Lecture 18: Continued Fractions II: Infinite Continued Fractions

William Stein

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1 The Continued Fraction Algorithm

Let $x \in \mathbb{R}$ and write

$$x = a_0 + t_0$$

with $a_0 \in \mathbb{Z}$ and $0 \leq t_0 < 1$. If $t_0 \neq 0$, write

$$\frac{1}{t_0} = a_1 + t_1$$

with $a_1 \in \mathbb{N}$ and $0 \leq t_1 < 1$. Thus $t_0 = \frac{1}{a_1+t_1} = [0, a_1 + t_1]$, which is a (nonintegral) continued fraction expansion of t_0. Continue in this manner so long as $t_n \neq 0$ writing

$$\frac{1}{t_n} = a_{n+1} + t_{n+1}$$

with $a_{n+1} \in \mathbb{N}$ and $0 \leq t_{n+1} < 1$. This process, which associates to a real number x the sequence of integers $a_0, a_1, a_2, \ldots$, is called the *continued fraction algorithm.*

Example 1.1. Let $x = \frac{8}{3}$. Then $x = 2 + \frac{2}{3}$, so $a_0 = 2$ and $t_0 = \frac{2}{3}$. Then $\frac{1}{t_0} = \frac{3}{2} = 1 + \frac{1}{2}$, so $a_1 = 1$ and $t_1 = \frac{1}{2}$. Then $\frac{1}{t_1} = 2$, so $a_2 = 2$, $t_2 = 0$, and the sequence terminates. Notice that

$$\frac{8}{3} = [2, 1, 2],$$

so the continued fraction algorithm produces the continued fraction of $\frac{8}{3}$.

Proposition 1.2. *For every n such that a_n is defined, we have*

$$x = [a_0, a_1, \ldots, a_n + t_n],$$

and if $t_n \neq 0$ then $x = [a_0, a_1, \ldots, a_n, \frac{1}{t_n}]$.

Proof. Use induction. The statements are both true when $n = 0$. If the second statement is true for $n - 1$, then

$$x = [a_0, a_1, \ldots, a_{n-1}, \frac{1}{t_{n-1}}] = [a_0, a_1, \ldots, a_{n-1}, a_n + t_n] = [a_0, a_1, \ldots, a_{n-1}, a_n, \frac{1}{t_n}].$$

Similarly, the first statement is true for n if it is true for $n - 1$.

□

Example 1.3. Let $x = \frac{1+\sqrt{5}}{2}$. Then

$$x = 1 + \frac{-1+\sqrt{5}}{2},$$

so $a_0 = 1$ and $t_0 = \frac{-1+\sqrt{5}}{2}$. We have

$$\frac{1}{t_0} = \frac{2}{-1+\sqrt{5}} = \frac{-2-2\sqrt{5}}{-4} = \frac{1+\sqrt{5}}{2}$$

so again $a_1 = 1$ and $t_1 = \frac{-1+\sqrt{5}}{2}$. Likewise, $a_n = 1$ for all n. Does the following crazy-looking equality makes sense??

$$\frac{1+\sqrt{5}}{2} = 1 + \frac{1}{1+\frac{1}{1+\frac{1}{1+\frac{1}{1+\frac{1}{1+\cdots}}}}}$$

Example 1.4. Next suppose $x = e$. Then

$$a_0, a_1, a_2, \ldots = 2, 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, 1, 1, \ldots$$

```
? contfrac(exp(1))
%1 = [2, 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, 1, 1, 10, 1, 1,
       12, 1, 1, 14, 1, 1, 16, 1, 1, 18, 1, 1, 20, 2]
? \\ to get more terms, increase the real precision:
? \p60
? contfrac(exp(1),[])
%12 = [2, 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, 1, 1, 10, 1, 1,
   12, 1, 1, 14, 1, 1, 16, 1, 1, 18, 1, 1, 20, 1, 1, 22, 1,
   1, 24, 1, 1, 26, 1, 1, 28, 1, 1, 30, 1, 1, 32, 1, 1, 34,
   1, 1, 36, 1, 1, 38, 1, 1, 40, 1, 1, 42, 2]
```

The following program uses a proposition we proved yesterday to compute the partial convergents of a continued fraction:

```
{convergents(v)=
    local(pp,qq,p,q,tp,tq,answer);
    pp=1; qq=0; p=v[1]; q=1;    \\ pp is p_{n-1} and p is p_n.
    answer = vector(length(v)); \\ put answer in this vector
    answer[1] = p/q;
    for(n=2,length(v),
       tp=p; tq=q; p=v[n]*p+pp; q=v[n]*q+qq; pp=tp; qq=tq;
       answer[n] = p/q;
    );
    return(answer);
}
```

