Long-ish Exact Sequence of Cohomology Sets

(Next G = cyclic)

$X \xrightarrow{\pi} Y$ map of pointed sets.

$x \in X$, $y \in Y$ (point)

$\text{kernel}(\pi) = \pi^{-1}(y) \subseteq X$.

Motivating example:

C curve$/_k$ with noncommutative $\mathrm{Aut}(C)$, e.g. ss curve over $\mathbb{F}_p$

$H^1(\mathrm{Gal}(\bar{k}/k), \mathrm{Aut}(C))$

$\cong$ twists of C over k

$= \{C'/_k : C'_{\bar{k}} \cong C_{\bar{k}}\}/\sim$

① Connecting map δ:

$$1 \to A \xrightarrow{i} B \xrightarrow{p} C \to 1 \quad \text{exact seq. of nonab } G\text{-modules.}$$

$$H^0(G,C) \xrightarrow{\delta} H^1(G,A)$$

Recall: $H^1(G,A) = \{ a_t : G \to A \text{ s.t. } a_{st} = a_s \cdot s(a_t) \}/\sim$

$H^0(G,A) = A^G$.

$a_s \sim b_s$ if $\exists\, a \in A$ with $b_s = a^{-1} a_s s(a)$

(sort of "twisted conjugation")

[See notes from yesterday for defn of δ.]

Prop: $1 \to H^0(G,A) \xrightarrow{i_0} H^0(G,B) \xrightarrow{p_0} H^0(G,C) \xrightarrow{\delta} H^1(G,A) \xrightarrow{i_1} H^1(G,B) \xrightarrow{p_1} H^1(G,C)$

is exact.

Proof: (we continue to identify $A \subset B$)

- at $H^0(G,A)$: i_0 injective because $A^G \hookrightarrow B^G$
- at $H^0(G,B)$:

  $p_0 \circ i_0 = 0$: We have $(p \circ i)(a) = 0$ for all $a \in A$ so same is true on A^G, B^G, C^G.

  $\ker(p_0) \subseteq \mathrm{im}(i_0)$: $b \in \ker(p_0) \Rightarrow b \in \ker(p) = \mathrm{imag}(i) = A$ ($\ker(p_0) \subseteq B^G$)

  $\Rightarrow b \in A \cap B^G = A^G$, so $b \in \mathrm{im}(i_0)$.

- at $H^0(G,C)$:

$$p_0(B^G) = \{ c \in C^G : \text{ c lifts to an invariant element of } B \}$$

$$\ker(\delta) = \{ c \in C^G : \text{ if } b \mapsto c \text{ then } a_s = b^{-1}s(b) \sim 1 \} \quad \text{in } H^1(G,A).$$

Suppose $c \in \ker(\delta)$, so $c \in C^G$ and $\exists\, b \mapsto c$ s.t. $a_s = b^{-1}s(b) \sim 1$.
Then a_s is trivial iff $a_s \sim 1$ i.e, $\boxed{\exists\, a \in A}$ with

$$a^{-1}b^{-1}.s(b)s(a) = 1 \quad \text{for all } s.$$

$$s(ba) = ba. \quad \text{for all } s$$

$$\text{so } ba \in B^G,$$

$$a_s \sim (ba)^{-1}s(ba) \text{ and } ba \in B^G.$$

But $a \in A$ so $p(ba) = p(b) = c$,
so $ba \longmapsto c$ and $ba \in B^G$ hence $c \in p_0(B^G)$.

- at $H^1(G,A)$: $C^G \xrightarrow{\delta} H^1(G,A) \xrightarrow{i_1} H^1(G,B)$

$\ker(i_1) \subseteq \operatorname{im}(\delta)$: $[a_s] \longmapsto$ trivial

So there's $b \in B$ s.t. $a_s \sim b^{-1}s(b)$ all $s \in G$.
Thus $[a_s] = \delta(p(b))$.

$\operatorname{im}(\delta) \subseteq \ker(i_1)$: $\delta(c)$ given by $a_s = b^{-1}s(b)$ some $b \mapsto c$.
But that means $i_1(a_s) \sim 1$.

- at $H^1(G,B)$: $H^1(G,A) \xrightarrow{i_1} H^1(G,B) \xrightarrow{p_1} H^1(G,C)$.
  - $\operatorname{im}(i_1) \subseteq \ker(p_1)$: $a_s \in H^1(G,A)$ maps to $s \mapsto p_1(i_1(a_s)) = 1$ since $p \circ i = 1$. ✓
  - $\ker(p_1) \subseteq \operatorname{im}(i_1)$: $b_s \in H^1(G,B)$, $(s \mapsto p_1(b_s)) \sim 1$.

Then there's $c \in C$ s.t. $p_1(b_s) = c^{-1}s(c)$.
Modify by lift of c so that $p_1(b_s) = 1$ for all s.
Then $b_s \in A$ for all s, so $[b_s] \in \operatorname{im}(i_1)$.