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Notes on étale cohomology of number fields

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NOTES ON ÉTALE COHOMOLOGY OF NUMBER FIELDS (1)

By Barry MAZUR

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In the setting of étale cohomology, M. Artin and J.-L. Verdier have proved a duality theorem for constructible abelian sheaves over the scheme Spec D, where D is the ring of integers in a number field (*see* [AV]). This duality theorem contains within it Tate duality for finite Galois modules over local and global fields, and is indeed the natural extension of Tate duality to the context of such schemes.

In what follows, I would like to poke at this result from various angles, and make what I hope to be enlightening remarks and computations, so as to convey in a concrete way a sense of the information contained in this theorem. Rather than give its proof completely, I shall empha-

(1) This is an expository paper. In its original form it was presented as a series of lectures at the Summer Conference on Algebraic Geometry, at Bowdoin, in 1966. It gives me pleasure to thank J.-P. Serre for his vigorous editing and his suggestions and corrections, which led to this revised version. I should also like to express my thanks to the Institut des Hautes Études Scientifiques for extending its hospitality to me during the rewriting, and to the NSF under grant 31359X—1.

size those aspects of the proof that relate closely to number theory, and I shall try to suppress much of the general sheaf-theoretic thecniques that go into it. I will assume some familiarity with étale cohomology, but whenever something may be done more explicitly (i. e. in the category of Galois modules over a field, or more generally over a discrete valuation ring) I shall do so and try to indicate the appropriate translation.

1. Remarks on duality statements over fields, and introduction to sheaves for the étale topology

Duality over fields. — Given a field k and a separable algebraic closure $\bar{k}$ of k, we consider the category S_k of Galois modules over the profinite group $G_k = G(\bar{k}/k)$. This is an abelian category. A most interesting module to consider is the G_k-module $\bar{k}^*$ itself, which generally plays an important role in the category S_k. For example :

(*a*) If k is a local field, and M a Galois module over k, set $\hat{M} = \text{Hom}(M, \bar{k}^*)$ which is given the structure of a Galois module by the action $(g\varphi)(a) = g(\varphi(g^{-1}a))$. One has *Tate's local duality*, which says that the cup-product

$$H^r(G_k, M) \times H^{2-r}(G_k, \hat{M}) \to H^2(G_k, \bar{k}^*) \overset{\approx}{\to} \mathbf{Q}/\mathbf{Z},$$

is a non degenerate pairing for all finite Galois modules M of order prime to the characteristic of k. Note that the isomorphism $H^2(G_k, \bar{k}^*) \to \mathbf{Q}/\mathbf{Z}$ comes from local class field theory, which also yields $H^r(G_k, \bar{k}^*) = 0$ for all $r \neq 0,2$. These facts are summarized in the statement that $\bar{k}^*$ is a dualizing module for G_k, and G_k has cohomological dimension 2 (*cf.* [CG], [P] or [Sh]). If M is of any order, one has the duality theorem of Shatz [Sh] (§ 6, Theorem 46).

(*b*) If k is a global field, there is again a duality statement (*Tate global duality*) relating the cohomology of M to the cohomology of $\hat{M}$ (if k is function field, one insists that the order of the finite module M not be divisible by the characteristic). This statement is not as simple as Tate local duality (*cf.* [T], [CG] or [P]).

(*c*) While we are on the subject of duality theorems for S_k, we should not shun the case of k a finite field. In this case the constant module $\mathbf{Q}/\mathbf{Z}$ plays the role of a one-dimensional dualizing module, and putting $\tilde{M} = \text{Hom}(M, \mathbf{Q}/\mathbf{Z})$, we get that

$$H^r(G_k, M) \times H^{1-r}(G_k, \tilde{M}) \to \mathbf{Q}/\mathbf{Z},$$

is a non degenerate pairing for all finite Galois modules M. This reduces immediately to Pontrjagin duality for finite abelian groups, using that G_k is isomorphic to the profinite completion of $\mathbf{Z}$.

Remark. — The global duality theorem of Artin-Verdier will be an elegant statement which puts together these three duality theorems. In order to discuss it, we must consider certain abelian categories which are slightly more complex than the categories S_k :

If $\tau : A \to B$ is a left exact functor of abelian categories, Artin constructs a new category $C = C_\tau$, which we may call the *mapping cylinder of* τ. The objects of C are triples (M, N, φ) where $M \in A$, $N \in B$ and $\varphi : N \to \tau M$ is a morphism in B. The morphisms are given by pairs

$$f: \ M \to M' \quad \text{and} \quad g: \ N \to N',$$

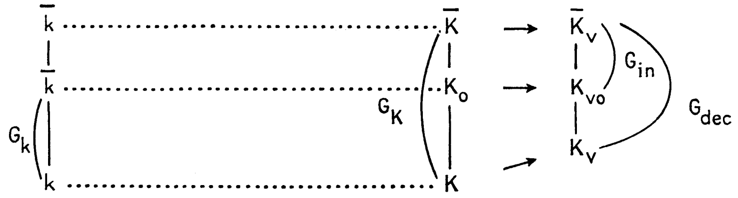
giving rise to commutative diagrams :

$$\begin{array}{ccc} N & \xrightarrow{\varphi} & \tau M \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle \tau} \\ N' & \xrightarrow{\varphi'} & \tau M' \end{array}$$

One checks that C is again an abelian category, using left exactness of τ, and that C has enough injectives if A and B do (exercise : describe injective, surjective morphisms, sums and products in C).

XAMPLE. — Let R be a discrete valuation ring, with field of fractions K, an a perfect residue field k. Choose a separable algebraic closure $\overline{K}$ of K and an extension v of our valuation to $\overline{K}$. Let the subscript v denote *completion with respect to* v. Let $K_0 \subset \overline{K}$ denote the maximal unramified subfield with respect to v. Its residue field $\overline{k}$ is an algebraic closure of k. We have the diagram of inertia and decomposition groups :

Diagram 1



and the standard relations between Galois groups :

$$G_K \supset G_{dec} \xrightarrow{\pi} G_{dec}/G_{in} \overset{\alpha}{\underset{\sim}{\leftarrow}} G_k,$$

with $G_{dec} = G_K$ in the important case where R is *complete* (or more generally henselian).

When R is *not* henselian, there is yet another field worth considering, along with those already introduced in diagram 1 : Let K_{hens} denote the subfield of K_0 fixed by G_{dec}. Then one has the diagram of fields

$$\begin{tikzcd}[column sep=small] K_0 \arrow[d, dash] & & \\ K_{hens} \arrow[d, dash] \arrow[rr, dash] & & K_v \\ K & & \end{tikzcd}$$

and the Hensel-closure of the discrete valuation ring R is precisely the integral closure of R in K_{hens}.

THE CATEGORY OF GALOIS MODULES OVER R. — Consider the following functor $\tau : S_K \to S_k$: if M is a Galois module for G_K, form $M^0 = M^{G_{in}}$ and regard this as a Galois module for G_{dec}/G_{in}. Take τM to be the Galois module M^0 regarded as a module for G_k via α. (Using the notation of sheaf theory, we might have written $\tau = \alpha^* \pi_*$, and τ is left exact since α^* is exact and π_* left exact.)

Let us define S_R, the *category of Galois modules over* R, to be the mapping cylinder category of τ. Thus a Galois module over R is given quite simply by a diagram $\varphi : N \to M$ where N is a G_k-module, M a G_K-module, φ a homomorphism of abelian groups which sends N into $M^0 \subset M$ and, when regarded as a map of N into M^0, is a homomorphism of G_k-modules.

THE FUNCTORS i^*, i_*, ... — For any left exact functor $\tau : A \to B$, there are various functors relating the categories A, B and $C = C_\tau$. To give them their sheaf-theoretic names, they are :

Diagram 2

$$\begin{tikzcd} A \arrow[r, shift left=2.5, "j_!"] \arrow[r, shift right=2.5, "j_*"'] & C \arrow[l, "j^*" description] \arrow[r, shift left=2.5, "i^*"] \arrow[r, shift right=2.5, "i^!"'] & B. \arrow[l, "i_*" description] \end{tikzcd}$$

One defines them as follows :

pull-backs :

$$i^* : \ (M, N, \varphi) \mapsto N; \qquad j^* : \ (M, N, \varphi) \mapsto M;$$

direct images :

$$i_* : \ N \mapsto (0, N, 0); \qquad j_* : \ M \mapsto (M, \tau M, \mathrm{id});$$

"*extension by zero over a closed point*" :

$$j_! : \ M \mapsto (M, 0, 0);$$

" *sections with support on a closed point* " :

$$i^! : \ (\mathrm{M}, \mathrm{N}, \varphi) \mapsto \operatorname{Ker} \varphi.$$

Each functor in diagram 2 is adjoint to the one below it; for instance,

$$\operatorname{Hom}(i^* \mathrm{X}, \mathrm{Y}) = \operatorname{Hom}(\mathrm{X}, i_* \mathrm{Y})$$

and

$$\operatorname{Hom}(i_* \mathrm{Y}, \mathrm{Z}) = \operatorname{Hom}(\mathrm{Y}, i^! \mathrm{Z}).$$

The functors i^*, i_*, j^*, $j_!$ are exact, j_* and $i^!$ are left exact; i_* and j_* are fully faithful. The derived functors of j_* are given by :

$$\mathrm{R}^q j_* \mathrm{M} = (\mathrm{R}^q \tau \mathrm{M}, 0, 0).$$

Examples. — In the category S_{R} of Galois modules over the discrete valuation ring R, consider the Galois module which may be called $\mathbf{G}_{m,\mathrm{R}}$:

$$\mathrm{U}_0 \to \overline{\mathrm{K}}^*,$$

where U_0 denotes the group of units in K_0, regarded as a G_k-module. Let us denote by $\mathbf{G}_{m,\mathrm{K}}$ the G_{K}-Galois module $\overline{\mathrm{K}}^*$. Then, we have an exact sequence in S_{R} :

$$\begin{array}{ccccccccc} 0 & \to & \mathrm{U}_0 & \longrightarrow & \mathrm{K}_0^* & \xrightarrow{v} & \mathbf{Z} & \to & 0 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 0 & \to & \overline{\mathrm{K}}^* & \longrightarrow & \overline{\mathrm{K}}^* & \longrightarrow & 0 & \to & 0 \end{array}$$

which may be rewritten, using our new terminology :

$$(\bigstar) \qquad 0 \to \mathbf{G}_{m,\mathrm{R}} \to j_* \mathbf{G}_{m,\mathrm{K}} \to i_* \mathbf{Z} \to 0.$$

Now, let us suppose that R is *complete* (or henselian) and its residue field k is *finite*. In this case, we have :

$$(1.1) \qquad \mathrm{R}^q j_* \mathbf{G}_{m,\mathrm{K}} = 0 \quad \text{for} \quad q > 0,$$

because

$$\mathrm{R}^q j_* \mathbf{G}_{m,\mathrm{K}} = (\mathrm{R}^q \tau \mathbf{G}_{m,\mathrm{K}}, 0, 0)$$

quite generally, and

$$\mathrm{R}^q \tau \mathbf{G}_{m,\mathrm{K}} = \mathrm{H}^q (\mathrm{G}_{\mathrm{In}}, \overline{\mathrm{K}}^*)$$

which is zero by local class field theory. (*Note.* — The latter statement is equivalent to the fact that every positive dimensional Galois cocycle in the multiplicative group splits in some unramified extension of K.)

