Homework Assignment 5

(Math 252: Modular Abelian Varieties)

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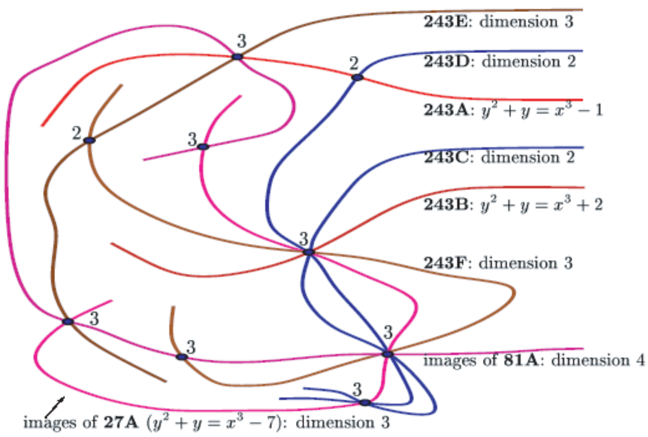
Oct. 15 (Due: Oct. 22)

There are two problems and parts of problems are of equal weight.

1. Let $T = V/T$ be a complex torus, and $T_1 = V_1/T_1$ and $T_2 = V_2/T_2$ be subtori of T such that $V_1 \cap V_2 = 0$. In this problem, you will learn how to compute $T_1 \cap T_2$, which will help you to understand how I designed the T-shirt with all the wavy lines for the MSRI conference on modularity of elliptic curves.



   (a) Prove that there is an exact sequence
   $$0 \to T_1 \cap T_2 \to T_1 \oplus T_2 \to T$$
   where the map $T_1 \oplus T_2 \to T$ is $(a, b) \mapsto a - b$.

   (b) Prove that there is an exact sequence
   $$0 \to T_1 \cap T_2 \to T_1 \oplus T_2 \to T' \to 0$$
   where $T' = T_1 + T_2 \subset T$.

   (c) Prove that $T' = (V_1 + V_2)/L'$, where $L' = (V_1 + V_2) \cap L$.

   (d) Conclude that
   $$T_1 \cap T_2 \cong L'/(L_1 + L_2),$$
   where L' is the saturation of $L_1 + L_2$ in L (i.e., $L' = (V_1 + V_2) \cap L$).

2. It is a fact (which you do not have to prove) that there is an abelian variety called $J_0(37) = V/L$ that contains the following two elliptic curves as abelian subvarieties:
$$E_1 : \qquad y^2 + y = x^3 - x$$
$$E_2 : \qquad y^2 + y = x^3 + x^2 - 23x - 50$$

There is a basis for L with respect to which the lattices L_1 and L_2 that define E_1 and E_2 have the following basis:

$$\begin{array}{ll} L_1: & [1,-1,1,0], \quad [1,-1,-1,1] \\ L_2: & [0,0,0,1], \quad [1,1,1,0] \end{array}$$

(a) Show that $E_1 \cap E_2 = \mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$.

(b) Part (1) shows that if $a_p(E_i) = p + 1 - \#E_i(\mathbf{F}_P)$, then for every prime p, we have $a_p(E_1) = a_p(E_2)$. You do not have to prove this fact to get full credit on this problem (though you are welcome to try, using, e.g., the nontrivial fact that $a_p(E_i)$ is the eigenvalue of the Hecke operator T_p acting on either of the basis vectors for L_i). Verify that $a_p(E_1) = a_p(E_2)$ for $p \leq 7$.

(c) By any means necessary, determine the rational numbers $L(E_i, 1)/\Omega_{E_i}$ for these two elliptic curves. Note that $L(E_1, 1)/\Omega_{E_1}$ is *not* congruent modulo 2 to $L(E_2, 1)/\Omega_{E_2}$. (Two rational numbers a and b are congruent modulo a prime p if $\mathrm{ord}_p(a - b) > 0$.) This example illustrates that although the coefficients that define the L-series are congruent modulo 2, the rational parts of their special values are not.