Lecture 10: Cup Product

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1 Cup Product

1.1 Introduction

We will define and construct the cup product pairing on Tate cohomology groups and describe some of its basic properties. The main references are §7 of Atiyah-Wall, §VIII.3 of Serre's *Local Fields*, Washington's paper *Galois Cohomology* (in Cornell-Silverman-Stevens), and §7 of Tate's *Galois Cohomology* (PCMI). The cusp product is absolutely central to Galois cohomology, in that many of the central theorems and constructions involve various types of *duality* results, which involve cup products at their core.

1.2 The Definition

Let G be a finite group.

Theorem 1.1. *There is a unique family of "cup product" homomorphisms*

$$\hat{\mathrm{H}}^p(G,A)\otimes\hat{\mathrm{H}}^q(G,B)\to\hat{\mathrm{H}}^{p+q}(G,A\otimes B)$$

$$a\otimes b\mapsto a\cup b,$$

for all $p,q\in\mathbb{Z}$ and G-modules A,B, such that:

1. *Cup product is functorial in A,B, e.g., if $A\to A'$, $B\to B'$ are G-module homomorphisms, then we have a commutative diagram (with vertical maps that I have not typeset below):*

$$\hat{\mathrm{H}}^p(G,A)\otimes\hat{\mathrm{H}}^q(G,B)\to\hat{\mathrm{H}}^{p+q}(G,A\otimes B)$$

$$\hat{\mathrm{H}}^p(G,A')\otimes\hat{\mathrm{H}}^q(G,B')\to\hat{\mathrm{H}}^{p+q}(G,A'\otimes B')$$

2. *When $p=q=0$ the cup product is induced by the natural map*

$$A^G\otimes B^G\to(A\otimes B)^G.$$

3. *A natural compatibility statement that allows for dimension shifting and ensure uniqueness (see Cassels-Frohlich or Serre for the exact statement).*

1.3 Existence

Let P_n be a complete resolution of G, e.g., P_n could be the standard resolution:

$$P_n = \begin{cases} \mathbb{Z}[G^{n+1}] & \text{if } n \geq 0, \\ \mathrm{Hom}(P_{|n|-1}, \mathbb{Z}) & \text{if } n < 0. \end{cases}$$

Recall that this fit together to form an exact sequence of free G-modules:

$$\cdots \xrightarrow{d} P_2 \xrightarrow{d} P_1 \xrightarrow{d} P_0 \xrightarrow{d} P_{-1} \xrightarrow{d} P_{-2} \xrightarrow{d} \cdots .$$

Moreover, we have

$$\hat{\mathrm{H}}^q(G, A) = \mathrm{H}^q(\mathrm{Hom}_G(P_*, A))$$

is the qth cohomology of the complex $\mathrm{Hom}_G(P_*, A)$. In particular

$$\hat{\mathrm{H}}^q(G, A) = \frac{\ker(\mathrm{Hom}_G(P_q, A) \to \mathrm{Hom}_G(P_{q+1}, A))}{\mathrm{im}(\mathrm{Hom}_G(P_{q-1}, A) \to \mathrm{Hom}_G(P_q, A))}.$$

To prove that the family of cup product morphisms exist, we will construct a G-module homomorphism from the complete resolution with certain properties.

Proposition 1.2. *There exist G-module homorphisms*

$$\varphi_{p,q} : P_{p+q} \to P_p \otimes Q_q$$

for all $p, q \in \mathbb{Z}$ such that

1. $\varphi_{p,q} \circ d = (d \otimes 1) \circ \varphi_{p+1,q} + (-1)^p (1 \otimes d) \circ \varphi_{p,q+1}$, *and*
2. $(\varepsilon \otimes \varepsilon) \circ \varphi_{0,0} = \varepsilon$, *where $\varepsilon : P_0 \to \mathbb{Z}$ is defined by $\varepsilon(g) = 1$ for all $g \in G$.*

Assume that the proposition has been proved. Then we define the cup product explicitly on the level of cochains as follows. Let

$$f \in \mathrm{Hom}_G(P_p, A), \qquad g \in \mathrm{Hom}_G(P_q, B)$$

be cochains (so elements of the kernel of d). Define the cochain

$$f \cup g \in \mathrm{Hom}_G(P_{p+q}, A \otimes B)$$

by

$$f \cup g = (f \otimes g) \circ \varphi_{p,q}.$$

Lemma 1.3. *We have*

$$f \cup g = (df) \cup g \, + \, (-1)^p f \cup (dg).$$

Corollary 1.4. *If f, g are cochains, then:*

1. *$f \cup g$ is a cochain*
2. *$f \cup g$ only depends on the classes of f and g.*

We conclude that we have a well-defined homomorphism

$$\hat{\mathrm{H}}^p(G, A) \otimes \hat{\mathrm{H}}^q(G, B) \to \hat{\mathrm{H}}^{p+q}(G, A \otimes B).$$

Proposition 1.5. *Condition 2 of Theorem 1.1 is satisfied.*

Proof. This uses from Proposition 1.2 that $(\varepsilon \otimes \varepsilon) \circ \varphi_{0,0} = \varepsilon$. □

