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Math 124 Problem Set 6

1. -389 is negative, and thus not the sum of two squares. Since $3||12345$, it is not the sum of two squares. Since $7||91210$, it is not the sum of two squares. $729 = 27^2$ is a perfect square. Since $7||1729$, it is not the sum of two squares. Finally, $151||68252$ and $151 \equiv 3(mod\ 4)$, so it is not the sum of two squares.

2i. On input n, the program breaks up n into two parts and looks for a sum of two squares representation.

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{squares(n) = local(y); for(x=1,floor(sqrt(n)), y=sqrt(n-x^2);
if(y-floor(y)==0, return([x,floor(y)])) ); return(0) }

f(n) = for(x=1,n, a=squares(x); b=squares(n-x);
if(a!=0 && b!=0, return([a,b])));}
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2ii. $2001 = 1^2 + 8^2 + 44^2$.

3. 625 There are two Pythagorean triples with 25 as the hypotenuse: $(7, 24, 25)$ and $(15, 20, 25)$. This gives two representations of 625 as the sum of two squares. Of course, 25^2 is a third.

4. The forward direction is trivial. For the opposite direction, suppose that n is the sum of two rational squares: $n = (\frac{a}{b})^2 + (\frac{c}{d})^2$, but it is not the sum of two integer squares. Then $p^r||n$, where $p \equiv 3(mod\ 4)$ is a prime factor and r is odd. Now, $nb^2d^2 = (ad)^2 + (bc)^2$, so nb^2d^2 is the sum of two integer squares. However, all the prime factors of b^2d^2 have even exponent, so $p^s||nb^2d^2$, where s is still odd. This is a contradiction; therefore n must be the sum of two integer squares.

5. Suppose $p = x^2 + 2y^2$, where p is an odd prime and x, y are integers. Then $x^2 + 2y^2 \equiv 0(mod\ p)$, so $(\frac{x}{y})^2 \equiv -2(mod\ p)$ (since Z_p is a field). From Lecture 13, $(\frac{2}{p}) = 1$ iff $p \equiv \pm 1(mod\ 8)$. Also, $(\frac{-1}{p}) = 1$ iff $p \equiv 1(mod\ 4) \Rightarrow p \equiv 1, -3(mod\ 8)$. Since $(\frac{-2}{p}) = (\frac{-1}{p}) \cdot (\frac{2}{p})$, we have $(\frac{-2}{p}) = 1$ iff $p \equiv 1, 3(mod\ 8)$. Conversely, suppose $p \equiv 1, 3(mod\ 8)$ is prime. Let r be such that $r^2 \equiv -2(mod\ p)$. Taking $n = \lfloor \sqrt{p} \rfloor$ and applying Lemma 1.3 from Lecture 21, there exist integers a, b with $0 < b < \sqrt{p}$ such that

$$\left| -\frac{r}{p} - \frac{a}{b} \right| \le \frac{1}{b(n+1)} < \frac{1}{b\sqrt{p}}.$$

Let $c = rb + pa$; then $|c| < \frac{pb}{b\sqrt{p}} = \sqrt{p}$, so $2b^2 + c^2 < 3p$. Since $c \equiv rb(mod\ p)$, we also have that $2b^2 + c^2 \equiv b^2(2 + r^2) \equiv 0(mod\ p)$. Therefore $2b^2 + c^2 = p$ or $2p$. If $2b^2 + c^2 = p$ we are done. If $2b^2 + c^2 = 2p$ then c must be even (else $2b^2 + c^2$ is odd). Put $c = 2d$; then

$$2p = 2b^2 + c^2 = 2b^2 + 4d^2 \Rightarrow p = b^2 + 2d^2,$$

as desired.

6. Let T_m be the mth triangular number. It is easy to see by induction that $T_m = \frac{m(m+1)}{2}$. Then

$$8T_m^2 = 2m^2(m+1)^2 = (2T_m)^2 + (2T_m)^2,$$

$$8T_m^2 + 1 = 2m^2(m+1)^2 + 1 = [(m-1)(m+1)]^2 + [m(m+2)]^2,$$

$$8T_m^2 + 2 = 2m^2(m+1)^2 + 2 = [m(m+1)-1]^2 + [m(m+1)+1]^2.$$

This shows that $8T_m$, $8T_m + 1$ and $8T_m + 1$ can be written as the sum of two squares.

7. Of any four consecutive integers, there is one n such that $n \equiv -1(mod\ 4)$. Since all odd prime factors are congruent to $\pm 1(mod\ 4)$, n must have some prime factor $p \equiv -1(mod\ 4)$ with odd exponent. This implies that n is not representable as the sum of two squares.

8. We first solve

$$13x^2 + 36xy + 25y^2 = (ax+by)^2 + (cx+dy)^2 = (a^2+c^2)x^2 + 2(ab+cd)xy + (b^2+d^2)y^2,$$

and then check that $ad - bc = 1$. By inspection, we try $a = 3, c = 2$. Then $b^2 + d^2 = 25$ and $2b + 3d = 18$, which again by inspection is satisfied by $b = 4, d = 3$. Since $ad - bc = 1$, the form is equivalent to $x^2 + y^2$. As above, we solve

$$58x^2 + 82xy + 29y^2 = (ax+by)^2 + (cx+dy)^2 = (a^2+c^2)x^2 + 2(ab+cd)xy + (b^2+d^2)y^2,$$

and then check that $ad - bc = 1$. By inspection, we try $a = 3, c = 7$. Then $b^2 + d^2 = 29$ and $7b + 3d = 41$, which again by inspection is satisfied by $b = 2, d = 5$. Since $ad - bc = 1$, the form is equivalent to $x^2 + y^2$. We know that $x = 17, y = 10$ satisfies $x^2 + y^2 = 389$. To find x, y such that $13x^2 + 36xy + 25y^2$, We use the transformation above and solve for

$$17 = 3x + 4y, \ 10 = 2x + 3y.$$

The solution to this system is **x= 11, y= -4**. Indeed, $13 \cdot 11^2 - 36 \cdot 11 \cdot 4 + 25 \cdot 4^2 = 389$.

9. The discriminants are equal: $-24 = 162^2 - 4 \cdot 199 \cdot 33 = 96^2 - 4 \cdot 35 \cdot 66$. However, the forms are not equivalent. To see this, we first show that $35x^2 = 96xy + 66y^2$ is equivalent to $2x^2 + 3y^2$. As above, this means solving

$$35x^2 = 96xy + 66y^2 = 2(ax+by)^2 + 3(cx+dy)^2 = (2a^2+3c^2)x^2 + 2(2ab+3cd)xy + (2b^2+3d^2)y^2$$

over the integers such that $ad - bc = 1$. By inspection we see that $a = 2, b = -3, c = 3, d = -4$ is a solution, so the forms are equivalent. Now we show the first form is not equivalent to $2x^2 + 3y^2$. If we try to solve for as above, we encounter the equation $33 = 2b^2 + 3d^2$, which has no solutions over the integers (we can just check $0 \leq b \leq 5$).