

582e: Lectures 4<sup>5</sup> Morphisms of Pairs.

§1. Morphisms of Pairs

$$(G, A) \longrightarrow (G', A')$$

$$G' \xrightarrow{f} G$$

$$A' \xleftarrow{g} A$$

such that  $g(f(s') \cdot a) = s' \cdot g(a)$   
for all  $s' \in G', a \in A$

induces:

$$H^q(G, A) \longrightarrow H^q(G', A') \quad \text{for all } q.$$

Why:  $f: G' \rightarrow G$  induces a morphism of the resolutions of  $\mathbb{Z}$  in  $\text{Mod}_G$  and  $\text{Mod}_{G'}$ .

$$\begin{array}{ccccccc} \dots & \longrightarrow & \mathbb{Z}[G' \times G'] & \longrightarrow & \mathbb{Z}[G'] & \longrightarrow & \mathbb{Z} \rightarrow 0 \\ & & \downarrow & & \downarrow f & & \\ \dots & \longrightarrow & \mathbb{Z}[G \times G] & \longrightarrow & \mathbb{Z}[G] & \longrightarrow & \mathbb{Z} \rightarrow 0 \end{array}$$

Hence get morphism of corresponding complexes (in other direction)

$$0 \rightarrow \text{Hom}_G(\mathbb{Z}[G], A) \rightarrow \text{Hom}_G(\mathbb{Z}[G \times G], A) \rightarrow \dots$$

$$0 \rightarrow \text{Hom}_{G'}(\mathbb{Z}[G'], A') \rightarrow \text{Hom}_{G'}(\mathbb{Z}[G' \times G'], A') \rightarrow \dots$$

$$\begin{array}{c} \mathbb{Z}[\varphi: \mathbb{Z}[G'] \rightarrow A] \longrightarrow (g \circ \varphi \circ f : \mathbb{Z}[G'] \xrightarrow{\varphi} A' \xrightarrow{g} A) \\ \searrow f \quad \swarrow g \\ \mathbb{Z}[G] \xrightarrow{\varphi} A \end{array}$$

Morphism of pairs  $(G, A) \rightarrow (G', A')$

induces morphism on cohomology  $H^q(G, A) \rightarrow H^q(G', A')$ .

Useful!!!

§2. Shapiro's Lemma:

Special case:

$G' \leq G$  subgroup

$A'$  —  $G'$ -module.

$A = \text{Hom}_{G'}(\mathbb{Z}[G], A')$

coinduction to make a left  $G$ -module from  $A'$ .

action:  $\varphi: \mathbb{Z}[G] \rightarrow A'$   
 $s \in G$

then  $s \cdot \varphi: t \mapsto \varphi(ts)$

$\varphi(ts)$   
NOT  $\varphi(ts^{-1})$

For fixed  $s$ ,  $\psi_s: \mathbb{Z}[G] \rightarrow \mathbb{Z}[G]$  given by  $t \mapsto ts^{-1}$  is a left  $G$ -module morphism.  
 ~~$\varphi \mapsto \varphi \circ \psi_s$  would thus give a right  $G$ -action since it is precomp, so  $\psi_s$  is a left action.~~

Shapiro's Lemma: For all  $q \geq 0$ ,

$H^q(G, A) \cong H^q(G', A')$

Note:  
Very important in making Eichler Shimura explicit and computable...

Proof:  $L^\bullet \rightarrow \mathbb{Z}$  free  $\mathbb{Z}[G]$ -resolution,

$\Rightarrow L^\bullet \rightarrow \mathbb{Z}$  also free  $\mathbb{Z}[G']$ -resolution, since  $G'$  is a subgroup

We have

$\text{Hom}_G(L^i, A) \cong \text{Hom}_{G'}(L^i, A')$

$\parallel$   
 $\text{Hom}_G(L^i, \text{Hom}_{G'}(\mathbb{Z}[G], A'))$

$b \mapsto \psi(b)(1) \in A'$   
 $s \cdot b \mapsto \psi(s \cdot b)(1)$   
 $= s' \psi(b)(1)$

$b \mapsto (s \mapsto \varphi(b \cdot s^{-1}))$

Conclusion: Same complexes  
So same cohomology.

§3: Inf and Res.

$$H \hookrightarrow G$$

$$H \leq G \quad \text{subgroup}$$

$A$  -  $G$ -module

Morphism of pairs:  $(G, A) \longrightarrow (H, A)$        $H \hookrightarrow G$   
 $A \xleftarrow{id} A$

induces:  $H^q(G, A) \xrightarrow{res_H} H^q(H, A).$

"Restriction homomorphism".

