Lecture 9: Complete Resolution, Res and Cores.

§1. Prelude

Let G be a finite group, and A a G-module. Then:

(1) $\hat{H}^q(G,A)$ is killed by $\#G$ for all q, (nice)

(2) A finitely generated $\Rightarrow$ $\hat{H}^q(G,A)$ finite for all q (good to know)

(3) If $S \subseteq G$ is a Sylow p-subgroup, then (very useful.)

$$\hat{H}^q(G,A)(p) \xhookrightarrow{\text{res}} \hat{H}^q(S,A) \quad \text{for all } q$$

means p-primary subgroup; injective

How on earth can the above facts be proved?

So far we do not have the tools. Soon we will!

§2. Complete Resolution of G:

Recall: G finite group

$$\hat{H}^q(G,A) = \begin{cases} H^q(G,A) & \text{for } q \geq 1 \\ A^G/N(A) & \text{for } q=0 \\ \mathrm{Ker}(N)/I_G A & \text{for } q=-1 \\ H_{|q|-1} & \text{for } q \leq -2 \end{cases}$$

$I_G = (s-1 : s \in G) \subseteq \mathbb{Z}[G]$ augmentation ideal.

$f : P_i \to \mathbb{Z}$

$f(g^{-1}x)$

$P_\bullet$ free $\mathbb{Z}[G]$-resolution of $\mathbb{Z}$:

Dualize $P_i^* = \mathrm{Hom}_{\mathbb{Z}}(P_i, \mathbb{Z})$ ← not over $\mathbb{Z}[G]$.

$$\cdots \to P_2 \to P_1 \to P_0 \xrightarrow{\varepsilon} \mathbb{Z} \to 0$$

$$\|$$

$$0 \to \mathbb{Z} \xrightarrow{\varepsilon^*} P_0^* \to P_1^* \to P_2^* \to \cdots$$

exact since each P_i is free.

Splice: $\boxed{P_{-n} = P_{n-1}^*:}$

$$\cdots \to P_2 \to P_1 \to P_0 \to P_{-1} \to P_{-2} \to \cdots$$ ← a "complete resolution".

Prop: $\hat{H}^q(G,A) =$ q-th cohomology of the complex

$$\cdots \leftarrow \mathrm{Hom}(P_1, A) \leftarrow \mathrm{Hom}(P_0, A) \leftarrow \mathrm{Hom}(P_{-1}, A) \leftarrow \mathrm{Hom}(P_{-2}, A) \leftarrow \cdots$$

Proof:

$q \geq 1$: By definition.

$q \leq -2$: By defn, $H_n(G,A) =$ nth homology of the complex $P_\bullet \otimes_{\mathbb{Z}[G]} A$

e.g. standard complex, though any proj. resolution of $\mathbb{Z}$ will work.

But for any f.g. free $\mathbb{Z}[G]$-module B,

$B^* = \mathrm{Hom}(B, \mathbb{Z})$

$$B \otimes A \xrightarrow{\cong} \mathrm{Hom}(B^*, A)$$

$$b \otimes a \longmapsto (f \mapsto f(c).a)$$

isom. of $\mathbb{Z}[G]$-modules.

So get isom:

$$\tau: B \otimes_{\mathbb{Z}[G]} A = (B \otimes A)/I_G(B \otimes A) \xrightarrow[\cong]{N^*} (B \otimes A)^G \longrightarrow \mathrm{Hom}(B^*, A)^G = \mathrm{Hom}_G(B^*, A)$$

$(s-1)(b \otimes a)$

$= s.b \otimes sa - b \otimes a$

$b \otimes sa$

$\parallel$

$b.s \otimes a$

$\parallel$

$s^{-1}b \otimes a$

$b \mapsto sb$

$sb \otimes sa$

$\parallel$

$b \otimes a$

N^* iso, since $B \otimes_{\mathbb{Z}} A$ is induced, since B is free, and tensor over $\mathbb{Z}$.

So: $\mathrm{Hom}_G(P_{-n}, A) = \mathrm{Hom}_G(\overline{\mathrm{Hom}(P_{n-1}, \mathbb{Z})}, A) \cong P_{n-1} \otimes_{\mathbb{Z}[G]} A$

so computing cohomology of same complex.

$q = 0, 1$: see book (Cassels–Frohlich, pg 103)

$*$ induced $= \mathbb{Z}[G] \otimes_{\mathbb{Z}} A$

Recall: $0 \to A' \to A_* \to A \to 0$

"Dimension shifting" $\hat{H}^q(G,A) \overset{\delta}{\cong} \hat{H}^{q+1}(G,A')$ for all q.

