Homework Assignment 8

(Math 252: Modular Abelian Varieties)

William A. Stein

Nov. 3 (Due: Nov. 12))

1. The principal congruence subgroup $\Gamma(N)$ of level N is the kernel of the reduction map $\mathrm{SL}_2(\mathbf{Z}) \to \mathrm{SL}(2, \mathbf{Z}/N\mathbf{Z})$. The subgroup $\Gamma_1(N)$ consists of matrices of the form $\left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right)$ modulo N. Let $\Gamma \subset \mathrm{SL}_2(\mathbf{Z})$ be a subgroup that contains $\Gamma(N)$ for some N. Show that there exists $g \in \mathrm{GL}(2, \mathbf{Q})$ such that the conjugate $g^{-1}\Gamma g$, which is a subgroup of $\mathrm{GL}(2, \mathbf{Q})$, contains $\Gamma_1(N^2)$. This shows that many problems about modular forms can be reduced to questions about modular forms for Γ_1.

2. Let $f \in S_k(\Gamma_1(N))$ be a nonzero modular form that is an eigenform for all the Hecke operators T_p and for the diamond bracket operators $\langle d \rangle$. Let
$$\varepsilon : (\mathbf{Z}/N\mathbf{Z})^* \to \mathbf{C}^*$$
be the character of f, so $\langle d \rangle f = \varepsilon(d) f$ for all $d \in (\mathbf{Z}/N\mathbf{Z})^*$. Show that f satisfies the following equation:
for any $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$,
$$f(z) = \varepsilon(d)(cz+d)^{-k} f\Big(\frac{az+b}{cz+d}\Big).$$

3. Let M be a positive integer and let p be a prime.
   (a) Show that there are two injective linear maps
   $$S_2(\Gamma_1(M)) \hookrightarrow S_2(\Gamma_1(pM))$$
   sending $f(q)$ to $f(q)$ and sending $f(q)$ to $f(q^p)$. (I proved this in class, and you can copy the proof from your notes.)
   (b) Is it ever the case that the intersection of the images of these two maps is nonzero?

4. Let M be an integer such that $S_2(\Gamma_1(M))$ has positive dimension, and let p be a prime (thus $M = 11$ or $M \geq 13$).
   (a) Let $f \in S_2(\Gamma_1(M))$ be an eigenvector for T_p with eigenvalue λ. Show that T_p acting on $S_2(\Gamma_1(Mp))$ preserves the two-dimensional subspace generated by f and $f(pz)$. (When p divides the level, we *define* $T_p(\sum a_n q^n) = \sum a_{np} q^n$.)
   (b) Show that if $\lambda^2 \neq 4p$ then T_p is diagonalizable on this 2-dimensional space. What are the eigenvalues of T_p on this space? In fact, one never has $\lambda^2 = 4p$; see [Coleman-Edixhoven, On the semisimplicity of the U_p operator on modular forms] for more details.
   (c) Show that for any $r > 2$, the Hecke operator T_p on $S_2(\Gamma_1(Mp^r))$ is not diagonalizable.
   (d) Deduce that for $r > 2$ the Hecke algebra $\mathbf{T}$ associated to $S_2(\Gamma_1(Mp^r))$ has nilpotent elements, so it is not a subring of a product of number fields.