E.g.: in terms of 1-cocycles

$$H^1(G, A) \xrightarrow{res_H} H^1(H, A)$$

$$[f] \longmapsto [f|_H] \quad \text{literally "restriction"}$$

$$f: G \rightarrow A$$

$$f(st) = f(s) + s f(t)$$

Next suppose  $H \trianglelefteq G$ .  $N_G H \rightarrow (G/H, A^H) \rightarrow (G, A)$        $G \rightarrow G/H$   
 $A \rightarrow A^H$

Induces the inflation homomorphism:

$$H^q(G/H, A^H) \xrightarrow{inf} H^q(G, A).$$

Theorem (for later):

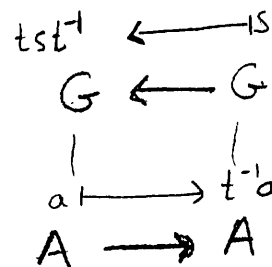
$$0 \rightarrow H^1(G/H, A^H) \xrightarrow{inf} H^1(G, A) \xrightarrow{res} H^1(H, A) \xrightarrow{G/H} H^2(G/H, A^H)$$

is exact.

Full Proof uses spectral sequences.

Suppose  $t \in G$

$s \mapsto tst^{-1}$  inner automorphism of  $G$



Morphism of pairs:  $(G, A) \longrightarrow (G, A)$

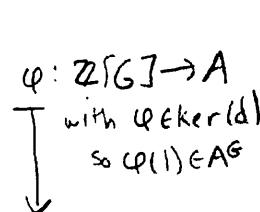
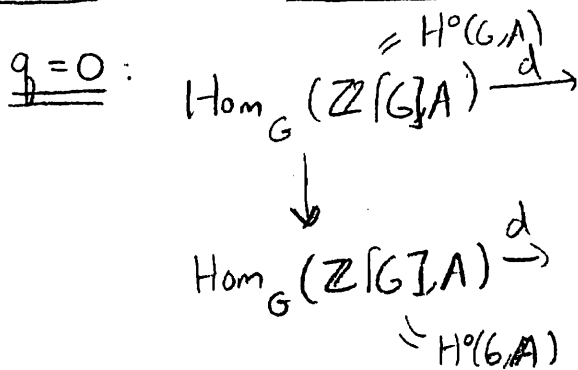
Get  $H^q(G, A) \longrightarrow H^q(G, A)$ .

i.e. a natural action of  $G$  on  $H^q(G, A)$ !

$t \cdot st^{-1}a$        $st^{-1}a$

Prop: This action is trivial, i.e. map  $t \in G$  induces identity map on  $H^q(G, A)$ .

Proof: Use dimension shifting.



$$\begin{aligned}
 & \text{so } (g \circ \varphi \circ f)(1) \\
 &= g(\varphi(t^{-1}t)) \\
 &= t^{-1}\varphi(t^{-1}t) = t^{-1}\varphi(1) \\
 &= \varphi(1) \\
 & \text{since } \varphi(1) \\
 & \text{is } G\text{-invariant.}
 \end{aligned}$$

so maps id for  $q=0$ .

$q > 0$ :  $0 \rightarrow A \rightarrow A^* \rightarrow A' \rightarrow 0$  with  $A^*$  co-induced.

$$H^q(G, A) \cong H^{q-1}(G, A') \text{ for } q \geq 2$$

$$\text{and } H^1(G, A) \cong \text{coker}(H^0(G, A^*) \rightarrow H^0(G, A'))$$

|| functorial.

By induction,  $H^{q-1}(G, A') \cong H^{q-1}(G, A')$

by conj. by  $t$ , so prop follows.

§5. An application of inner automorphisms.

$k$  field of char  $\neq 2$ .

$G = GL_n(k)$ ,  $A = k^n$  equipped with natural  $G$ -action

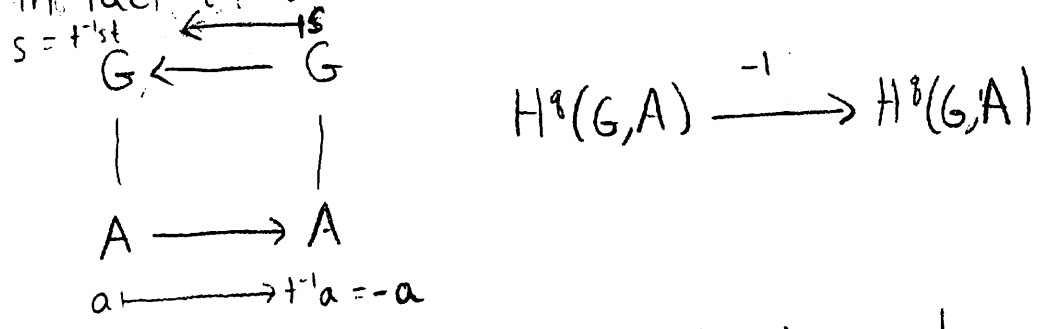
Prop:  $H^q(G, A) = 0$  for all  $q$ . // Finally, an example!

