Homework Assignment 2
Math 252: Modular Abelian Varieties

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Sep. 24 (Due: Oct. 1)

1. Prove that $\mathfrak{h}^* = \mathfrak{h} \cup \mathbf{Q} \cup \{\infty\}$ is Hausdorff.

2. Compute the length of the geodesic path from -1 to 1 in the upper half plane with respect to the Poincaré metric. (It's OK to compute a numerical approximation—I just want you to play around.)

3. (a) Suppose $E = \mathbf{C}/\Lambda$ is an elliptic curve and $\lambda : \mathbf{C} \to \mathbf{C}$ defines an automorphism of E. Prove that λ lies in a quadratic imaginary extension of $\mathbf{Q}$.

   (b) Suppose E is an elliptic curve and $\mathrm{Aut}(E) \neq \{\pm 1\}$. Prove that E is isomorphic to E_τ with $\tau = i$ or $\tau = e^{2\pi i/3}$.

4. Let Γ be a congruence subgroup of $\mathrm{SL}_2(\mathbf{Z})$, and let $X = \Gamma\backslash\mathfrak{h}^*$ be the corresponding compact Riemann surface. Prove that the degree of the natural map $X \to X(1)$ equals the index in $\mathrm{PSL}_2(\mathbf{Z})$ of the image of Γ in $\mathrm{PSL}_2(\mathbf{Z})$.

5. Find explicit basis for each of the following homology groups, along with all explicit natural maps you can think of between these groups:

$$\mathrm{H}_1(X_0(11), \mathbf{Z}), \quad \mathrm{H}_1(X_0(11), \mathrm{cusps}, \mathbf{Z}),$$

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6. If you read Lemma 1.41–1.42 on page 23 of Shimura's book you'll find a way to compare the cusps for $\Gamma(N)$, i.e., the orbits for the action of $\Gamma(N)$ on $\mathbf{P}^1(\mathbf{Q})$. Shimura proves that two cusps a/b and c/d (reduced fractions) are equivalent if and only if $\pm(a, b) = (c, d) \pmod{N}$.

   (a) Use Shimura's result to prove that the cusps for $\Gamma(N)$ are in bijection with the following set: The vectors $\pm(a, b)$, where $a, b \in \mathbf{Z}/N\mathbf{Z}$ and $\gcd(a, b, N) = 1$, and the $\pm$ means that we identify (a, b) with $(-a, -b)$.

   (b) Deduce that for $N > 2$, the number of $\Gamma(N)$ cusps is $N^2/2 \cdot \prod_{p|N}(1 - p^{-2})$.