Math 581e, Fall 2012, Homework 7

William Stein (wstein@uw.edu)

Due: Friday, November 16, 2012

There are 4 problems. Turn your solutions in Friday, November 16, 2012 in class. You may work with other people and can find the LaTeX of this file at http://wstein.org/edu/2012/ant/hw/. If you use Sage to solve a problem, include your code in your solution. I have office hours 12:30–2:00 on Wednesdays in Padelford C423.

For any Dedekind domain R at all, define the *class group* $\mathrm{Cl}(R)$ to be the group of fractional ideals modulo the subgroup of principal fractional ideals. This definition makes sense for an arbitrary Dedekind domain.

Warning: I just made up all of these problems from scratch, so if something seems wrong or impossible, ask me!

1. Let $\mathcal{O}_K$ be the ring of integers of a number field and let n be a positive integer.
   (a) Prove that $R = \mathcal{O}_K[\frac{1}{n}]$ is a Dedekind domain.
   (b) Prove that $\mathrm{Cl}(R)$ is finite.
   (c) Describe (with proof) a useful relationship between $\mathrm{Cl}(R)$ and $\mathrm{Cl}(\mathcal{O}_K)$.

2. Let $R = k[t]$, where k is an algebraically closed field. Of course, R is a Dedekind domain.
   (a) Prove that $\mathrm{Cl}(R)$ is trivial (of order 1).
   (b) Prove that the group U_R of units in R is not finitely generated.

3. Let k be a *finite field* of characteristic $\neq 2$, and consider the Dedekind domain $R = k[x, y]/(y^2 - x^3 - x)$. [You can use anything you know from outside of class from algebra or algebraic geometry on this problem.]
   (a) Let $I \subset R$ be a nonzero ideal. Prove that the norm $N(I) = \#(R/I)$ is finite.
   (b) Let B be a positive integer. Prove that there are finitely many nonzero ideals I of R such that $N(I) \leq B$.
   (c) Prove that the unit group $U_R = R^*$ is a finite cyclic group.
   (d) Nonetheless, prove that $\mathrm{Cl}(R)$ is *infinite*. [[In fact, this part of the problem is completely *wrong* – the class group is in bijection with the group of rational points on the elliptic curve over the finite field, which is finite.]]

4. (a) Prove that if K is any number field, then the torsion subgroup of the group $U_K = \mathcal{O}_K^*$ has even order.
   (b) Prove that if $K = \mathbb{Q}(\sqrt{D})$, with $D \leq -1$ square free, is a quadratic imaginary field, then the unit group U_K of $\mathcal{O}_K$ has order 2, 4, or 6.
   (c) Prove that if K is a number field of odd degree, then the torsion subgroup of U_K has order 2

(d) What is the torsion subgroup of the unit group of the 389th cyclotomic field $\mathbb{Q}(\zeta_{389})$.