While we are studying this case, let us compute the higher derived functors of the only other nonexact functor among the six described in diagram 2 above :

$$(1.2) \qquad \begin{aligned} \mathrm{R}^q i^! \mathbf{G}_{m,\mathrm{R}} &= 0 \quad \text{if} \quad q \neq 1, \\ &= \mathbf{Z} \quad \text{if} \quad q = 1. \end{aligned}$$

To see this, use our exact sequence (★), and compute the other two terms in it :

(1.3) $$(\mathrm{R}^q\, i^!)\, j_*\, \mathbf{G}_{m,\mathrm{K}} = 0 \qquad \text{for all } q.$$

[Since j_* preserves injectives, we may apply the spectral sequence for composite functors :

$$(\mathrm{R}^q\, i^!)\,(\mathrm{R}^p\, j_*)\, \mathbf{G}_{m,\mathrm{K}} \Rightarrow \mathrm{R}^{q+p}\,(i^!\, j_*)\, \mathbf{G}_{m,\mathrm{K}},$$

and (1.3) follows from (1.1) together with the observation that $i^! j_* = 0$ [2]] :

(1.4) $$\begin{aligned}(\mathrm{R}^q\, i^!)\, i_*\, \mathbf{Z} &= \mathbf{Z} \quad \text{if } q = 0,\\ &= 0 \quad \text{if } q \neq 0.\end{aligned}$$

[Since i_* is exact and preserves injectives, we have

$$(\mathrm{R}^q\, i^!)\, i_* = \mathrm{R}^q\,(i^!\, i_*) = \mathrm{R}^q\,(\mathrm{id}),$$

hence (1.4).]

We have thus established (1.2). Let us use this fact to compute the *cohomology* and *compact cohomology* of $\mathbf{G}_m$. We define cohomology by :

$$\mathrm{H}^q\,(\mathrm{R}, \mathrm{F}) = \mathrm{Ext}^q_{\mathrm{S}_\mathrm{R}}\,(\mathbf{Z}, \mathrm{F}) = (\mathrm{R}^q\, \Gamma_\mathrm{R})\,(\mathrm{F})$$

where

$$\Gamma_\mathrm{R}\,(\mathrm{F}) = \mathrm{Hom}_{\mathrm{S}_\mathrm{R}}\,(\mathbf{Z}, \mathrm{F})$$

is the section functor over R and

$$\mathrm{H}^q_{\mathrm{comp}}\,(\mathrm{R}, \mathrm{F}) = \mathrm{Ext}^q_{\mathrm{S}_\mathrm{R}}\,(i_*\, \mathbf{Z}, \mathrm{F}).$$

To compute $\mathrm{H}^q\,(\mathrm{R}, \mathbf{G}_{m,\mathrm{R}})$, apply $\mathrm{H}^q\,(\mathrm{R},\ )$ to (★) to obtain a long exact sequence, and compute the cohomology of the other two terms first. The results are :

(1.5) $$\begin{aligned}\mathrm{H}^q\,(\mathrm{R}, j_*\, \mathbf{G}_{m,\mathrm{K}}) = \mathrm{H}^q\,(\mathrm{G}_\mathrm{K}, \overline{\mathrm{K}}^*) &= \mathrm{K}^* \quad \text{for } q = 0,\\ &= 0 \quad \text{» } q = 1,\\ &= \mathbf{Q}/\mathbf{Z} \quad \text{» } q = 2,\\ &= 0 \quad \text{» } q > 2.\end{aligned}$$

Proof of (1.5). — Consider the section-functors on S_R and S_K respectively :

$$\Gamma_{\mathrm{S}_\mathrm{R}}\,(\mathrm{F}) = \mathrm{Hom}_{\mathrm{S}_\mathrm{R}}\,(\mathbf{Z}, \mathrm{F}) \qquad \text{and} \qquad \Gamma_{\mathrm{S}_\mathrm{K}}\,(\mathrm{M}) = \mathrm{Hom}_{\mathrm{S}_\mathrm{K}}\,(\mathbf{Z}, \mathrm{M}).$$

These functors are left-exact, and $\Gamma_{\mathrm{S}_\mathrm{K}} = \Gamma_{\mathrm{S}_\mathrm{R}} \circ j_*$.

---

(2) Note that, if a functor has an exact left adjoint, then it preserves injectives; this fact will serve us very well for the construction of spectral sequences, save in two instances which will be signaled.

We have already seen that j_* preserves injectives and therefore we have a spectral sequence

$$R^p \Gamma_{S_R} \circ R^q j_* \Rightarrow R^{p+q} \Gamma_{S_K}$$

which is just

$$H^p(R, R^q j_* M) \Rightarrow H^{p+q}(G_K, M).$$

By (1.1), $R^q j_* \mathbf{G}_{m,K}$ vanishes for $q > 0$, and therefore the above spectral sequence collapses when $M = \mathbf{G}_{m,K}$, giving :

$$H^p(R, j_* \mathbf{G}_{m,K}) = H^p(G_K, \overline{K}^*) \quad \text{for all } p.$$

The groups $H^p(G_K, \overline{K}^*)$ are the Galois cohomology groups over K with coefficients in $\overline{K}^*$ and are calculated by local class field theory. This accounts for the list occuring in (1.5).

(1.6)
$$\begin{aligned} H^q(R, i_* \mathbf{Z}) = H^q_{\text{comp}}(R, i_* \mathbf{Z}) &= \mathbf{Z} && \text{for } q = 0,\\ &= 0 && \text{» } q = 1,\\ &= \mathbf{Q}/\mathbf{Z} && \text{» } q = 2,\\ &= 0 && \text{» } q > 2. \end{aligned}$$

Proof of (1.6). — Consider the short exact sequence of Galois modules over S_R :

$$0 \to j_! \mathbf{Z} \to \mathbf{Z} \to i_* \mathbf{Z} \to 0.$$

CLAIM :

$$\operatorname{Ext}^q_{S_R}(j_! \mathbf{Z}, i_* N) = 0 \quad \textit{for all } q,$$

and any Galois module N *over* S_k.

Proof of claim :

(*a*) $\operatorname{Hom}_{S_R}(j_! \mathbf{Z}, i_* N) = 0$ for all N.

(*b*) i_* is exact, and preserves injectives.

It follows from the above claim, and consideration of the above exact sequence that

$$H^q(R, i_* N) = H^q_{\text{comp}}(R, i_* N) \quad \text{for all } q,$$

and any Galois module N over S_k. This establishes the first equality of (1.6).

The remainder of (1.6) quickly reduces to a question of cohomology of G_k as follows :

$$H^q_{\text{comp}}(R, i_* \mathbf{Z}) = \operatorname{Ext}^q_{S_R}(i_* \mathbf{Z}, i_* \mathbf{Z}) = \operatorname{Ext}^q_{S_k}(\mathbf{Z}, \mathbf{Z}) = H^q(G_k, \mathbf{Z}),$$

where one has the middle equality above by virtue of the fact that i_* is exact and preserves injectives.

To obtain the list occurring on the right-hand side of (1.6), note that

$$\mathrm{H}^0(\mathrm{G}_k, \mathbf{Z}) = \mathbf{Z} \quad \text{and} \quad \mathrm{H}^1(\mathrm{G}_k, \mathbf{Z}) = \mathrm{Hom}(\mathrm{G}_k, \mathbf{Z}) = 0.$$

To make the evaluation for $q \geqq 2$, note that the coboundary map coming from the short exact sequence

$$0 \to \mathbf{Z} \to \mathbf{Q} \to \mathbf{Q}/\mathbf{Z} \to 0,$$

induces an isomorphism,

$$\mathrm{H}^{q-1}(\mathrm{G}_k, \mathbf{Q}/\mathbf{Z}) \underset{\approx}{\to} \mathrm{H}^q(\mathrm{G}_k, \mathbf{Z}) \quad \text{for} \quad q \geqq 2.$$

Since

$$\mathrm{H}^1(\mathrm{G}_k, \mathbf{Q}/\mathbf{Z}) = \mathrm{Hom}(\hat{\mathbf{Z}}, \mathbf{Q}/\mathbf{Z}) = \mathbf{Q}/\mathbf{Z},$$

and since $\mathrm{G}_k = \hat{\mathbf{Z}}$ is of cohomological dimension $\leqq 1$ for torsion module coefficients, one obtains the desired list.

(1.7 *a*)
$$\begin{aligned} \mathrm{H}^q(\mathrm{R}, \mathbf{G}_{m,\mathrm{R}}) &= \mathrm{R}^* && \text{if} \quad q = 0, \\ &= 0 && \text{otherwise;} \end{aligned}$$

(1.7 *b*)
$$\begin{aligned} \mathrm{H}^q_{\mathrm{comp}}(\mathrm{R}, \mathbf{G}_{m,\mathrm{R}}) &= 0 && \text{for} \quad q = 0, \\ &= \mathbf{Z} && \text{»} \quad q = 1, \\ &= 0 && \text{»} \quad q = 2, \\ &= \mathbf{Q}/\mathbf{Z} && \text{»} \quad q = 3, \\ &= 0 && \text{»} \quad q > 3. \end{aligned}$$

Proof of (1.7) :

(*a*) Apply (1.5), (1.6) to (★), noting that the intervening maps are given by :

$$v:\ \mathrm{K}_0^* \to \mathbf{Z} \quad (\text{for } q = 0) \qquad \text{and} \qquad \mathrm{id}:\ \mathbf{Q}/\mathbf{Z} \to \mathbf{Q}/\mathbf{Z} \quad (\text{for } q = 2).$$

(*b*) Since i_* is left-adjoint to $i^!$ and since i_* is exact, $i^!$ preserves injectives. One therefore has the spectral sequence

$$\mathrm{Ext}^p_{\mathrm{S}_k}(\mathbf{Z}, \mathrm{R}^q\, i^!\, \mathbf{G}_{m,\mathrm{R}}) \Rightarrow \mathrm{Ext}^{p+q}_{\mathrm{S}_\mathrm{R}}(i_*\mathbf{Z}, \mathbf{G}_{m,\mathrm{R}}).$$

But by the definition of $\mathrm{H}^r_{\mathrm{comp}}$ and by (1.3) above, this spectral sequence collapses, and yields isomorphisms

$$\mathrm{Ext}^p_{\mathrm{S}_k}(\mathbf{Z}, \mathbf{Z}) \cong \mathrm{H}^{p+1}_{\mathrm{comp}}(\mathrm{R}, \mathbf{G}_{m,\mathrm{R}}).$$

The groups

$$\mathrm{Ext}^p_{\mathrm{S}_k}(\mathbf{Z}, \mathbf{Z}) = \mathrm{H}^p(\mathrm{G}_k, \mathbf{Z})$$

have been already computed in the course of the proof of (1.6). Thus the list occurring on the right of (1.7 *b*) is just the list occurring in (1.6), shifted by one dimension. Our last computation indicates that compact cohomology of $\mathbf{G}_{m,\mathrm{R}}$ is an excellent recipient for a duality pairing. Consider, then, the Yoneda pairing :

$$\mathrm{H}^r_{\mathrm{comp}}(\mathrm{R}, \mathrm{F}) \times \mathrm{Ext}^{3-r}_{\mathrm{S}_\mathrm{R}}(\mathrm{F}, \mathbf{G}_{m,\mathrm{R}}) \to \mathrm{H}^3_{\mathrm{comp}}(\mathrm{R}, \mathbf{G}_{m,\mathrm{R}}) = \mathbf{Q}/\mathbf{Z}.$$

The formulation of local duality given by Artin and Verdier is the following :

Local duality. — *The above pairing is* non degenerate *for all finite Galois modules* F *over* R, *and all q, and furthermore the intervening groups are* finite.

Remarks :

(*a*) A Galois module $\mathrm{F} = (\mathrm{M}, \mathrm{N}, \varphi)$ over R is said to be *finite* if M and N are finite abelian groups.

