Math 581g, Fall 2011, Homework 6

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There are 4 problems. Turn your solutions in on **November 18**. You may work with other people and you can find the LaTeX of this file at http://wstein.org/edu/2011/581g/hw/. I will have office hours 11–3 in Padelford C423 on November 17.

1. **(Warm up)** Using the formula from class (or the book), compute the genus of the modular curve $X(54)$. Be prepared: what is the genus of $X(2012)$?

2. Consider the map $j : X(N) \to \mathbf{P}^1_{\mathbf{C}}$ for $N \geq 3$. Following the argument presented in class, prove that
$$\#j^{-1}(1728) = \frac{\#\,\mathrm{SL}_2(\mathbf{Z}/N\mathbf{Z})}{4}.$$

3. Explicitly compute the sets $\Gamma_0(N)\backslash\mathbf{P}^1(\mathbf{Q})$ for $N = 3$, $N = 9$, and $N = 54$, using the method I described in class. [**Hint: You should double check your work with Sage:** `Gamma0(N).cusps()`**, but don't just get the answer this way.**]

4. Let N be a positive integer. Prove that
$$\#\,\mathrm{SL}_2(\mathbf{Z}/N\mathbf{Z}) = N^3 \cdot \prod_{p\mid N}\left(1 - \frac{1}{p^2}\right),$$
where the product is over the prime divisors of N. [**Hint: First reduce to the prime power case, by noting that $\mathrm{SL}_2(\mathbf{Z}/N\mathbf{Z}) \cong \prod_{p\mid N}\mathrm{SL}_2(\mathbf{Z}/p^{\nu_p}\mathbf{Z})$. Next, compute the cardinality of $\mathrm{GL}_2(\mathbf{Z}/p^n\mathbf{Z})$ using the exact sequence**
$$1 \to K \to \mathrm{GL}_2(\mathbf{Z}/p^n\mathbf{Z}) \to \mathrm{GL}_2(\mathbf{Z}/p\mathbf{Z}) \to 1,$$
where K is by definition the kernel, which has a simple description as
$$K = \left\{\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + A \,:\, A \in pM_{2\times 2}(\mathbf{Z}/p^n\mathbf{Z})\right\},$$
so it is easy to compute $\#K$. Finally, relate the cardinality of $\mathrm{SL}_2(\mathbf{Z}/p^n\mathbf{Z})$ to that of $\mathrm{GL}_2(\mathbf{Z}/p^n\mathbf{Z})$ and simplify the big mess you get to obtain the desired formula.]