Homework Assignment 9
Due **Wednesday** December 11

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Math 124 HARVARD UNIVERSITY **Fall 2002**

This is the last homework assignment.

Your projects are due on December 13 in print in class. You will presents your projects on December 16 and 18, and your presentations will (hopefully) be videotaped. Your presentations will not affect your project grade. Incidentally, if you're excited about your project and feel it could be developed in more depth, consider doing a 91r (Supervised Reading and Research) with me next semester.

1. (1 point) Let $S_2(\Gamma_0(N))$ denote the set of cuspidal modular forms of level N. Prove that $S_2(\Gamma_0(N))$ forms a $\mathbb{C}$-vector space under addition.

2. (3 points) Suppose $y^2 = x^3 + ax + b$ with $a, b \in \mathbb{Q}$ defines an elliptic curve. Show that there is another equation $Y^2 = X^3 + AX + B$ with $A, B \in \mathbb{Z}$ whose solutions are in bijection with the solutions to $y^2 = x^3 + ax + b$. (Hint: Multiply both sides of $y^2 = x^3 + ax + b$ by a power of a common denominator, and "absorb" powers into x and y.)

3. (a) (3 points) Use the modularity and dimension theorems from the notes to deduce that there is no elliptic curve $y^2 = x^3 + ax + b$ (with $a, b \in \mathbb{Z}$) that has discriminant ± 16.
   (b) (2 points) The point $(12, 36)$ lies on the elliptic curve $y^2 = x^3 - 432$. Use this fact and elementary algebra to find a rational solution (a, b) to $4a^3 + 27b^2 = -1$, and hence exhibit an elliptic curve over $\mathbb{Q}$ with discriminant 16.

4. (5 points) One can prove that the function

$$f = q \prod_{n=1}^{\infty} (1 - q^n)^2 (1 - q^{11n})^2 = \sum_{n=1}^{\infty} a_n q^n$$

   spans $S_2(\Gamma_0(11))$, and that the following three matrices generate the subgroup $\Gamma_0(11)$ of $\mathrm{SL}_2(\mathbb{Z})$:

$$S = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad T = \begin{pmatrix} 3 & -2 \\ 11 & -7 \end{pmatrix} \qquad U = \begin{pmatrix} 4 & -3 \\ 11 & -8 \end{pmatrix}.$$

   Using the above product expression for f, compute f to some large precision then give numerical evidence that $f(z)$ satisfies the defining equation for an element of $S_2(\Gamma_0(11))$.

5. (2 points) Show that if Fermat's last theorem is true for prime exponents, then it is true for all exponents.

6. (3 points) Let R be a ring. Say that Fermat's last theorem is false in R if there exists $x, y, z \in R$ and $n \in \mathbb{Z}$ with $n \geq 3$ such that $x^n + y^n = z^n$ and $xyz \neq 0$. For which prime numbers p is Fermat's last theorem false in the ring $\mathbb{Z}/p$?