(*b*) (Some general facts about the Yoneda pairing.) The above gives us maps

$$m^r(\mathrm{F}) : \quad \mathrm{H}^r_{\mathrm{comp}}(\mathrm{R}, \mathrm{F}) \to \mathrm{Ext}^{3-r}(\mathrm{F}, \mathbf{G}_{m,\mathrm{R}})^{\sim},$$

where $^{\sim}$ denotes Pontrjagin dual, and, by one of the principal properties of Yoneda pairing, the $m^r(\mathrm{F})$'s make up a morphism of cohomological functors. Consequently, if we are given an exact sequence

$$0 \to \mathrm{F}_1 \to \mathrm{F}_2 \to \mathrm{F}_3 \to 0,$$

then, if we show that $m^r(\mathrm{F}_i)$ are isomorphisms for all r and two out of three of the F_i, we have shown it for all the F_i.

(*c*) Any Galois module $\mathrm{F} = (\mathrm{M}, \mathrm{N}, \varphi)$ over R lives in an exact sequence

$$\begin{array}{ccccccccc} 0 & \to & 0 & \to & \mathrm{N} & \to & \mathrm{N} & \to & 0 \\ & & \downarrow & & \downarrow{\scriptstyle\varphi} & & \downarrow & & \\ 0 & \to & \mathrm{M} & \to & \mathrm{M} & \to & 0 & \to & 0 \end{array}$$

which, of course, may be written

$$0 \to j_! j^* \mathrm{F} \to \mathrm{F} \to i_* i^* \mathrm{F} \to 0.$$

By remark (*b*), this shows that, if we prove duality for sheaves of the form $i_* \mathrm{N}$ and $j_! \mathrm{M}$, we have duality in general.

(*d*) For Galois modules of the form $i_* \mathrm{N}$ (which we call *punctual Galois modules*) the above local duality statement reduces to duality in the residue field k as follows :

$$\begin{array}{ccccc} \mathrm{H}^r_{\mathrm{comp}}(\mathrm{R}, i_* \mathrm{N}) & \otimes & \mathrm{Ext}^{3-r}(i_* \mathrm{N}, \mathbf{G}_{m,\mathrm{R}}) & \to & \mathrm{H}^3_{\mathrm{comp}}(\mathrm{R}, \mathbf{G}_{m,\mathrm{R}}) \\ \wr\wr & & \wr\wr & & \wr\wr \\ \mathrm{H}^r(\mathrm{G}_k, \mathrm{N}) & \otimes & \mathrm{Ext}^{2-r}_{\mathrm{G}_k}(\mathrm{N}, \mathbf{Z}) & \to & \mathrm{H}^2(\mathrm{G}_k, \mathbf{Z}), \end{array}$$

using (1.2).

(*e*) For Galois modules of the form $j_! \mathrm{M}$, the above local duality becomes Tate local duality after the identifications :

(i) $$\mathrm{H}^p_{\mathrm{comp}}(\mathrm{R}, j_! \mathrm{M}) \xrightarrow{\sim} \mathrm{H}^{p-1}(\mathrm{G}_\mathrm{K}, \mathrm{M})$$

(ii) $$\mathrm{Ext}^q(j_! \mathrm{M}, \mathbf{G}_{m, \mathrm{R}}) \xrightarrow{\sim} \mathrm{Ext}^q_{\mathrm{G}_\mathrm{K}}(\mathrm{M}, \overline{\mathrm{K}}^*) \xrightarrow{\sim} \mathrm{H}^q(\mathrm{G}_\mathrm{K}, \tilde{\mathrm{M}}).$$

To see (i), use the relative cohomology exact sequence

$$\ldots \to \mathrm{H}^p_{\mathrm{comp}}(\mathrm{R}, \mathrm{F}) \to \mathrm{H}^p(\mathrm{R}, \mathrm{F}) \to \mathrm{H}^p(\mathrm{G}_\mathrm{K}, j^* \mathrm{F}) \to \ldots,$$

[which may be obtained by applying Ext (, F) to the exact sequence

$$0 \to j_! j^* \mathrm{Z} \to \mathbf{Z} \to i_* i^* \mathbf{Z} \to 0],$$

and showing that $\mathrm{H}^p(\mathrm{R}, j_! \mathrm{M}) = 0$ for all p. [To do this, one is tempted to use that $j_!$ is exact, and that $\mathrm{H}^0(\mathrm{R}, j_!)$ is the zero functor. However, one must check that $j_! \mathrm{I}$ is acyclic for cohomology over R, whenever I is injective. To do this, consider

$$\begin{array}{ccccccccc} 0 & \to & 0 & \to & \mathrm{I}^0 & \to & \mathrm{I}^0 & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \mathrm{I} & \to & \mathrm{I} & \to & 0 & \to & 0 \end{array}$$

as an exact sequence of Galois modules over R, which may also be described as

$$0 \to j_! \mathrm{I} \to j_* \mathrm{I} \to i_* i^* j_* \mathrm{I} \to 0.$$

Now, this is an injective resolution (the thing to check is that i^* preserves injectives, which is true since

$$\alpha : \quad \mathrm{G}_k \to \mathrm{G}_{\mathrm{dec}}/\mathrm{G}_{\mathrm{in}}$$

is an isomorphism). Computing $\mathrm{H}^q(\mathrm{R}, j_! \mathrm{I})$ using this resolution gives us the desired acyclicity.]

To see (ii), use that $j_!$ and j^* are adjoint, together with the fact that they are both exact, to get the first isomorphism; the second comes from the ordinary spectral sequence for Galois modules, relating Ext with cohomology, using that $\overline{\mathrm{K}}^*$ is an injective abelian group to achieve degeneration of that spectral sequence.

Etale topology. — To discuss global versions of the above theorems, we must now use the étale sheaf-theoretic language. We will only consider affine schemes of the form $\mathrm{X} = \mathrm{Spec}(\mathrm{D})$ where D is a Dedekind domain. If $\mathrm{D} \subset \mathrm{E}$ is an extension of Dedekind domains, such that E is of finite type as a D-module, and $f \in \mathrm{E}$ is $\neq 0$, form $\mathrm{D}' = \mathrm{E}_f$ (E localized away from f) and consider the extension $\mathrm{D} \subset \mathrm{D}'$. If every prime ideal of D' is unramified over D, say that the extension $D \subset D'$ is an *étale connected extension.* In our situation, the étale connected morphisms $Y \to X$ are exactly those coming from such étale connected extensions $Y = \mathrm{Spec}(D')$. By the " étale topology " for our scheme X, we shall mean the topology T_X whose underlying category is the category Et (X) of étale schemes Y over X, coverings being the surjective families. We refer to [GT] or [SGAA] for the theory of étale topologies. It may be useful to point out that, for the schemes X that we consider, the numerous variants of the topolgy T_X given in [GT] yield equivalent categories of sheaves.

Let S_X denote the category of abelian sheaves for the topology T_X. Some relevant facts about S_X are : it is an abelian category with enough injectives, possessing products and sums. Given any morphism $f : X \to Y$ of schemes, we have the pullback and direct image functors

$$f^* : \quad S_Y \to S_X, \qquad f_* : \quad S_X \to S_Y,$$

which are adjoint :

$$\mathrm{Hom}(A, f_* B) = \mathrm{Hom}(f^* A, B).$$

Furthermore, f^* is exact and f_* is left exact (and preserves injectives since it has an exact left adjoint). Suppose $i : X \to Y$ is a closed subscheme of dimension zero and let $j : U \to Y$ denote the open complement. Consider the left exact functor $h : S_U \to S_X$ given by $h = i^* j_*$. Artin shows ([GT], cor. 2.5) that certain easily definable functors from S_Y to the mapping cylinder category C_h, and back again, establish an equivalence between these two categories. This result (which we will refer to as the *decomposition lemma*) enables one to think of the category S_Y as " nothing more than " the mapping cylinder category associated to $h : S_U \to S_X$, giving a firm intuitive hold on abelian sheaves for the étale topology.

What abelian sheaves are there over an arbitrary scheme X ?

(*a*) *Representable sheaves.* — Recall that a set-valued functor F on any category $\mathcal{C}$ is said to be *represented by an object* Y of $\mathcal{C}$ if one gives an isomorphism of set-valued functors on $\mathcal{C}$:

$$F(-) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(-, Y).$$

If Y represents the functor F, Y is unique up to canonical isomorphism in the caterogy $\mathcal{C}$. A sheaf is called *representable* (in $\mathcal{C}$) if it can be represented by an object of $\mathcal{C}$.

By a *representable sheaf* we shall mean a representable abelian group functor on the category of schemes over X. These *are* actually sheaves, as follows from descent theory [Gr].

(*b*) *Constant sheaves.* — A subclass of the representable sheaves. Given any abelian group A, define the sheaf $\overline{A}$ by $\overline{A}(U) = A$ for all connected étales $U \to X$.

(*c*) *Locally constant sheaves.* — To widen the above class a bit, consider an étale surjective morphism $\pi : Y \to X$, with Y connected. Say that a sheaf F on X is *split by* π if $\pi^* F$ is constant as a sheaf on Y, in which case $\pi^* F = \overline{F(Y)}$. Suppose Y is Galois over X with finite Galois group G. Then F (Y) can may be given the structure of a G-module by letting $g \in G$ act as $F(g^{-1})$. In this manner we obtain a functor from the full subcategory of sheaves on X split by $\pi : Y \to X$ to the category of G-modules. This is an equivalence of categories, and one may obtain a functor in the reverse direction as follows : if M is a G-module, consider $M \otimes Y = \coprod_{m \in M} Y$ and let G operate on $M \otimes Y$ by the diagonal action. Then, one may show that $(M \otimes Y)/G$ is an abelian group scheme over X, and represents a sheaf split by π. We say that a sheaf is *locally constant* if it is split by such an étale surjective π. More generally, we may consider inductive limits of locally constant sheaves.

EXAMPLES :

1. Take $X = \operatorname{Spec}(K)$, for K a field. Then any abelian sheaf on X is an inductive limit of locally constant sheaves. Choosing a separable algebraic closure $\overline{K}$ of K, we may obtain a functor $S_X \to S_K$ by associating to any sheaf F the G_K-module $\varinjlim F(\operatorname{Spec} L)$ where the limit is taken over all finite extensions L/K in $\overline{K}$, and $\gamma \in G_K$ operates by $F(\operatorname{Spec} \gamma)$. This functor is an equivalence of categories.

2. Take $X = \operatorname{Spec}(R)$. The following functor

$$\begin{aligned} S_X &\dashrightarrow S_R, \\ F &\mapsto (M_F, N_F, \varphi_F), \end{aligned}$$

is an equivalence of categories :

(*a*) Take M_F to be the module obtained by the composition of functors

$$\begin{aligned} S_X &\xrightarrow{j^*} S_{\operatorname{Spec}(K)} \to S_K, \\ F &\longmapsto M_F, \end{aligned}$$

where the second functor is the one described in Example (1) above. Explicitly, $M_F = \varinjlim_{L/K} F(\operatorname{Spec} L)$ where the limit is taken over all finite extensions L/K in $\overline{K}$. The abelian group M_F is regarded as G_K-module by the action : $\gamma \in G_K$ operates by $F(\operatorname{Spec} \gamma)$.

(*b*) The G_K-module N_F is defined as $N_F = \varinjlim_{L/K} F(R_L)$ where L/K ranges through all finite subextensions of K_0/K, and R_L denotes the ring of integers in L. We may regard N_F as a G_k-module since Gal (K_0/K) operates on N_F in a natural way, and we have the homomorphism

$$G_k \overset{\alpha}{\underset{\sim}{\to}} G_{\mathrm{dec}}/G_{\mathrm{in}} \to \mathrm{Gal}\,(K_0/K).$$

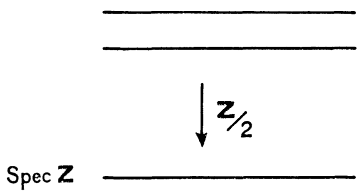
(*c*) The map $\varphi_F : N_F \to M_F$ is the evident one. It is induced from the natural homomorphism $F(R_L) \to F(L)$ for L/K any finite subextension of K_0/K, by passage to the direct limit.

