Homework Assignment 6

(Math 252: Modular Abelian Varieties)

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Oct. 22 (Due: Oct. 29)

1. **Formation of Néron models commutes with products:**
   Let A and B be abelian varieties over $\mathbf{Q}$. Show that if $\mathcal{A}$ and $\mathcal{B}$ are the Néron models of A and B, respectively, then the Néron model of $A \times_{\mathbf{Q}} B$ is $\mathcal{A} \times_{\mathbf{Z}} \mathcal{B}$. [Hint: Use the universal property of products–see Hartshorne's book, Section 3, page 87. You can just work formally, if you are not familiar with schemes. Also, I chose the base to be $\mathbf{Q}$ for concreteness; the statement is true in general.]

2. **The Néron models functor:**

   (a) Suppose $\varphi : A \to B$ is a homomorphism of abelian varieties. Prove that there is a homomorphism $\mathcal{A} \to \mathcal{B}$ of Néron models whose generic fiber is φ.

   (b) Suppose $0 \to A \to B \to C \to 0$ is an exact sequence of abelian varieties. Prove that there is a complex of Néron models $0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$ whose generic fiber is $0 \to A \to B \to C \to 0$. [Hint: By "complex" I mean that the sequence of Néron models need not be exact. The only condition is that the composition of $\mathcal{A} \to \mathcal{B}$ with $\mathcal{B} \to \mathcal{C}$ is the 0 map.]

3. **Component groups over $\mathbf{F}_p$ and $\overline{\mathbf{F}}_p$:**
   Give an example of an abelian variety A and a prime p such that

   $$\Phi_{A,p}(\overline{\mathbf{F}}_p) \neq \Phi_{A,p}(\mathbf{F}_p).$$

   [Hint: Your A should be an elliptic curve. You may use Corollary 15.2.1, which is on page 359 of Silverman's book *Arithmetic of Elliptic Curves*, which we now recall. Let K be a finite extension of $\mathbf{Q}_p$ with valuation v normalized so that $v(\pi) = 1$, where π is a generator of the maximal ideal of the ring of integers of K, and let k be the residue class field of K. **Corollary 15.2.1.** *Let E be an elliptic curve over K. If E has split multiplicative reduction, then $\Phi_{E,v}(k)$ is cyclic of order $-\mathrm{ord}_v(j(E))$. In all other cases, $\Phi_{E,v}(k)$ has order at most* 4. Now choose an E with nonsplit multiplicative reduction over $\mathbf{Q}_p$, which attains split multiplicative reduction over an unramified extension of $\mathbf{Q}_p$.]