Proof: Let  $t = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \in G$ . Then

We proved above that  $(G, A) \longrightarrow (G, A)$   
induced by conj by  $t$  induces the identity map:

$$H^q(G, A) \longrightarrow H^q(G, A)$$

But in fact  $t$  induces mult. by  $-1$  on  $A$ :



Conclusion: On  $k$ -vector space,  $H^q(G, A)$ , we have  $-1 = 1$ ,  
so  $H^q(G, A) = 0$  for all  $q$ . □

Remark: In the proof we can replace  $G$  by any subgroup  
of  $GL_n(k)$  that contains a nonidentity scalar  $t$ .

Application Elliptic Curves.

$$H^q(\text{Gal}(K(E[p])/K), E[p]) = 0 \quad \forall q$$

whenever  $\bar{\rho}_{E,p} : G_K \twoheadrightarrow \text{Aut}(E[p])$ .

# §6. The Restriction-Inflation Sequence

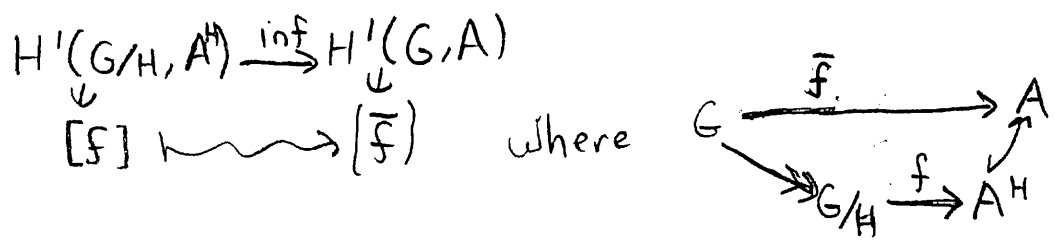
Prop:  $H \triangleleft G$  normal subgroup  
 $A$   $G$ -module

Then get exact sequence:

$$0 \rightarrow H^1(G/H, A^H) \xrightarrow{\text{inf}} H^1(G, A) \xrightarrow{\text{res}} H^1(H, A)$$

Proof: Direct check on 1-cocycles, following book (Atiyah & Wall).

Exactness at  $H^1(G/H, A^H)$ , i.e. injectivity of inf.



$[\bar{f}] = 0 \Rightarrow \bar{F}$  is a coboundary  
 $\Rightarrow a \in A$  with  $\bar{F}(s) = sa - a$ , all  $s \in G$ .

$\bar{F}$  is constant on each coset of  $H$  in  $G$ , so

$$\begin{array}{ccc}
 \bar{F}(s) = \bar{F}(st) & , & \text{all } t \in H \\
 \text{"} & \text{"} & \\
 sa - a & \text{sta} - a & \Rightarrow sa = sta \Rightarrow a = ta \Rightarrow a \in A^H.
 \end{array}$$

Thus  $f$  is also a coboundary (attached to  $a \in A^H$ ), hence  $[f] = 0$ .

Res ∘ Inf = 0:  $H^1(G/H, A^H) \rightarrow H^1(G, A) \rightarrow H^1(H, A)$

i.e.  $\text{im}(\text{Inf}) \in \ker(\text{res})$   $\quad [f] \mapsto [\bar{F}] \mapsto [\bar{F}|_H]$   $f(1,1) = f(1) + 1 \cdot f(1) = f(1) + f(1) \Rightarrow f(1) = 0$

But  $\bar{F}$  constant on cosets of  $H$ , so  $\bar{F}|_H(t) = \bar{F}|_H(1) = 0$ .

continued

Exactness at  $H^1(G, A)$ ; i.e.  $\ker(\text{res}) \subseteq \text{im}(\text{inf})$

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$$H^1(G, A) \longrightarrow H^1(H, A)$$

$$[f] \longmapsto [0]$$

$$f: G \rightarrow A, \exists a \in A \text{ s.t. } f(t) = ta - a \text{ all } t \in H.$$

$$[f] = [f - \varphi_a] \text{ where } \varphi_a(s) = sa - a, \text{ all } s \in G.$$

So may assume  $f|_H = 0$ .

If  $t \in H$ , then

$$f(st) = f(s) + s \cdot \varphi(t) = f(s) \quad (\text{so constant on cosets})$$

so  $f$  defines  $\tilde{f}: G/H \rightarrow A$ , function.

If  $s \in H, t \in G$ , then

$$f(t) = f(st) = 0 + s \cdot f(t) \Rightarrow f(t) \in A^H.$$

↑  
since constant  
on cosets

So  $\tilde{f}: G/H \rightarrow A^H$ , so  $[\tilde{f}] \longmapsto [f]$ .  $\square$

Prop: Let  $q \geq 1$ .

Suppose  $H^i(H, A) = 0$  all  $1 \leq i \leq q-1$ .

Then  $0 \rightarrow H^q(G/H, A^H) \xrightarrow{\text{inf}} H^q(G, A) \xrightarrow{\text{res}} H^q(H, A)$  is exact.

Proof: Dimension shifting. See book.

Next: Cohomology Sets, i.e. case when  $A$  is nonabelian.

See pg's 123-126 of Serre's "Local Fields".