The fact that the above functor is an equivalence of categories follows directly from the decomposition lemma ([GT] cor. 2.5).

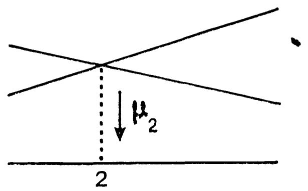
Representable sheaves which play key roles in the duality theory are :

(*a*) the multiplicative group $\mathbf{G}_m$, represented by the abelian group scheme Spec $(\mathbf{Z}[t, t^{-1}])$;

(*b*) the group $\boldsymbol{\mu}_n$ of nth roots of unity, which may be taken to be the kernel of the nth power map of $\mathbf{G}_m$, or more explicitly as the scheme Spec $(Z[t]/(t^n - 1))$. Of course, as a functor, $\boldsymbol{\mu}_n(Y)$ is simply the group of elements in $\mathcal{O}^*(Y)$ whose nth power is 1. If a primitive nth root of unity is rational over the scheme X, then $\boldsymbol{\mu}_n \simeq \mathbf{Z}/n$, as sheaves for the étale topology over X. Thus $\boldsymbol{\mu}_2 \simeq \mathbf{Z}/2$ as sheaves over any scheme of characteristic different from 2, e. g. over Spec $(\mathbf{Z})$. If one draws the two schemes $\boldsymbol{\mu}_2$ and $\mathbf{Z}/2$ over Spec $(\mathbf{Z})$, one gets the picture :



and



This is a most elementary example of the following phenomenon : two distinct abelian group schemes may represent the same étale sheaf. Of course, to achieve uniqueness, one has to restrict attention to the category E_X of étale abelian group schemes over X [i. e. the abelian group objects of Et (X) itself], where, by the general facts of life concerning representable functors, one has that the canonical functor $E_X \to S_X$ is fully faithful. One does not have to go far to find étale abelian sheaves which are not representable. [Consider the injection i : Spec $(\mathbf{Z}/p) \to$ Spec $(\mathbf{Z}_p)$.

Choose some non constant sheaf A over $\mathbf{Z}/p\,\mathbf{Z}$, for example $\boldsymbol{\mu}_n$, where n does not divide $p-1$. Then i_* A is not representable over Spec ($\mathbf{Z}_p$) (³).]

Having thus mastered sheaves over fields and discrete valuation rings, we turn to X = Spec (D), with D a general Dedekind domain. A very useful class of sheaves are the *constructible* ones. Explicitly, in our context, a sheaf F over X is constructible if there are a finite number of closed points $x_1, \ldots, x_n$ of X such that, if we denote by U the open complement $X - x_1 - \ldots - x_n$, the pullbacks of F to $x_1, \ldots, x_n$ and to U are locally constant finite abelian sheaves over those subschemes. Any finite abelian group scheme over X represents a constructible sheaf. If F is constructible, the *Cartier dual* $\hat{F}$ of F, defined as the sheaf $\underline{\mathrm{Hom}}$ (F, $\mathbf{G}_m$), is again constructible. [Note, however, that the class of locally constant sheaves is *not* closed under the operation of taking Cartier dual, as $(\mathbf{Z}/n)^\wedge \simeq \boldsymbol{\mu}_n$ shows.] The category of constructible sheaves has remarkably good properties, and only two failings :

(*a*) There aren't enough injectives.

(*b*) $\mathbf{G}_m$ is not constructible.

Finally, some remarks about the mechanics of the *homological algebra* to be used :

(*a*) All spectral sequences will be spectral sequences of composite functors
$$R^p\,FR^q\,G \Rightarrow R^{p+q}\,FG,$$
where G will preserve injectives (generally because G has an exact left adjoint), save in one instance :
$$H^p\,(X, \underline{\mathrm{Ext}}^q\,(A, B)) \Rightarrow \mathrm{Ext}_X^{p+q}\,(A, B),$$
and this spectral sequence exists since $\underline{\mathrm{Hom}}$ (A,) sends injective sheaves to flask sheaves ([SGAA], V, prop. 4.10). Indeed, one of the important uses of the Čech theory for étale cohomology (as well as for classical sheaf theory) is to provide a class of acyclic sheaves (the flask ones) broader than the class of injective sheaves, so as to facilitate homological algebraic computations.

(*b*) A second important use of the Čech theory (combined with descent theory) is to give a " down-to-earth " interpretation of H^1 (X, F) : for any étale group scheme F, we may identify H^1 (X, F) with the group of isomorphism classes of principal homogeneous spaces Y/X with respect

(³) It is, however, " locally representable for the étale topology ".

to F, which are locally trivial for the étale topology; this comes from [Gr] [which interprets $H^1(X, F)$ as above] combined with [GT], II, 3.6 [which identifies the Čech $\check{H}^1(X, F)$ with $H^1(X, F)$].

(*c*) For the record, if X is a scheme and $i : V \hookrightarrow X$ a closed immersion, define the *relative* cohomology groups by :

$$H^q_V(X, F) = \mathrm{Ext}^q_X(\mathbf{Z}_V, F).$$

If V is empty, one of course gets the absolute cohomology groups. The relative and absolute groups are connected by the long exact cohomology sequence

$$\ldots \to H^q_V(X, F) \to H^q(X, F) \to H^q(V, i^* F) \to \ldots$$

cf. [GT], III, 2.11.

There is also an excisive result, a particular case of which is the following :

TERMINOLOGY :

— $X = \mathrm{Spec}(D)$.

— $i : p \to X$ a closed point.

— D_p = localisation of D at p.

— $X_p = \mathrm{Spec}(D_p)$.

— $\tilde{D}_p$ = Hensel-closure of $\tilde{D}_p$. [*Note :* we do not mean the strict Hensel-closure. Thus $\tilde{D}_p$ has the same residue field as D_p, and is defined to be the direct limit of local rings $\varinjlim D'_p$ taken over the category of finite extension with a lifting of p :

$$\begin{tikzcd} & \mathrm{Spec}(D'_p) \arrow[d, "\pi"] \\ p \arrow[ur] \arrow[r] & \mathrm{Spec}(D_p) \end{tikzcd}$$

where π is *étale* at p.]

— $\tilde{X}_p = \mathrm{Spec}(\tilde{D}_p)$.

— $f : \tilde{X}_p \to X$ the natural morphism.

PROPOSITION (Excision). — *Let* F *be a sheaf on* X. *There is a natural isomorphism :*

$$H^n_p(\tilde{X}_p, f^* F) \xrightarrow{\sim} H^n_p(X, F).$$

Proof. — We shall first establish

$$H^n_p(X_p, f^* F) \xrightarrow{\sim} H^n_p(X, F).$$

Let U denote any open subscheme of X containing p, and let $g : U \to X$ be the inclusion. Consider the functor $g_!$ which extends sheaves on U to sheaves on X " by zero ". Since $g_!$ is adjoint to g^*, we have

$$\mathrm{Ext}^n_U(A, g^* F) = \mathrm{Ext}^n_X(g_! A, F),$$

using that g^* is exact. Now take A to be the sheaf $\mathbf{Z}_p = i_* \mathbf{Z}$ on U. Then $g_! A$ is just the sheaf $\mathbf{Z}_p$ on X and the above formula gives (upon passage to the limit) :

$$H^n_p(X_p, g^* F) \xrightarrow{\sim} H^n_p(X, F).$$

Now consider a finite extension

$$\begin{tikzcd} & \mathrm{Spec}(D'_p) = X'_p \arrow[dd, "\pi"] \\ p \arrow[ur] \arrow[dr] & \\ & \mathrm{Spec}(D_p) = X_p \end{tikzcd}$$

which is étale at p.

CLAIM 1 :

$$\mathrm{Hom}_{X'_p}(\mathbf{Z}_p, \pi^* G) \xrightarrow{\sim} \mathrm{Hom}_{X_p}(\mathbf{Z}_p, G),$$

where G *is any Galois module over* D_p.

Proof of Claim 1. — If the Galois module G is represented by the triple (G_1, G_2, φ), then both sides of the above formula are given by Ker φ, the map being the identity.

CLAIM 2 :

$$H^n_p(X'_p, \pi^* G) \to H^n_p(X_p, G)$$

is an isomorphism for any sheaf G.

(This follows immediately from Claim 1 and the fact that π^* is exact and preserves injectives.)

Our proposition then follows from claim 2 by passage to the limit.

Q. E. D.

2. The duality theorem and the norm theorem

Let D be the ring of integers in a number field K, $X = \mathrm{Spec}(D)$ and $j : x \to X$ the generic point, so $x = \mathrm{Spec}(K)$. For minor reasons we will asumes that K is *totally imaginary*.

There is a global version of the exact sequence of sheaves (★) of paragraph 1 :

$$(\star\star)\qquad 0 \to \mathbf{G}_{m,X} \to j_* \mathbf{G}_{m,x} \to \coprod_p \mathbf{Z}/_p \to 0,$$

where the direct sum is taken over all closed points p of X (i.e. non zero prime ideals of D), and $\mathbf{Z}/_p$ means the sheaf $\mathbf{Z}$ concentrated at the closed point p (*cf.* [GT], IV, 1.4).

We have :

$$(2.1)\qquad R^q j_* (\mathbf{G}_{m,x}) = 0 \quad \text{if} \quad q > 0.$$

To show this, use that $R^q j_* (G)$ is (in close analogy with classical sheaf theory) the sheaf associated with the presheaf whose value on any étale $U \to X$ is given by $H^q (j^{-1} U, G)$, *cf.* [GT], II, 4.7. It is then immediate that the stalk of this sheaf vanishes on the generic point, and alos on all closed points for reasons identical to those giving rise to (1.1) above. But, if all the stalks of a sheaf are zero, the sheaf is trivial ([GT], III, 1.8).

$$(2.2)\qquad H^q (X, j_* \mathbf{G}_{m,x}) \xrightarrow{\approx} H^q (x, \mathbf{G}_{m,x}) = H^q (\;/K).$$

For, by (2.1), the spectral sequence

$$H^p (X, R^q j_* \mathbf{G}_{m,x}) \Rightarrow H^{p+q} (x, \mathbf{G}_{m,x})$$

degenerates.

(2.3) Let us evaluate the cohomology exact sequence coming from (★★) and (2.2) :

(*a*) Starting with H^0, and using Hilbert's theorem 90 $[H^1 (x, \mathbf{G}_{m,x}) = 0]$ one obtains

$$0 \to D^* \to K^* \to \bigoplus_p \mathbf{Z} \to \operatorname{Pic} X \to 0.$$

(*b*) Using (1.6) we have

$$H^q \left(X, \coprod_p \mathbf{Z}/_p\right) = 0 \quad \text{for} \quad q = 1, q > 2.$$

$$H^2 \left(X, \coprod_p \mathbf{Z}/_p\right) = \bigoplus_p \mathbf{Q}/\mathbf{Z}$$

and so,

$$0 \to H^2 (X, \mathbf{G}_{m,X}) \to H^2 (x, \mathbf{G}_{m,x}) \xrightarrow{\eta} \bigoplus_p \mathbf{Q}/\mathbf{Z} \to H^3 (X, \mathbf{G}_{m,X}) \to H^3 (x, \mathbf{G}_{m,x}) \to 0,$$

$$H^q (X, \mathbf{G}_{m,X}) \xrightarrow{\approx} H^q (x, \mathbf{G}_{m,x}) \qquad \text{for} \quad q \geqq 4.$$

(*c*) At this point one is forced to use the results of global class field theory to complete our evaluation.