It remains to construct the maps $\varphi_{p,q}$. These maps are constructed in a natural way in terms of the standard complete resolution P_n mentioned above, as follows. First note that if $q \geq 1$, then $P_{-q} = P_{q-1}^* = \mathrm{Hom}(P_{q-1}, \mathbb{Z})$ has a $\mathbb{Z}$-module basis consisting of all $(g_1^*, \ldots, g_q^*)$, where $(g_1^*, \ldots, g_q^*)$ maps $(g_1, \ldots, g_q) \in P_{q-1}$ to $1 \in \mathbb{Z}$, and every other basis element of P_{q-1} to 0. The map $d : P_{-q} \to P_{-q-1}$ is then

$$d(g_1^*, \ldots, g_q^*) = \sum_{s \in G} \sum_{i=0}^{q} (-1)^i (g_1^*, \ldots, g_i^*, s^*, g_{i+1}^*, \ldots g_q^*).$$

and $d : P_0 \to P_{-1}$ is given by $d(g_0) = \sum_{s \in G} s^*$.

If $p \geq 0$ and $q \geq 0$, then

$$\varphi_{p,q}(g_0, \ldots g_{p+q}) = (g_0, \ldots, g_p) \otimes (g_p, \ldots g_{p+q}),$$

and if $p, q \geq 1$ then

$$\varphi_{-p,-q}(g_1^*, \ldots, g_{p+q}^*) = (g_1^*, \ldots g_p^*) \otimes (g_{p+1}^*, \ldots, g_{p+q}^*),$$

and similar definitions in other cases, when one of p, q is positive and the other is negative. (Again, see Cassels-Frohlich for more details.) The moral of all this is that one can construct the cup product by simply following your nose.

1.4 Properties

Proposition 1.6. *The cup product has these properties:*

1. $(a \cup b) \cup c = a \cup (b \cup c)$
2. $\mathrm{res}(a \cup b) = \mathrm{res}(a) \cup \mathrm{res}(b)$
3. $\mathrm{cores}(a \cup \mathrm{res}(b)) = \mathrm{cores}(a) \cup b$.

The above properties are proved by proving them when $p = q = 0$, then using dimension shifting.

Finally, notice that if $A \otimes B \to C$ is a G-homomorphism, then cup product induces

$$\hat{\mathrm{H}}^p(G, A) \otimes \hat{\mathrm{H}}^q(G, B) \to \hat{\mathrm{H}}^{p+q}(G, C).$$

See Tate's paper *Galois Cohomology* for an explicit description of the cup product for $p, q \geq 0$ on cocycles, which would make computation of the cup product of classes represented by cocycles explicit.

2 Cohomology of a Cyclic Group

Suppose that $G = \langle s \rangle$ is a finite cyclic group. In this section we give a quick summary of the basic facts about $\hat{\mathrm{H}}^q(G, A)$.

Let $K_i = \mathbb{Z}[G]$ and define maps $d : K_{i+1} \to K_i$ by multiplication by $s - 1$ if i is even and multiplication by $N = \sum_{t \in G} t$ if i is odd. Then

$$\cdots \xrightarrow{d} K_i \xrightarrow{d} K_{i-1} \xrightarrow{d} K_{i-2} \xrightarrow{d} \cdots$$

is a complete resolution of G, since

$$\ker(T) = \mathbb{Z}[G]^G = N(\mathbb{Z}[G]) = \text{image}(N),$$

and since $\hat{\mathrm{H}}^0(G, \mathbb{Z}[G]) = 0$,

$$\ker(N) = I_G\mathbb{Z}[G] = \text{image}(T).$$

Then $\mathrm{Hom}_G(K_\bullet, A)$ is

$$\cdots \leftarrow A \xleftarrow{N} A \xleftarrow{T} A \xleftarrow{N} \cdots .$$

Proposition 2.1. *For every integer q we have*

$$\hat{\mathrm{H}}^{2q}(G, A) \cong \hat{\mathrm{H}}^0(G, A) = A^G/N(A)$$

and

$$\hat{\mathrm{H}}^{2q+1}(G, A) \cong \hat{\mathrm{H}}^{-1}(G, A) = \ker(N_A)/I_G(A).$$

If $n = \#G$, then we have

$$\hat{\mathrm{H}}^2(G, \mathbb{Z}) \cong \hat{\mathrm{H}}^0(G, \mathbb{Z}) \cong \mathbb{Z}^G/N(\mathbb{Z}) = \mathbb{Z}/n\mathbb{Z}.$$

Theorem 2.2. *Cup product by a generator of $\hat{\mathrm{H}}^2(G, \mathbb{Z})$ induces an isomorphism*

$$\hat{\mathrm{H}}^q(G, A) \xrightarrow{\cong} \hat{\mathrm{H}}^{q+2}(G, A)$$

for all $q \in \mathbb{Z}$ and all G-modules A.

For the proof, see Cassels-Frohlich, Section 8.

NEXT: We will finally start a systematic study of *Galois Cohomology...* so finally have lots of examples! This will begin on Friday of next week.