Ex: $H \subseteq G$ gives $\hat{H}^q(G,A') \xrightarrow{\text{res}} \hat{H}^q(H,A')$ for $q \geq 1$

$$\begin{tikzcd} \hat{H}^q(G,A') \arrow[r, "\text{res}"] \arrow[d, phantom, "\wr\|"] & \hat{H}^q(H,A') \arrow[d, phantom, "\wr\|"] \\ \hat{H}^{q-1}(G,A) \arrow[r, dashed, "\text{res}"] & \hat{H}^{q-1}(H,A) \end{tikzcd}$$

so get res for all q, defined by dim. shifting.

So get res for all q.

Cores: For homology, morphism of pairs

$$\begin{tikzcd} G' \arrow[r] \arrow[d, dash, dotted] & G \arrow[d, dash, dotted] \\ A' & A \arrow[l] \end{tikzcd}$$

induces $H_q(G',A') \to H_q(G,A)$

$\cdots \to P'_{i+1} \to P'_i \to \cdots$
$\downarrow \quad \downarrow$
$\cdots \to P_{i+1} \to P_i \to \cdots$

} tensor with A' & A to compute homology

$\cdots \to P'_{i+1} \otimes_G A' \to P'_i \otimes_G A' \to \cdots$
$\Downarrow \quad \Downarrow$
$\cdots \to P_{i+1} \otimes_G A \to P_i \otimes_G A \to \cdots$

So for $H \hookrightarrow G$ get

$$H_q(H,A) \longrightarrow H_q(G,A) \quad \text{all } q \leq 0.$$

So: $0 \to A \to A^* \to A' \to 0$

$$\begin{tikzcd} \hat{H}^q(G,A) \arrow[r, phantom, "\cong"] & \hat{H}^{q-1}(G,A') \\ \hat{H}^q(H,A) \arrow[u, dashed, "\text{cores}"] \arrow[r, phantom, "\cong"] & \hat{H}^{q-1}(H,A') \arrow[u, "\text{cores}"] \end{tikzcd}$$

all q

Prop:
- $\hat{H}^0(G,A) \xrightarrow{\text{res}} \hat{H}^0(H,A)$ is given by $A^G/N(A) \longrightarrow A^H/N(A)$ induced by inclusion $A^G \hookrightarrow A^H$.
- $\hat{H}^{-1}(G,A) \xrightarrow{\text{res}} \hat{H}^{-1}(H,A)$ induced by $A/I_G(A) \xrightarrow{N'_{G/H}} A/I_H(A)$
  $N'_{G/H}(a) = \sum s_i^{-1} a$ where $G/H = \cup s_i H$.
- $\hat{H}^0(H,A) \xrightarrow{\text{cores}} \hat{H}^0(G,A)$ induced by $A^H/N_H(A) \longrightarrow A^G/N_G(A)$
  $N_{G/H}(a) = \sum s_i a$

Proof: See [CF]

Prop: $\mathrm{cor}_{G/H} \circ \mathrm{res}_H : \hat{H}^q(G,A) \longrightarrow \hat{H}^q(G,A)$ for all q

$\mathrm{cor} \circ \mathrm{res} =$ mult. by $[G:H]$.

Proof: Check for $q=0$. General statement follows by dimension shifting.

$$\hat{H}^0(G,A) = A^G / N(A)$$

res = induced by $A^G \subseteq A^H$

$$\hat{H}^0(H,A) = A^H / N(A)$$

cor = induced by $N_{G/H}(a) = \sum s_i a$

coset reps.

trivial action.

For $a \in A^G$, $\mathrm{cor}(\mathrm{res}(a)) = \left[\sum s_i a\right] = n \cdot a$

where $n = [G:H]$

Corollaries:

(1) $\hat{H}^q(G,A)$ killed by $\#G$ for all q: **Proof.** Take $H = \{1\} \leq G$ above and note that $\hat{H}^q(\{1\}, A) = 0$ for all q.

(2) A f.g. $\Rightarrow \hat{H}^q(G,A)$ finite for all q:

Proof: Calculation of $\hat{H}^q(G,A)$ using standard resolution $\Rightarrow$ they are finitely generated.

But they are killed by $\#G$. So finite.

(3) $S \leq G$ p-Sylow subgroup (so $\#S = p^n \| \#G$). Then $\hat{H}^q(G,A)(p) \hookrightarrow \hat{H}^q(S,A)$; for all q.

p-primary part

Proof:

$$x \in \hat{H}^q(G,A) \xrightarrow{\mathrm{res}} \hat{H}^q(S,A), \quad \#G = p^n \cdot m$$

$\#S$

with order$(x) = p$-power, and $\mathrm{res}(x) = 0$. Then $(\mathrm{cor} \circ \mathrm{res})(x) = 0 = m \cdot x \Rightarrow x = 0$. ∎