A. Student

Math 124 Problem Set 7

1. D=-155 There are four elements: $[[1, 1, 39], [3, -1, 13], [3, 1, 13], [5, 5, 9]]$.

By the structure theorem, $\mathcal{C}_{-155}$ is isomorphic to either $C_2 \text{x} C_2$ or C_4. It is easy to verify that $[1, 1, 39]$ is the identity. From this we find that $[3, -1, 13]$ has order 4, so it must be that $\mathcal{C}_{-155} \simeq C_4$.

D=-231 There are twelve elements: $[1, 1, 58], [2, -1, 29], [2, 1, 29], [3, 3, 20], [4, -3, 15], [4, 3, 15],$ $[5, -3, 12], [5, 3, 12], [6, -3, 10], [6, 3, 10], [7, 7, 10], [8, 5, 8]$. Therefore $\mathcal{C}_{-231} \simeq C_{12}$ or $C_2 \text{x} C_6$. The identity is $[1, 1, 58]$. Both $[2, -1, 29]$ and $[2, 1, 29]$ have order 6, which is impossible in C_{12}, so $\mathcal{C}_{-231} \simeq C_2 \text{x} C_6$.

D=-660 There are eight elements: $[1, 0, 165], [10, 10, 19], [11, 0, 15], [13, 4, 13], [2, 2, 83], [3, 0, 55], [5, 0, 33],$ $[6, 6, 29]$. The first element is the identity, and all others have order 2. Therefore $\mathcal{C}_{-660} \simeq C_2 \text{x} C_2 \text{x} C_2$.

D=-12104 There are forty-eight elements: (listed in an email from Professor Stein). By the structure theorem, $\mathcal{C}_D \simeq C_{48}$, $C_4 \text{x} C_{12}$, or $C_2 \text{x} C_{24}$. The identity element is $[1, 0, 3026]$, and using it we find two elements of order four: $[45, -26, 71]$ and $[50, -36, 67]$, eliminating everything but $C_4 \text{x} C_{12}$.

D=-10015 There are fifty-four elements (listed in an email from Professor Stein). Therefore $\mathcal{C}_D \simeq C_3 \text{x} C_{18}$ or C_{54}. The identity is $[1, 1, 2504]$; from this we find two elements with order 9: $[10, -5, 251]$ and $[10, 5, 251]$. Therefore the group cannot be C_{54}, so $\mathcal{C}_D \simeq C_3 \text{x} C_{18}$.

2. The three graphs are on the next page, plotted in MAPLE.

3. Differentiating implicitly, the slope of the tangent at (x, y) is $\frac{3x^2}{2y}$. At $(3, 5)$, the slope is $\frac{27}{10}$, and the tangent line has equation $y = \frac{27x-31}{10}$. Substituting into the relation $y^2 - x^3 = -2$, we have $(\frac{27x-31}{10})^2 - x^3 = -2$, which simplifies to the polynomial

$$100x^3 - 729x^2 + 1674x - 1161 = 0.$$

This polynomial has a double root at $x = 3$, so it factors into $(x-3)^2(100x - 129)$, giving a rational root with $x = 1.29$. Therefore $(1.29, .383)$ is a rational solution to the original equation.