One has (compare [AV]) :

$$0 \to H^2(x, \mathbf{G}_{m,x}) \xrightarrow{r_x} \bigoplus_p \mathbf{Q}/\mathbf{Z} \xrightarrow{\text{sum}} \mathbf{Q}/\mathbf{Z} \to 0,$$

$$H^q(x, \mathbf{G}_{m,x}) = 0 \quad \text{for} \quad q \geqq 3.$$

Thus we obtain the following list :

$$H^q(X, \mathbf{G}_m) = \begin{cases} D^*, & \text{for } q = 0, \\ \operatorname{Pic} X = \text{ideal class group of } D, & \text{for } q = 1, \\ 0, & \text{for } q = 2, \\ \mathbf{Q}/\mathbf{Z}, & \text{for } q = 3, \\ 0, & \text{for } q > 3. \end{cases}$$

Remarks :

1. The above computation looks slightly different when K is not totally imaginary.

2. Since we have used class field theory in the computation of $H^q(X, \mathbf{G}_m)$ it is not at all surprising that many of the basic results of class field theory may be reread from the ensuing duality theorem, A reasonable question is whether one can, by some direct method (using, to be sure, the special nature of global fields) compute $H^q(X, \mathbf{G}_m)$. This is missing at present for number fields, however, for function fields of one variable over finite fields, Tsen's theorem leads directly to such a computation.

(2.4) Statement of the duality theorem. — To state the duality theorem of Artin and Verdier, let F be an abelian sheaf over X; by Yoneda pairing of Ext's we get :

$$H^r(X, F) \times \operatorname{Ext}_X^{3-r}(F, \mathbf{G}_m) \to H^3(X, \mathbf{G}_m) = \mathbf{Q}/\mathbf{Z}.$$

Global duality theorem. — *The cohomology groups* $H^r(X, F)$ *are* finite, *and the above pairing is* non degenerate, *for all constructible sheaves* F *and all integers r.*

We may translate the local duality theorem discussed in paragraph 1 into the language of étale cohomology :

Local duality theorem. — *The cohomology groups* $H_p^r(X_p, F)$ *are* finite, *and the Yoneda pairing :*

$$H_p^r(X_p, F) \times \operatorname{Ext}_{X_p}^{3-r}(F, \mathbf{G}_m) \to H_p^3(X_p, \mathbf{G}_m) = \mathbf{Q}/\mathbf{Z}$$

is non degenerate *for all constructible sheaves* F *and all integers r.*

Remarks :

(*a*) Note that the global duality theorem is especially fine since the scheme X over which things are occurring has residue fields of arbitrary characteristics, and yet the duality result is exact. This should be compared with e. g. the Poincaré duality, where one must ignore p-primary components for those p occurring as residue characteristics of the base scheme. This feature of the global duality theorem reflects the mystery and success of class field theory which classifies *all* abelian extensions whether or not they have ramification of order divisible by a residue characteristic.

(*b*) This Yoneda pairing is *not* a Poincaré-type duality, in the sense that $\mathrm{Ext}_X^m(F, \mathbf{G}_m)$ will not, in general, reduce to $H^m(X, \hat{F})$. Indeed, this is the key to the fineness of the theorem as described in Remark (*a*) above. We will return to analyze the spectral sequence relating $\mathrm{Ext}_X^m(F, \mathbf{G}_m)$ and $H^m(X, \hat{F})$ in paragraph 3 below.

(*c*) If we apply $\mathrm{Ext}_X(\ , \mathbf{G}_m)$ to the exact sequence

$$0 \to \mathbf{Z} \to \mathbf{Z} \to \mathbf{Z}/n \to 0$$

and use the computations of

$$H^q(X, \mathbf{G}_{m,X}) = \mathrm{Ext}_X^q(\mathbf{Z}, \mathbf{G}_m)$$

given in (2.3) we may calculate $\mathrm{Ext}_X^q(\mathbf{Z}/n, \mathbf{G}_m)$.

Explicitly,

$$\mathrm{Ext}_X^q(\mathbf{Z}/n, \mathbf{G}_m) = \begin{cases} \boldsymbol{\mu}_n(D) & \text{if } q = 0, \\ \mathrm{Pic}\, X/n & \text{» } q = 2, \\ \mathbf{Z}/n & \text{» } q = 3, \\ 0 & \text{» } q > 3, \end{cases}$$

where $\boldsymbol{\mu}_n(D)$ denotes the n th roots of 1 in D and where we have left out the value $q = 1$ because the best that we can say about $\mathrm{Ext}_X^1(\mathbf{Z}/n, \mathbf{G}_m)$ is that it lives in a short exact sequence

$$0 \to D^*/D^{*n} \to \mathrm{Ext}_X^1(\mathbf{Z}/n, \mathbf{G}_m) \to {}_n\mathrm{Pic}\, X \to 0,$$

where ${}_n\mathrm{Pic}\, X$ denotes the subgroup of elements of order n.

If we apply the duality theorem for $F = \mathbf{Z}/n$ we obtain that

$$H^q(X, \mathbf{Z}/n) = \mathrm{Ext}_X^{3-q}(\mathbf{Z}/n, \mathbf{G}_m)^{\sim},$$

where $^{\sim}$ denotes Pontrjagin duality.

Thus we obtain the following list of values for the groups $H^q(X, \mathbf{Z}/n)$:

$$H^q(X, \mathbf{Z}/n) = \begin{cases} \mathbf{Z}/n & \text{if } q = 0, \\ (\mathrm{Pic}\, X/n)^{\sim} & \text{» } q = 1, \\ \mathrm{Ext}_X^1(\mathbf{Z}/n, \mathbf{G}_m)^{\sim} & \text{» } q = 2, \\ \boldsymbol{\mu}_n(D)^{\sim} & \text{» } q = 3, \\ 0 & \text{» } q > 3. \end{cases}$$

Of course, for $q = 1$, we are told something recognizable. Namely, the group of isomorphism classes of cyclic unramified extensions of X of order n is dual to the cokernel of the n th power map on the ideal class group of D.

(2.5) Classification at extensions with prescribed ramification data except at a finite set of primes $p_1, \ldots, p_t$. — Let us explore further the computations that can be made. If $V = \{p_1, \ldots, p_t\}$ is a closed subscheme of X of dimension zero, one gets the exact sequence

$$\ldots \to \mathrm{H}^r(\mathrm{X}, \mathrm{F}) \to \mathrm{H}^r(\mathrm{X}-\mathrm{V}, \mathrm{F}) \to \coprod_{i=1}^{t} \mathrm{Ext}^{3-r-1}_{\mathrm{X}_{p_i}}(\mathrm{F}, \mathbf{G}_m)^{\sim} \to \ldots$$

by pooling the relative cohomology exact sequence, excision, and local duality.

It is usually an easy matter to evaluate the local terms which occur in the above exact sequence. By way of example, take $\mathrm{F} = \mathbf{Z}/n$. Since the group

$$\mathrm{Ext}^q_{\mathrm{X}_p}(\mathbf{Z}, \mathbf{G}_m) = \mathrm{H}^q(\mathrm{X}_p, \mathbf{G}_m)$$

has already been evaluated (1.7 a) we may evaluate $\mathrm{Ext}^q_{\mathrm{X}_0}(\mathbf{Z}/n, \mathbf{G}_m)$ by applying

$$0 \to \mathbf{Z} \xrightarrow{n} \mathbf{Z} \to \mathbf{Z}/n \to 0.$$

One obtains the following list :

$$\mathrm{Ext}^q_{\mathrm{X}_p}(\mathbf{Z}/n, \mathbf{G}_m) = \begin{cases} {}_n\mathrm{U}_p & \text{if } q = 0, \\ \mathrm{U}_p/\mathrm{U}_p^n & \text{»} \ \ q = 1, \\ 0 & \text{»} \ \ q > 1, \end{cases}$$

where $\mathrm{U}_p = \mathrm{D}_p^*$ is the group of local units, and the pre-subscript n denotes the kernel of the n th power map.

If we now use the calculations of $\mathrm{H}^q(\mathrm{X}, \mathbf{Z}/n)$ made in (2.4), the above long exact sequence enables us to find $\mathrm{H}^q(\mathrm{X}-\mathrm{V}, \mathbf{Z}/n)$. If the reader unravels this computation for $q = 1$ [which involves identifying the morphisms of the diagram of exact sequences

$$\begin{tikzcd}
 & & & & 0 \arrow[d] \\
 & & & & {}_n\mathrm{Pic}\,\mathrm{X}^{\sim} \arrow[d] \\
0 \arrow[r] & (\mathrm{Pic}\,\mathrm{X}/n)^{\sim} \arrow[r] & \mathrm{H}^1(\mathrm{X}-\mathrm{V}, \mathbf{Z}/n) \arrow[r] & \prod_{i=1}^{t} (\mathrm{U}_{p_i}/\mathrm{U}_{p_i}^n)^{\sim} \arrow[r] \arrow[dr] & \mathrm{Ext}^1_{\mathrm{X}}(\mathbf{Z}/n, \mathbf{G}_m)^{\sim} \arrow[d] \\
 & & & & (\mathrm{D}^*/\mathrm{D}^{*n})^{\sim} \arrow[d] \\
 & & & & 0
\end{tikzcd}$$

an amusing exercise !] he will of course find that the description of the group of isomorphism classes of cyclic extensions of K, unramified outside V, agrees with the classical description, coming from class field theory.

(2.6) THE HILBERT SYMBOL. — The classical Hilbert symbols and global reciprocity law are related to the following isomorphisms :

$$\mathrm{H}^3_p(\mathrm{X}_p, \boldsymbol{\mu}_n) \xrightarrow{\sim} \mathrm{Ext}^0_{\mathrm{X}_p}(\boldsymbol{\mu}_n, \mathbf{G}_m) = \mathbf{Z}/n,$$
$$\mathrm{H}^3(\mathrm{X}, \boldsymbol{\mu}_n) \xrightarrow{\sim} \mathrm{Ext}^0_{\mathrm{X}}(\boldsymbol{\mu}_n, \mathbf{G}_m) = \mathbf{Z}/n.$$

Let us fix a pairing $\alpha : \mathrm{F}_1 \times \mathrm{F}_2 \to \boldsymbol{\mu}_n$ of finite sheaves over X into $\boldsymbol{\mu}_n$. If $x_i \in \mathrm{H}^1(\mathrm{X}_p - p, \mathrm{F}_i)$ $(i = 1, 2)$ define the *Hilbert symbol* (*at* p) (associated to the pairing α) :

$$(x_1, x_2)_p \in \mathbf{Z}/n$$

to be the image of the cup-product of x_1 and x_2 computed via the pairing α. If U_i $(i = 1, 2)$ are open nonempty subschemes of X and

$$x_i \in \mathrm{H}^1(\mathrm{U}_i, \mathrm{F}_i) \qquad (i = 1, 2)$$

let us denote by the same symbol and name $(x_1, x_2)_p$ the Hilbert symbol at p of the *images* of x_i in $\mathrm{H}^1(\mathrm{X}_p - p, \mathrm{F}_i)$. Note that if both U_1 and U_2 contain p, then the Hilbert symbol at p vanishes

$$(x_1, x_2)_p = 0$$

for classes $x_i \in \mathrm{H}^1(\mathrm{U}_i, \mathrm{F}_i)$. This follows from the fact that the cup-product then factors through the group $\mathrm{H}^2(\mathrm{X}_p, \boldsymbol{\mu}_n)$ which is zero.

If α is a *nondegenerate* pairing, then the Hilbert symbol at p is a nondegenerate pairing

$$\mathrm{H}^1(\mathrm{X}_p - p, \mathrm{F}_1) \times \mathrm{H}^1(\mathrm{X}_p - p, \mathrm{F}_2) \to \mathbf{Z}/n.$$

Let $\mathrm{V} = \{p_1, \ldots, p_t\} \subset \mathrm{X}$ be a closed nonempty subscheme of dimension zero. One has the diagram

$$\begin{tikzcd}
 & \coprod_{p \in \mathrm{V}} \mathbf{Z}/n \arrow[r, "\mathrm{sum}"] \arrow[d, "\cong"] & \mathbf{Z}/n \arrow[d, "\cong"] & \\
\mathrm{H}^2(\mathrm{X} - \mathrm{V}, \boldsymbol{\mu}_n) \arrow[r] & \mathrm{H}^3_{\mathrm{V}}(\mathrm{X}, \boldsymbol{\mu}_n) \arrow[r] & \mathrm{H}^3(\mathrm{X}, \boldsymbol{\mu}_n) \arrow[r] & 0
\end{tikzcd}$$

where the bottom line is exact.