Let's try this with π:

```
? contfrac(Pi)
%26 = [3, 7, 15, 1, 292, 1, 1, ...]
? convergents([3,7,15])
%27 = [3, 22/7, 333/106]
? convergents([3,7,15,1,292])
%28 = [3, 22/7, 333/106, 355/113, 103993/33102]
? %[5]*1.0
%29 = 3.1415926530119026040...
? % - Pi
%30 = -0.0000000005778906034...
```

2 Infinite Continued Fractions

Theorem 2.1. *Let $a_0, a_1, a_2, \ldots$ be a sequence of integers such that $a_n > 0$ for all $n \geq 1$, and for each $n \geq 0$, set $c_n = [a_0, a_1, \ldots a_n]$. Then $\lim_{n\to\infty} c_n$ exists.*

Proof. For any $m \geq n$, the number c_n is a partial convergent of $[a_0, \ldots, a_m]$. Recall from the previous lecture that the even convergents c_{2n} form a strictly *increasing* sequence and the odd convergents c_{2n+1} form a strictly *decreasing* sequence. Moreover, the even convergents are all $\leq c_1$ and the odd convergents are all $\geq c_0$. Hence $\alpha_0 = \lim_{n\to\infty} c_{2n}$ and $\alpha_1 = \lim_{n\to\infty} c_{2n+1}$ both exist and $\alpha_0 \leq \alpha_1$. Finally, by a proposition from last time

$$|c_{2n} - c_{2n-1}| = \frac{1}{q_{2n} \cdot q_{2n-1}} \leq \frac{1}{2n(2n-1)} \to 0,$$

so $\alpha_0 = \alpha_1$. □

We define

$$[a_0, a_1, \ldots] = \lim_{n\to\infty} c_n.$$

Example 2.2. We use PARI to illustrate the convergence of the theorem for $x = \pi$.

```
? a = contfrac(Pi)
%38 = [3, 7, 15, 1, 292, 1, 1, 1, 2, 1, 3, 1, 14, 2, 1, 1, 2, 2, ...]
? c = convergents(a)
%39 = [3, 22/7, 333/106, 355/113, 103993/33102, 104348/33215, ...]
? \p9                         \\ so we can see.
   realprecision = 9 significant digits
? [c[1]*1.0, c[3]*1.0, c[5]*1.0, c[7]*1.0]  \\ odd ones converge up to pi
%43 = [3.00000000, 3.14150943, 3.14159265, 3.14159265]
? [c[2]*1.0,c[4]*1.0,c[6]*1.0,c[8]*1.0]     \\ even ones swoop down on pi.
%44 = [3.14285714, 3.14159291, 3.14159265, 3.14159265]
```

Theorem 2.3. *Let $x \in \mathbb{R}$ be a real number. Then*

$$x = [a_0, a_1, a_2, \ldots],$$

where $a_0, a_1, a_2, \ldots$ is the sequence produced by the continued fraction algorithm.

Proof. If the sequence is finite then some $t_n = 0$ and the result follows by Proposition 1.2. Suppose the sequence is infinite. By Proposition 1.2,

$$x = [a_0, a_1, \ldots, a_n, \frac{1}{t_n}].$$

By a proposition from the last lecture[1],

$$x = \frac{\frac{1}{t_n} p_n + p_{n-1}}{\frac{1}{t_n} q_n + q_{n-1}}.$$

Thus if $c_n = [a_0, a_1, \ldots, a_n]$, then

$$\begin{aligned} x - c_n &= x - \frac{p_n}{q_n} \\ &= \frac{\frac{1}{t_n} p_n q_n + p_{n-1} q_n - \frac{1}{t_n} p_n q_n - p_n q_{n-1}}{q_n \left(\frac{1}{t_n} q_n + q_{n-1}\right)}. \\ &= \frac{p_{n-1} q_n - p_n q_{n-1}}{q_n \left(\frac{1}{t_n} q_n + q_{n-1}\right)} \\ &= \frac{(-1)^n}{q_n \left(\frac{1}{t_n} q_n + q_{n-1}\right)}. \end{aligned}$$

Thus

$$\begin{aligned} |x - c_n| &= \frac{1}{q_n \left(\frac{1}{t_n} q_n + q_{n-1}\right)} \\ &< \frac{1}{q_n (a_{n+1} q_n + q_{n-1})} \\ &= \frac{1}{q_n \cdot q_{n+1}} \leq \frac{1}{n(n+1)} \to 0. \end{aligned}$$

(In the inequality we use that a_{n+1} is the integer part of $\frac{1}{t_n}$, and is hence $\leq \frac{1}{t_n}$.) □

[1] Which we apply in a case when the partial quotients of the continued fraction are not integers!