathbf{Z}\}$. If A and A^* are abelian varieties satisfying $A(\mathbf{C}) \cong V/L$ and $A^*(\mathbf{C}) \cong V^*/L^*$, then A^* is called the dual abelian variety of A [**Rosen**, §4]. Note that A is isomorphic to $(A^*)^*$.

Now let C be a Riemann surface and let W denote the complex vector space of holomorphic differentials on C. Consider the complex torus V/L where $V = \mathrm{Hom}(W,\mathbf{C})$ and L is the image of the map $H_1(C,\mathbf{Z}) \to \mathrm{Hom}(W,\mathbf{C})$ defined by integration. Note that the cotangent space of V/L at the origin may be naturally identified with W. The intersection pairing on $H_1(C,\mathbf{Z})$ can be used to define a nondegenerate Riemann form on V/L, and the resulting abelian variety J is called the Jacobian of C [**Mil3**, §2]. Moreover this Riemann form gives rise to a canonical isomorphism $J \cong J^*$.

Another interpretation of the Jacobian of C is provided by the Picard functor Pic^0 (see [**Mil3**, §1]). Let $\mathrm{Div}^0(C)$ denote the group of divisors on C of degree zero, and let $\mathrm{Pic}^0(C)$ denote $\mathrm{Div}^0(C)$ modulo the group of principal divisors. Integration then defines a natural map $\mathrm{Div}^0(C) \to V/L$ which, according to the Abel-Jacobi theorem, induces a natural isomorphism of groups $\mathrm{Pic}^0(C) \cong J(\mathbf{C})$. Now choose a base-point P in C and define a mapping $C \to \mathrm{Pic}^0(C)$ by sending Q to the divisor $Q - P$. The resulting map $C \to V/L$ is analytic and induces an isomorphism $H^0(J(\mathbf{C}),\Omega^1) \to H^0(C,\Omega^1) = W$ which is independent of the base-point. Moreover the isomorphism is compatible with the natural identification of W with the cotangent space of $J(\mathbf{C}) \cong V/L$ at the origin.

To describe the Jacobian of a curve over any field, or indeed an arbitrary base scheme S, we use the Picard functor [**Mil3**, §8], [**BLRa**, Chapter 8]. For a morphism of schemes $s : X \to S$, Grothendieck [**Gro1**] defines a relative Picard

functor $\mathrm{Pic}_{X/S}$ on S-schemes by "sheafifying" the functor which sends T to the group of isomorphism classes of invertible sheaves on $X_T = X \times_S T$. Under quite general hypotheses (see Chapters 8 and 9 of [**BLRa**]) this contravariant functor is represented by a group scheme over S, and we denote its identity component $\mathrm{Pic}^0_{X/S}$. The definition is functorial in X, so that a morphism $Y \to X$ of S-schemes gives rise to a natural transformation $\mathrm{Pic}_{X/S} \to \mathrm{Pic}_{Y/S}$ and consequently a morphism $\mathrm{Pic}^0_{X/S} \to \mathrm{Pic}^0_{Y/S}$. We remark also that formation of $\mathrm{Pic}^0_{X/S}$ commutes with base change, meaning that $\mathrm{Pic}^0_{(X_T)/T}$ is naturally isomorphic to $\mathrm{Pic}^0_{X/S} \times_S T$.

If $X \to S$ is a relative curve, meaning that it is smooth and proper and its geometric fibers are curves, then $\mathrm{Pic}^0_{X/S}$ is an abelian scheme which we denote $J_{X/S}$ and call the Jacobian of X (over S), [**BLRa**, §9.2]. If also $S = \mathrm{Spec}\, k$ for an algebraically closed field k, then $\mathrm{Pic}_{X/S}(S)$ may be identified with the group of invertible sheaves on X, or equivalently, with $\mathrm{Div}(X)$ modulo the group of principal divisors. Then $\mathrm{Pic}^0_{X/S}(S)$ may be identified with $\mathrm{Pic}^0(X)$, the group $\mathrm{Div}^0(X)$ modulo the group of principal divisors. Moreover if $k = \mathbf{C}$, then the isomorphism $V/L \cong J(\mathbf{C}) \cong J_{X/\mathbf{C}}(\mathbf{C})$ is analytic, so our two descriptions of the Jacobian in this case are equivalent.

The relative Picard functor also provides a general construction of the dual of an abelian scheme. If A is an abelian scheme over S, then $\mathrm{Pic}_{A/S}$ is representable by a scheme, and $\mathrm{Pic}^0_{A/S}$ is an abelian scheme, [**BLRa**, §8.4, Theorem 5], [**FaCh**, I.1]. We write A^* for $\mathrm{Pic}^0_{A/S}$ and call it the dual abelian scheme of A. Again there is a natural isomorphism $A \cong (A^*)^*$. For a relative curve X over S there is a general construction of a "Θ-divisor" on $J_{X/S}$ which gives rise to an isomorphism $\phi_{X/S}$ of $J_{X/S}$ with $J^*_{X/S}$, [**BLRa**, §9.4]. The constructions of the dual abelian scheme, its biduality and the autoduality of the Jacobian are compatible with base-change. They are also compatible with the descriptions given above in the case $S = \mathrm{Spec}\,\mathbf{C}$.

A morphism $\pi : Y \to X$ of relative curves over S induces by Picard functoriality a homomorphism of abelian schemes $\pi^* : J_{X/S} \to J_{Y/S}$. We obtain also a homomorphism $\pi_* : J_{Y/S} \to J_{X/S}$ defined by the composite $\phi^{-1}_{Y/S} \circ (\pi^*)^* \phi_{X/S}$ where $(\pi^*)^* : J^*_{Y/S} \to J^*_{X/S}$ is again defined by Picard functoriality. We thus have two functors from the category of relative curves over S to the category of abelian schemes over S; the contravariant Picard functor Pic^0 defined by $\mathrm{Pic}^0(X) = J_{X/S}$ and $\mathrm{Pic}^0(\pi) = \pi^*$, and the covariant Albanese functor Alb defined by $\mathrm{Alb}(X) = J_{X/S}$ and $\mathrm{Alb}(\pi) = \pi_*$, [**Mil3**, §6]. If $S = \mathrm{Spec}\, k$ for an algebraically closed field k, then π^* on $J_{X/S}(S)$ is induced by the map $\mathrm{Div}(X) \to \mathrm{Div}(Y)$ defined by pull-back of divisors; a point $x \in X(S)$ is sent to $\Sigma_{y \in \pi^{-1}(x)} e_{y/x} y$ where $e_{y/x}$ is the ramification degree. On the other hand, π_* on $J_{Y/S}(S)$ is induced by the map $\mathrm{Div}(Y) \to \mathrm{Div}(X)$ which sends $y \in Y(S)$ to $\pi(y)$. Note that $\pi_* \circ \pi^*$ is simply multiplication by the degree of π.

There is in general a natural isomorphism of $s_*\Omega^1_{X/S}$ with the cotangent sheaf $i^*\Omega^1_{J_{X/S}/S}$ along the zero section $i : S \to J_{X/S}$. For $S = \mathrm{Spec}\, k$, this can be viewed as an isomorphism $H^0(X,\Omega^1_{X/S}) \cong \mathrm{Cot}_0(J_{X/S})$ (see [**Mil3**, Proposition 2.2]). Consider now the maps induced by π^* and π_* on the cotangent spaces at