From this discussion we obtain the following " reciprocity " formula :

$$\sum_p (x_1, x_2)_p = \sum_{p \in \mathrm{V}} (x_1, x_2)_p = 0$$

for $x_i \in \mathrm{H}^1(\mathrm{X} - \mathrm{V}, \mathrm{F}_i)$ $(i = 1, 2)$.

(2.7) THE NORM. — How does the classical norm map fit into this formulation? Some hints: let $\pi : Y \to X$ be a finite morphism, where $Y = \operatorname{Spec}(D_Y)$, $X = \operatorname{Spec}(D_X)$, the D's being Dedekind domains. The classical *norm* defines a map of sheaves

$$\pi_* \mathbf{G}_{m,Y} \to \mathbf{G}_{m,X}.$$

We have the standard property that the composition

$$\mathbf{G}_{m,X} \to \pi_* \pi^* \mathbf{G}_{m,X} = \pi_* \mathbf{G}_{m,Y} \xrightarrow{\text{norm}} \mathbf{G}_{m,X}$$

is just multiplication by the degree of π, and a deeper property: the diagram

$$\begin{array}{ccccc} H^3(Y, \mathbf{G}_{m,Y}) & \longrightarrow & H^3(X, \pi_* \mathbf{G}_{m,Y}) & \xrightarrow{\text{norm}} & H^3(X, \mathbf{G}_{m,X}) \\ {\scriptstyle\text{inv}}\downarrow\wr & & & & {\scriptstyle\text{inv}}\downarrow\wr \\ \mathbf{Q}/\mathbf{Z} & & \xrightarrow{\qquad\text{id}\qquad} & & \mathbf{Q}/\mathbf{Z} \end{array}$$

is commutative. To see this, one must recall the computation of $H^r(\ ,\mathbf{G}_m$ via ($\star\star$) and then invoke the analogous diagrams in local and global class field theory which relate the norm map to inv.

For any sheaf F on Y, the norm map gives us a map

$$N: \quad \operatorname{Ext}_Y^r(F, \mathbf{G}_m) \to \operatorname{Ext}_X^r(\pi_* F, \mathbf{G}_m)$$

as the composition

$$\operatorname{Ext}_Y^r(F, \mathbf{G}_{m,Y}) \to \operatorname{Ext}_X^r(\pi_* F, \pi_* \mathbf{G}_{m,Y}) \xrightarrow{\text{norm}} \operatorname{Ext}_X^r(\pi_* F, \mathbf{G}_{m,X}),$$

the first map owing its existence to the fact that $R^q \pi_* = 0$ for $q > 0$ since π is finite. Another application of the vanishing of $R^q \pi_*$ for $q > 0$ is that the natural map

$$\nu: \quad H^r(X, \pi_* F) \to H^r(Y, F)$$

is an isomorphism.

One checks by diagram-chasing that the maps N and ν are compatible with the Yoneda pairings in the sense that, if

$$\alpha \in H^r(X, \pi_* F) \qquad \text{and} \qquad \beta \in \operatorname{Ext}_Y^{3-r}(F, \mathbf{G}_m),$$

then

$$\operatorname{inv}\langle \nu\alpha, \beta \rangle = \operatorname{inv}\langle \alpha, N\beta \rangle.$$

NORM THEOREM. — *Let* $\pi : Y \to X$ *be a finite morphism. If* F *is a constructible sheaf on* Y, *then*

$$N: \quad \operatorname{Ext}_Y^r(F, \mathbf{G}_m) \to \operatorname{Ext}_X^r(\pi_* F, \mathbf{G}_m)$$

is an isomorphism for all r.

Remarks :

(*a*) The norm theorem behaves well with respect to restriction to open subschemes. For, if

$$\begin{tikzcd} V \arrow[r, "\bar{j}"] \arrow[d, "\pi'"'] & Y \arrow[d, "\pi"] \\ U \arrow[r, "j"] & X \end{tikzcd}$$

is a cartesian diagram, with j an open immersion, then

$$j_! \pi'_* = \pi_* \bar{j}_!$$

and the diagram

$$\begin{tikzcd} \operatorname{Ext}^r_Y(\bar{j}_! F, \mathbf{G}_m) \arrow[r, "N"] \arrow[d, "\sim"'] & \operatorname{Ext}^r_X(\pi_* j_! F, \mathbf{G}_m) \arrow[d, "\sim"'] \\ \operatorname{Ext}^r_V(F, \mathbf{G}_m) \arrow[r, "N"] & \operatorname{Ext}^r_U(\pi'_* F, \mathbf{G}_m) \end{tikzcd}$$

is commutative.

(*b*) As with the duality theorem, if

$$0 \to F_1 \to F_2 \to F_3 \to 0$$

is an exact sequence of sheaves, the norm theorem for two out of three implies the norm theorem for the third, by the five-lemma (using that π_* is exact, since π is finite).

(*c*) The norm theorem follows from global duality since, by the previous remarks, N and ν are adjoint with respect to Yoneda pairings, and ν is an isomorphism. The analogous local theorem (replacing X by X_p and Y by Y_p) also follows from local duality. Actually one proves the global duality theorem by proving the norm theorem first.

(*d*) A sheaf F will be called *punctual* if there is a closed subscheme $V \subset X$ such that F vanishes on $X - V$. Equivalently, $F = i_* E$ for some sheaf E on V. By the *punctual duality theorem* (resp. *punctual norm theorem*) we will mean the duality theorem (resp. the norm theorem) for punctual constructible sheaves. Clearly the punctual duality theorem implies the punctual norm theorem. But excision reduces the punctual duality theorem to the corresponding statement for Galois modules over discrete valuation rings, proved in paragraph 1. Consequently, the punctual norm theorem is established.

(*e*) If π is étale, the norm theorem is true. For in this case there is a beautiful morphism $F \to \pi^* \pi_* F$ which (using the exactness of π^*) provides an inverse to N.

(*f*) PROOF OF THE NORM THEOREM. — Let $j : U \to X$ be an open nonempty subscheme of X which does not contain any ramification point of π. Denote by $\bar{j}$ the open inclusion of $V = \pi^{-1} U$ into Y. Consider the exact sequence

$$0 \to \bar{j}_! \bar{j}^* F \to F \to F_0 \to 0$$

and let us combine some of the above remarks. Since $\pi' : V \to U$ is étale and finite, the norm theorem is true for $\bar{j}^* F$, consequently by (*a*) it is true for $\bar{j}_! \bar{j}^* F$. Since F_0 is punctual the norm theorem is true for it by (*d*), and now use (*b*) to establish the norm theorem in general.

(*g*) One application of the norm theorem is to show that, if the duality theorem is true for F (over Y), then it is true for $\pi_* F$ (over X).

3. Proof of the duality theorem : Reduction to a special case

The sheaf-theoretic techniques of paragraphs 1 and 2 quickly reduce the question of duality for all constructible sheaves to the question of duality for constant sheaves, using the norm theorem. In fact, one reduces to the simplest situation : $F = \mathbf{Z}/n$, with n prime, and X a global scheme over which ζ, a primitive n th root of unity, is rational. More precisely

PROPOSITION A. — *Suppose*

$$m^r(F) : \quad H^r(X, F) \to \operatorname{Ext}_X^{3-r}(F, \mathbf{G}_m)^\sim$$

is an isomorphism for all $r < r_0$ *and all constructible sheaves. Suppose furthermore that* $m^r(\mathbf{Z}/n)$ *is an isomorphism for* $r = r_0$ *for all prime numbers* n, *and all* X *such that a primitive* n th-*root of unity is rational over* X. *Then* $m^r(F)$ *is an isomorphism for all constructible sheaves* F *and* $r \leq r_0$.

Proof of proposition A. — It is convenient to have the following terminology :

If

$$0 \to F \to F' \to F'' \to 0$$

is a short exact sequence of sheaves such that F'' is *punctual*, say that F and F' are *elementarily punctually equivalent*. This notion generates an equivalence relation which we will refer to as *punctual equivalence*.

It is a consequence of the local duality theorem and of the punctual duality theorem [remark (d)] and the five-lemma that if

$$m^r(\mathrm{F}): \quad \mathrm{H}^r(\mathrm{X}, \mathrm{F}) \to \mathrm{Ext}_{\mathrm{X}}^{3-r}(\mathrm{F}, \mathbf{G}_m)^{\sim}$$

is an isomorphism for all r and a given sheaf $\mathrm{F} = \mathrm{F}_1$, it is also an isomorphism for all r and any sheaf $\mathrm{F} = \mathrm{F}_2$ which is punctually equivalent to F_1.

Note that any sheaf F is punctually equivalent to a sheaf of the form $j_! \mathrm{E}$ where $j : \mathrm{U} \to \mathrm{X}$ is an open immersion of a nonempty scheme U, and E is *locally constant* on U. One may even suppose that the residual characteristics of the closed points of U do not divide the order e of the underlying abelian group E. Fixing such an F, let $\pi : \mathrm{Y} \to \mathrm{X}$ be a finite morphism with these properties :

(a) A primitive e th root of unity is rational over Y.

(b) $\pi^{-1}\mathrm{U} = \overline{\mathrm{U}} \xrightarrow{\pi} \mathrm{U}$ is a finite étale morphism and $\mathrm{C} = \pi^* \mathrm{E}$ is a *constant* sheaf on $\overline{\mathrm{U}}$.

One checks easily that if $\mathrm{F} = j_! \mathrm{E}$ as above, the natural map $\mathrm{F} \to \pi_* \mathrm{C}$ is an injection of sheaves ([4]). Form the short exact sequence

$$0 \to \mathrm{F} \to \pi_* \mathrm{C} \to \mathrm{G} \to 0$$

which gives rise to the following diagram which is commutative up to sign :

$$(\dagger)\quad \begin{array}{ccccccccc}
\mathrm{H}^{r_0-1}(\pi_* \mathrm{C}) & \longrightarrow & \mathrm{H}^{r_0-1}(\mathrm{G}) & \longrightarrow & \mathrm{H}^{r_0}(\mathrm{F}) & \longrightarrow & \mathrm{H}^{r_0}(\pi_* \mathrm{C}) & \longrightarrow & \mathrm{H}^{r_0}(\mathrm{G}) \\
\downarrow{\scriptstyle\approx} & & \downarrow{\scriptstyle\approx} & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\approx} & & \downarrow{\scriptstyle\beta} \\
\mathrm{Ext}^{4-r_0}(\pi_* \mathrm{C}, \mathbf{G}_m)^{\sim} & \to & \mathrm{Ext}^{4-r_0}(\mathrm{G}, \mathbf{G}_m)^{\sim} & \to & \mathrm{Ext}^{3-r_0}(\mathrm{F}, \mathbf{G}_m)^{\sim} & \to & \mathrm{Ext}^{3-r_0}(\pi_* \mathrm{C}, \mathbf{G}_m)^{\sim} & \to & \mathrm{Ext}^{3-r_0}(\mathrm{G}, \mathbf{G}_m)
\end{array}$$

where all the vertical morphisms are m^r's. The first two vertical morphisms are isomorphisms by the hypothesis of proposition A. To show that the fourth is an isomorphism, we use (2.7) remark (g). By that remark, we are reduced to showing that

$$m^{r_0}(\mathrm{C}): \quad \mathrm{H}^{r_0}(\mathrm{Y}, \mathrm{C}) \to \mathrm{Ext}_{\mathrm{Y}}^{3-r_0}(\mathrm{C}, \mathbf{G}_{m,\mathrm{Y}})^{\sim}$$

is an isomorphism. But since C is a constant sheaf one is reduced to showing that $m^{r_0}(\mathbf{Z}/n)$ is an isomorphism for all primes n dividing e which is again the case, thanks to the hypotheses of proposition A.

