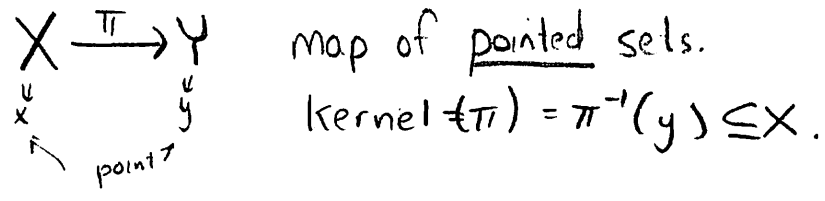


# Long-ish Exact Sequence of Cohomology Sets

(Next  $G = \text{cyclic}$ )



Motivating example:  
 $C$  curve with noncommutative  $\text{Aut}(C)$ , e.g. ss curve over  $\mathbb{F}_p$   
 $H^1(\text{Gal}(\bar{k}/k), \text{Aut}(C))$   
 $\cong$  twists of  $C$  over  $k$   
 $= \{ \mathbb{F}_k \cdot C_{\bar{k}} \cong C_{\bar{k}} \} / \sim$

## ① Connecting map $\delta$ :

$$1 \rightarrow A \xrightarrow{i} B \xrightarrow{p} C \rightarrow 1 \quad \text{exact seq. of nonab } G\text{-modules.}$$

$$H^0(G, C) \xrightarrow{\delta} H^1(G, A)$$

Recall:  $H^1(G, A) = \{ a_t : G \rightarrow A \text{ s.t. } a_{st} = a_s \cdot s(a_t) \} / \sim$

$$H^0(G, A) = AG.$$

$$a_s \sim b_s \text{ if } \exists a \in A \text{ with } b_s = a^{-1} a_s s(a)$$

[See notes from yesterday for defn of  $\delta$ .]

(sort of "twisted conjugation")

Prop:  $1 \rightarrow H^0(G, A) \xrightarrow{i_0} H^0(G, B) \xrightarrow{p_0} H^0(G, C) \xrightarrow{\delta} H^1(G, A) \xrightarrow{i_1} H^1(G, B) \xrightarrow{p_1} H^1(G, C)$   
 is exact.

Proof: (we continue to identify  $AG \subset BG$ )  
 • at  $H^0(G, A)$ :  $i_0$  injective because  $AG \hookrightarrow BG$

• at  $H^0(G, B)$ :  
 $p_0 \circ i_0 = 0$ : We have  $(p \circ i)(a) = 0$  for all  $a \in A$  so same is true on  $AG, B^G, C^G$ .  
 $\ker(p_0) \subseteq \text{im}(i_0)$ :  $b \in \ker(p_0) \Rightarrow b \in \ker(p) = \text{im}(i) = A \Rightarrow b \in A \cap B^G = A^G$ , so  $b \in \text{im}(i_0)$ .

• at  $H^0(G, C)$ :

$$p_0(BG) = \{c \in C^G : c \text{ lifts to an invariant element of } B\}$$

$$\ker(\delta) = \{c \in C^G : \text{if } b \mapsto c \text{ then } a_s = b^{-1}s(b) \sim 1 \text{ } \leftarrow \text{in } H^1(G, A)\}$$

Suppose  $c \in \ker(\delta)$  so  $c \in C^G$  and  $\exists b \mapsto c$  s.t.  $a_s = b^{-1}s(b) \sim 1$ ,

Then  $a_s$  is trivial iff  $a_s \sim 1$  i.e.,  $\exists a \in A$  with

$$a^{-1}b^{-1}s(b)s(a) = 1 \text{ for all } s.$$

$$s(ba) = ba \text{ for all } s$$

$$\text{so } ba \in BG,$$

$$a_s \sim (ba)^{-1}s(ba) \text{ and } ba \in BG.$$

But  $a \in A$  so  $p(ba) = p(b) = c$ ,

so  $ba \mapsto c$  and  $ba \in BG$  hence  $c \in p_0(BG)$ .

• at  $H^1(G, A)$ :  $C^G \xrightarrow{\delta} H^1(G, A) \xrightarrow{i_1} H^1(G, B)$

$\ker(i_1) \subseteq \text{im}(\delta)$ :  $[a_s] \mapsto \text{trivial}$

so there's  $b \in B$  s.t.  $a_s \sim b^{-1}s(b)$  all  $s \in G$ ,

Thus  $[a_s] = \delta(p(b))$ .

$\text{im}(\delta) \subseteq \ker(i_1)$ :

$\delta(c)$  - given by  $a_s = b^{-1}s(b)$  some  $b \mapsto c$ .

But that means  $i_1(a_s) \sim 1$ .

• at  $H^1(G, B)$ :  $H^1(G, A) \xrightarrow{i_1} H^1(G, B) \xrightarrow{p_1} H^1(G, C)$ .

•  $\text{im}(i_1) \subseteq \ker(p_1)$ :  $a_s \in H^1(G, A)$  maps to  $s \mapsto p_1(i_1(a_s)) = 1$   
since  $p_1 = 1$ . ✓

•  $\ker(p_1) \subseteq \text{im}(i_1)$ :  $b_s \in H^1(G, B)$ ,  $(s \mapsto p_1(b_s)) \sim 1$ .

Then there's  $c \in C$  s.t.  $p_1(b_s) = c^{-1}s(c)$ .

Modify by lift of  $c$  so that  $p_1(b_s) = 1$  for all  $s$ .

Then  $b_s \in A$  for all  $s$ , so  $[b_s] \in \text{im}(i_1)$ . □