([4]) Compare with the general result : proposition 2.4 (ii), Exposé IX SGA 4.

We now consider the two remaining vertical morphisms of diagram (†). They are labelled α and β. A straightforward diagram chase establishes that α is injective.

That is, $m^{r_0}(\mathrm{F})$ is injective for any of the sheaves F of the form that we are considering (i. e. $\mathrm{F} = j_! \mathrm{E}$ where E is locally constant on U). Since *any* sheaf F′ is elementarily punctually equivalent to such an F :

$$0 \to \mathrm{F} \to \mathrm{F}' \to \mathrm{F}'' \to 0$$

and since $m^r(\mathrm{F}'')$ is an isomorphism for all r, another diagram chase in

$$\begin{tikzcd}
\mathrm{H}^{r_0-1}(\mathrm{F}') \arrow[r] \arrow[d, "\approx"'] & \mathrm{H}^{r_0-1}(\mathrm{F}'') \arrow[r] \arrow[d, "\approx"'] & \mathrm{H}^{r_0}(\mathrm{F}) \arrow[r] \arrow[d, "\subseteq"'] & \mathrm{H}^{r_0}(\mathrm{F}') \arrow[r] \arrow[d] & \mathrm{H}^{r_0}(\mathrm{F}'') \arrow[d, "\approx"'] \\
\mathrm{Ext}^{4-r_0}(\mathrm{F}', \mathbf{G}_m)^{\sim} \arrow[r] & \mathrm{Ext}^{4-r_0}(\mathrm{F}'', \mathbf{G}_m)^{\sim} \arrow[r] & \mathrm{Ext}^{3-r_0}(\mathrm{F}, \mathbf{G}_m)^{\sim} \arrow[r] & \mathrm{Ext}^{3-r_0}(\mathrm{F}', \mathbf{G}_m)^{\sim} \arrow[r] & \mathrm{Ext}^{3-r_0}(\mathrm{F}'', \mathbf{G}_m)^{\sim}
\end{tikzcd}$$

establishes that $m^{r_0}(\mathrm{F}')$ is injective for *any* F′. In particular the morphism β in diagram (†) is injective. One now returns to diagrams (†) with this extra information, and a final diagram chase establishes that α is also surjective.

Q. E. D.

Another general push-pull technique which is something of a companion to proposition A is the following :

Proposition B. — *Suppose* $m^r(\mathrm{F})$ *is an isomorphism for all* $r < r_0$ *and all constructible sheaves* F. *Then* $m^{r_0}(\mathrm{F})$ *is injective for all constructible sheaves* F.

To prove this, one makes use of effaceability of $\mathrm{H}^r(\mathrm{X}, -)$ in the category of constructible sheaves. Namely, given any $x \in \mathrm{H}^r(\mathrm{X}, \mathrm{F})$ for G constructible, there is an exact sequence

$$0 \to \mathrm{F} \to \mathrm{F}' \to \mathrm{K} \to 0$$

of *constructible* sheaves such that x goes to 0 in $\mathrm{H}^r(\mathrm{X}, \mathrm{F}')$ ([5]).

The strenght of propositions A and B is the following : They can be used to help us prove that $m^r(\mathrm{F})$ is an isomorphism by ascending induction on r, where the additional input at each stage is surjectivity of $m^r(\mathbf{Z}/n)$. Of course, to begin the inductive process we must have that $m^r(\mathrm{F})$ is an isomorphism if r is sufficiently negative. That is, we need finite dimensionality of Ext. In the proposition below we state something stronger than we need.

([5]) Any torsion sheaf of abelian groups is a filtered direct limit of constructible sheaves (SGA 4, Exposé IX).

PROPOSITION C. — *Let* $U \subset X$ *be an open subscheme and* F *a constructible sheaf on* U. *Then* $H^r(U, F)$ *and* $\operatorname{Ext}_U^r(F, \mathbf{G}_m)$ *vanish for* $r > 3$.

Proof. — Consider cohomology first. If $U' \subset U$ is a nonempty open with $V = U - U'$ we have

$$H_V^r(U, F) \to H^r(U, F) \to H^r(U', F) \to H_V^{r+1}(U, F).$$

By the local duality theorem we may suppose the two end groups vanish if $r > 3$. Thus for $r > 3$:

$$H^r(U, F) \underset{\approx}{\to} H^r(U', F)$$

for any nonempty open $U' \subset U$. Consequently, passing to the limit

$$H^r(U, F) \simeq H^r(\operatorname{Gal}(\overline{K}/K), F(\overline{K})) = 0 \qquad \text{for} \quad r > 3,$$

because $\operatorname{Gal}(\overline{K}/K)$ has cohomological dimension 2.

Now consider Ext. Note that $\operatorname{Ext}_U^r(F, \mathbf{G}_m) = 0$ for all nonempty opens $U \subset X$, all *punctual* constructible sheaves F on U and all $r > 3$, by the local duality theorem. It follows that for $r > 3$, the group $\operatorname{Ext}_U^r(F, \mathbf{G}_m)$ depends only on the punctual equivalence class of F. Moreover, since

$$\operatorname{Ext}_{U_1}^r(j_! F, \mathbf{G}_m) = \operatorname{Ext}_{U_2}^r(F, \mathbf{G}_m) \quad \text{for all } r,$$

where $j : U_2 \to U_1$ is an inclusion of nonempty opens in X, it follows that

$$\operatorname{Ext}_{U_1}^r(F, \mathbf{G}_m) \approx \operatorname{Ext}_{U_2}^r(F, \mathbf{G}_m) \quad \text{for all } r > 3,$$

and all constructible sheaves F on U_1.

This discussion implies that it is sufficient to prove

$$\operatorname{Ext}_U^r(F, \mathbf{G}_m) = 0 \qquad \text{for} \quad r > 3,$$

and all locally constant sheaves F such that the residual characteristics of U do not divide the order of the underlying abelian group of F. In this situation, the sheaves of local exts vanish :

$$(\star) \qquad \underline{\operatorname{Ext}}_U^r(F, \mathbf{G}_m) = 0 \qquad \text{for} \quad r > 0.$$

[*Proof.* — Since F is locally constant, and we are dealing with *local* ext's one is reduced to considering the special case where F is constant. After consideration of a Jordan-Hölder sequence of F one is reduced to the case $F = \mathbf{Z}/p$ where p is a prime. Now write the long exact sequence obtained from the functor $\underline{\operatorname{Ext}}_U^r(-, \mathbf{G}_m)$ applied to the exact sequence

$$0 \to \mathbf{Z} \xrightarrow{p} \mathbf{Z} \to \mathbf{Z}/p \to 0.$$

Since

$$\underline{\mathrm{Ext}}^r_{\mathrm{U}}(\mathbf{Z}, \mathbf{G}_m) = 0 \quad \text{for} \quad r > 0,$$

and since

$$\begin{array}{ccc} \underline{\mathrm{Hom}}_{\mathrm{U}}(\mathbf{Z}, \mathbf{G}_m) & \xrightarrow{p} & \underline{\mathrm{Hom}}_{\mathrm{U}}(\mathbf{Z}, \mathbf{G}_m) \\ \downarrow \approx & & \downarrow \approx \\ \mathbf{G}_m & \xrightarrow[p]{} & \mathbf{G}_m \end{array}$$

is a *surjective map* of étale sheaves (p is not equal to any residual characteristic of U) one obtains that $\underline{\mathrm{Ext}}^r_{\mathrm{U}}(\mathbf{Z}/p, \mathbf{G}_m)$ vanishes for $r > 0$.]

Using the spectral sequence

$$\mathrm{H}^r(\mathrm{U}, \underline{\mathrm{Ext}}^s_{\mathrm{U}}(\mathrm{F}, \mathbf{G}_m)) \Rightarrow \mathrm{Ext}^{r+s}_{\mathrm{U}}(\mathrm{F}, \mathbf{G}_m)$$

and ($\star$) above, we conclude the proof of proposition C.

Proposition D. — *Let* $\mathrm{U} \subset \mathrm{X}$ *be an open subscheme and* F *a constructible sheaf on* U. *Then the groups* $\mathrm{H}^r(\mathrm{U}, \mathrm{F})$ *and* $\mathrm{Ext}^r_{\mathrm{U}}(\mathrm{F}, \mathbf{G}_m)$ *are* finite *abelian groups.*

Proof. — Again, since the above proposition is true for punctual constructible sheaves F we are at liberty to retract or expand U, and replace F by any sheaf punctually equivalent to it. Thus, as in the proof of Proposition C, we may reduce finiteness of $\mathrm{Ext}^r_{\mathrm{U}}(\mathrm{F}, \mathbf{G}_m)$ to finiteness of H^r.

We concentrate on showing that $\mathrm{H}^r(\mathrm{U}, \mathrm{F})$ is finite, where F is locally constant. By considering the kernel of multiplication by p for p an arbitrary prime, we may reduce ourselves to the case where p annihilates F. That is, F is a vector space over $\mathbf{F}_p$. Let G be a finite quotient group of $\mathrm{Gal}(\overline{\mathrm{K}}/\mathrm{K})$ through which $\mathrm{Gal}(\overline{\mathrm{K}}/\mathrm{K})$ acts on $\mathrm{F}(\overline{\mathrm{K}})$. Let L/K be the galois extension " classified " by G. That is : L is the fixed field of the kernel of $\mathrm{Gal}(\overline{\mathrm{K}}/\mathrm{K}) \to \mathrm{G}$. Let $\mathrm{G}_p \subset \mathrm{G}$ be a p-Sylow subgroup and $\mathrm{K}' \subset \mathrm{L}$ the fixed field of G_p. Let $\mathrm{U}' \xrightarrow{\pi} \mathrm{U}$ be *the* finite (étale) extension such that

$$\mathrm{U}' \times_{\mathrm{U}} \mathrm{Spec}(\mathrm{K}) = \mathrm{Spec}(\mathrm{K}').$$

Then the degree of π is $[\mathrm{G} : \mathrm{G}_p]$ which is prime to p. Since the composition

$$\mathrm{H}^r(\mathrm{U}, \mathrm{F}) \to \mathrm{H}^r(\mathrm{U}', \pi^* \mathrm{F}) = \mathrm{H}^r(\mathrm{U}, \pi_* \pi^* \mathrm{F}) \xrightarrow{\text{norm}} \mathrm{H}^r(\mathrm{U}, \mathrm{F})$$

is multiplication by $[\mathrm{G} : \mathrm{G}_p]$ the first arrow is an injection and we are reduced to showing that $\mathrm{H}^r(\mathrm{U}', \pi^* \mathrm{F})$ is finite.

The above discussion allows us, then, to assume that F is an $\mathbf{F}_p$-vector space, and as a locally free sheaf it is split by a finite étale Galois extension with Galois group a p-group. By the well known result concerning the action of p-groups on $\mathbf{F}_p$-vector spaces it follows that F must contain the constant group $\mathbf{Z}/p$ as subsheaf. Working by *induction* on the dimension of the $\mathbf{F}_p$-vector space F, one is reduced to the case $F = \mathbf{Z}/p$. After another possible base change of U we may suppose that U contains a p th root of 1, and therefore $\mathbf{Z}/p$ is isomorphic to the sheaf $\boldsymbol{\mu}_p$. We are thus reduced to showing that $H^r(U, \boldsymbol{\mu}_p)$ is finite. But, as in the argument of proposition C, since p is not a residual characteristic of U,

$$H^r(U, \boldsymbol{\mu}_p) = \mathrm{Ext}^r_U(\mathbf{Z}/p, \mathbf{G}_m).$$

Now recall the very beginning discussion of the proof of this proposition and notice that we are reduced to showing that $\mathrm{Ext}^r_X(\mathbf{Z}/p, \mathbf{G}_m)$ is finite for all r. But we have, in fact, calculated these groups [remark (c) of (2.4)] and they are indeed finite.

Q. E. D.

We summarize the work done in this section by the following corollary :

COROLLARY :

(a) $m^r(F)$ *is an isomorphism for* $r < 0$, *and all constructible sheaves over any* X.

(b) *Suppose that* $m^r(F)$ *is an isomorphism for all constructible sheaves* F *over any* X, *for all* r *less than a fixed integer* r_0. *Suppose further :*

(ε_r) *For all primes* p *and bases* X *over which a primitive* p th *root of unity is rational,*

$$\# H^{r_0}(X, \mathbf{Z}/p) \geq \# \mathrm{Ext}^{3-r_0}_X(\mathbf{Z}/p, \mathbf{G}_m).$$

Then $m^r(F)$ *is an isomorphism for all constructible sheaves over any* X *and all* $r \leq r_0$; $m^{r_0+1}(F)$ *is an injection.*

4. Proof of the duality theorem : conclusion

STEP 1 : $r_0 = 0$.

Since $\# H^0(X, \mathbf{Z}/p) = p$, and by the table of (2.4) remark (c) $\# \mathrm{Ext}^3_X(\mathbf{Z}/p, \mathbf{G}_m) = p$, we have the condition (ε_0) of the corollary of paragraph 3.

STEP 2 : $r_0 = 1$.

Since $H^1(X, \mathbf{Z}/p)$ classifies cyclic unramified p-coverings,

$$H^1(X, \mathbf{Z}/p) = \mathrm{Hom}(\mathrm{Pic}\, X, \mathbf{Z}/p)$$

and since $\mathrm{Ext}^2_X(\mathbf{Z}/p, \mathbf{G}_m) = \mathrm{Pic}\, X/p$ (2.4) remark (c) we have the condition (ε_1) of the corollary of paragraph 3.

Thus $m^r(\mathrm{F})$ is an isomorphism for $r \leqq 1$ and an injection for $r = 2$.

STEP 3 — A study of the sheaf $\mathbf{Z}/n$.

Using the exact sequence of sheaves over X :

$$0 \to \mathbf{Z} \xrightarrow{n} \mathbf{Z} \to \mathbf{Z}/n \to 0,$$

we obtain

$$(4.0) \qquad \begin{cases} \underline{\mathrm{Ext}}^1_{\mathrm{X}}(\mathbf{Z}/n, \mathbf{G}_m) = \mathbf{G}_m/(\mathbf{G}_m)^n, \\ \underline{\mathrm{Ext}}^q_{\mathrm{X}}(\mathbf{Z}/n, \mathbf{G}_m) = 0 \qquad \text{if} \quad q > 1. \end{cases}$$

The sheaf $\mathrm{E} = \mathbf{G}_m/(\mathbf{G}_m)^n$ is 0 outside the points of X whose residue characteristic divides n. Thus E is punctual, and may be written $\mathrm{E} = \bigoplus \mathrm{E}_p$, where E_p is the sheaf concentrated at the point $p \in \mathrm{X}$ whose pullback to p is the Galois module $\mathrm{U}_{p,0}/\mathrm{U}^n_{p,0}$. Here we are using the terminology of paragraph 1 diagram 1 : $\mathrm{K} = \mathrm{K}_p$ is the function field of X endowed with the discrete valuation v coming from the point p; $\mathrm{U}_{p,0}$ is the group of units in the ring of integers of $\mathrm{K}_{p,0}$. If we pass to the *completion* K_v of K with respect to v, one has :

$$\mathrm{U}_{v,0}/\mathrm{U}^n_{v,0} \cong \mathrm{U}_{p,0}/\mathrm{U}^n_{p,0}$$

as $\hat{\mathbf{Z}}$-modules, and therefore our analysis of the sheaf E_p becomes a study of the *complete* discrete valued field K_v.

We procede by considering the exact sequence of $\hat{\mathbf{Z}}$-modules :

$$(4.1) \qquad 0 \to \mathbf{Z}/n \to \mathrm{K}^*_{v,0} \xrightarrow{n} \mathrm{K}^*_{v,0} \to \mathrm{K}^*_{v,0}/\mathrm{K}^{*n}_{v,0} \to 0,$$

where the inclusion of $\mathbf{Z}/n$ in $\mathrm{K}^*_{v,0}$ is by sending 1 to ζ, a *chosen* primitive nth root of 1, which lives in $\mathrm{K}_{v,0}$, by assumption.

From (4.1) we deduce the exact sequence :

$$(4.2) \qquad 0 \to \mathrm{H}^1(\mathbf{Z}, \mathbf{Z}/n) \to \mathrm{K}^*_v/\mathrm{K}^{*n}_v \to \mathrm{H}^0(\mathbf{Z}, \mathrm{K}^*_{v,0}/\mathrm{K}^{*n}_{v,0}) \to 0.$$

Moreover, using the fact that $\mathrm{H}^q(\hat{\mathbf{Z}}, \mathrm{U}_{v,0})$ vanishes for $q \geqq 1$ (CL XII 3, p. 193 bot.) one checks :

$$(4.3) \qquad \mathrm{H}^q(\hat{\mathbf{Z}}, \mathrm{K}^*_{v,0}/\mathrm{K}^{*n}_{v,0}) \xrightarrow[\approx]{\text{val mod } n} \mathrm{H}^q(\hat{\mathbf{Z}}, \mathbf{Z}/n)$$

is an isomorphism for $q \geqq 1$.

In (4.2) $\mathrm{H}^1(\hat{\mathbf{Z}}, \mathbf{Z}/n) \subset \mathrm{K}^*_v/\mathrm{K}^{*n}_v$ may be identified with the one-dimensional vector space over $\mathbf{Z}/n$ consisting of those $w \in \mathrm{K}^*_v/\mathrm{K}^{*n}_v$ which go to 0 in $K^*_{v,0}/K^{*n}_{v,0}$. We also have the diagram

$$\begin{array}{ccccccc}
 & & & & 0 & & 0 \\
 & & & & \uparrow & & \uparrow \\
 & & & & \mathbf{Z}/n & \longrightarrow & \mathbf{Z}/n \\
 & & & & \uparrow & & \uparrow \\
0 \rightarrow & H^1(\hat{\mathbf{Z}}, \mathbf{Z}/n) & \rightarrow & & K^*_v/K^{*n}_v & \rightarrow & H^0(\hat{\mathbf{Z}}, K^*_{v,0}/K^{*n}_{v,0}) \rightarrow 0 \\
 & & & & \uparrow & & \uparrow \\
 & & & & U_v/U^n_v & \longrightarrow & H^0(\hat{\mathbf{Z}}, U_{v,0}/U^n_{v,0}) \rightarrow 0 \\
 & & & & \uparrow & & \uparrow \\
 & & & & 0 & & 0
\end{array}$$

Now consider the Hilbert symbol pairing

$$K^*_v/K^{*n}_v \times K^*_v/K^{*n}_v \rightarrow \mathbf{Z}/n$$

and note that the subspaces $H^1(\hat{\mathbf{Z}}, \mathbf{Z}/n)$ and U_v/U^n_v of K^*_v/K^{*n}_v are the annihilators of one another. [This is so because, if u, w are units such that $u^{1/n}$ determines an unramified extension of K_p, then $(u, w)_p = 0$. Now use that the pairing is non degenerate, and count dimensions to get that they are *exact* annihilators.]

After this discussion, it is apparent that the Hilbert symbol determines a non degenerate pairing of $H^0(\hat{\mathbf{Z}}, U_{v,0}/U^n_{v,0})$ with itself, into $\mathbf{Z}/n \subset \mathbf{Q}/\mathbf{Z}$. Passing to $E = \bigoplus E_p$, these pairings give a non degenerate pairing

$$H^0(X, E) \times H^0(X, E) \rightarrow \mathbf{Q}/\mathbf{Z}.$$

Furthermore, (4.3) shows that $H^q(X, E) = 0$ for $q \geqq 1$. This allows us to use the spectral sequence

$$H^p(X, \underline{\mathrm{Ext}}^q(\mathbf{Z}/n, \mathbf{G}_m)) \Rightarrow \mathrm{Ext}^{p+q}_X(\mathbf{Z}/n, \mathbf{G}_m)$$

to obtain the following exact sequence

(4.4) $$\begin{aligned} 0 &\rightarrow H^1(X, \boldsymbol{\mu}_n) \rightarrow \mathrm{Ext}^1_X(\mathbf{Z}/n, \mathbf{G}_m) \\ &\rightarrow H^0(X, E) \rightarrow H^2(X, \boldsymbol{\mu}_n) \rightarrow \mathrm{Ext}^2_X(\mathbf{Z}/n, \mathbf{G}_m) \rightarrow 0, \end{aligned}$$

and the isomorphisms

(4.5) $$H^q(X, \boldsymbol{\mu}_n) \xrightarrow{\sim} \mathrm{Ext}^q_X(\mathbf{Z}/n, \mathbf{G}_m) \qquad \text{for} \quad q \geqq 3.$$

If we use our primitive root of unity ζ to identify $\mathbf{Z}/n$ with $\boldsymbol{\mu}_n$, we obtain a diagram which is commutative up to sign :

(4.6) $$\begin{array}{ccccccccccc}
0 \rightarrow & H^1(X, \boldsymbol{\mu}_n) & \rightarrow & \mathrm{Ext}^1_X(\mathbf{Z}/n, \mathbf{G}_m) & \rightarrow & H^0(X, E) & \rightarrow & H^2(X, \boldsymbol{\mu}_n) & \rightarrow & \mathrm{Ext}^2_X(\mathbf{Z}/n, \mathbf{G}_m) & \rightarrow 0 \\
 & {\scriptstyle m^1}\downarrow\approx & & {\scriptstyle \tilde{m}^2}\downarrow & & \downarrow\simeq & & {\scriptstyle m^2}\downarrow & & {\scriptstyle \tilde{m}^1}\downarrow\approx & \\
0 \rightarrow & \mathrm{Ext}^2_X(\mathbf{Z}/n, \mathbf{G}_m)^{\sim} & \rightarrow & H^2(X, \boldsymbol{\mu}_n)^{\sim} & \rightarrow & H^0(X, E)^{\sim} & \rightarrow & \mathrm{Ext}^1_X(\mathbf{Z}/n, \mathbf{G}_m)^{\sim} & \rightarrow & H^1(X, \boldsymbol{\mu}_n)^{\sim} & \rightarrow 0
\end{array}$$

where the middle isomorphism comes from our non degenerate pairing. The vertical arrows labelled $\tilde{m}^1$, $\tilde{m}^2$ are the Pontrjagin duals of m^1, m^2

and so we have the indicated isomorphisms, injections and surjections. One easily checks that this implies *all* vertical morphisms are isomorphisms. This establishes condition (ε_2). Condition (ε_3) comes from (4.5) together with the calculation of $\mathrm{Ext}^3_X(\mathbf{Z}/n, \mathbf{G}_m)$ given in (2.4) remark (*c*). This establishes the duality theorem.

Note. — This differs from Artin-Verdier's original proof of the theorem in that it uses the nondegeneracy of the local Hilbert symbol (i. e. local duality) in the proof of global duality. Artin and Verdier got around that by showing that, in a certain weak sense, $\mathrm{Ext}^1(\mathbf{Z}/n, \mathbf{G}_m)$ has a coeffaceability property which allows one to dualize the induction technique stemming from propositions A, B, C to obtain that m^2 is surjective, which also suffices to establish duality for the sheaf $\mathbf{Z}/n$, and hence for all constructible sheaves.

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