582e: Lectures 4, 5 Morphisms of Pairs

§1. Morphisms of Pairs

$$(G,A) \longrightarrow (G',A')$$

$$\therefore\quad G' \xrightarrow{f} G \qquad \text{such that } g(f(s').a) = s'.g(a)$$

$$A' \xleftarrow{g} A \qquad \text{for all } s' \in G',\ a \in A$$

induces:

$$H^q(G,A) \longrightarrow H^q(G',A') \qquad \text{for all } q.$$

Why: $f: G' \to G$ induces a morphism of the resolutions of $\mathbb{Z}$ in Mod_G and $\mathrm{Mod}_{G'}$.

$$\begin{tikzcd}
\cdots \arrow[r] & \mathbb{Z}[G'\times G'] \arrow[r] \arrow[d] & \mathbb{Z}[G'] \arrow[r] \arrow[d, "f"] & \mathbb{Z} \arrow[r] & 0 \\
\cdots \arrow[r] & \mathbb{Z}[G\times G] \arrow[r] & \mathbb{Z}[G] \arrow[r] & \mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$

Hence get morphism of corresponding complexes (in other direction).

$$\begin{tikzcd}
\cdots \mathrm{Hom}_G(\mathbb{Z}[G], A) \arrow[r] \arrow[d] & \mathrm{Hom}_G(\mathbb{Z}[G\times G], A) \arrow[r] \arrow[d] & \cdots \\
\cdots \mathrm{Hom}_{G'}(\mathbb{Z}[G'], A') \arrow[r] & \mathrm{Hom}_{G'}(\mathbb{Z}[G'\times G'], A') \arrow[r] & \cdots
\end{tikzcd}$$

$$(\varphi: \mathbb{Z}[G] \to A) \longmapsto \left(g\circ\varphi\circ f : \mathbb{Z}[G'] \longrightarrow A'\right)$$

$$\begin{tikzcd}
\mathbb{Z}[G'] \arrow[rr] \arrow[d, bend right, "f"'] & & A' \\
\mathbb{Z}[G] \arrow[r, "\varphi"] & A \arrow[ur, "g"'] &
\end{tikzcd}$$

Morphism of pairs $(G,A) \longrightarrow (G',A')$ induces morphism on cohomology $H^q(G,A) \longrightarrow H^q(G',A')$.

Useful!!!

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§2. Shapiro's Lemma:

Special case:

$G' \le G$ subgroup

A' — G'-module.

$A = \mathrm{Hom}_{G'}(\mathbb{Z}[G], A')$ coinduction to make a left G-module from A'.

action: $\varphi : \mathbb{Z}[G] \to A'$

$s \in G$

then $s.\varphi : t \mapsto \varphi(ts)$

$\varphi(ts)$ NOT $\varphi(ts^{-1})$

~~For fixed s, $\psi_s : \mathbb{Z}[G] \to \mathbb{Z}[G]$ given by $t \mapsto ts^{-1}$ is a left G-module morphism, $\varphi \mapsto \varphi \circ \psi_s$ would thus give a right G action~~ since it is precomp., so is a left action.

Note: Very important in making Eichler Shimura explicit and computable...

Shapiro's Lemma: For all $q \ge 0$,

$$H^q(G, A) \cong H^q(G', A').$$

Proof: $L^\bullet \twoheadrightarrow \mathbb{Z}$ free $\mathbb{Z}[G]$-resolution,

$\Rightarrow L^\bullet \twoheadrightarrow \mathbb{Z}$ also free $\mathbb{Z}[G']$-resolution, since G' is a subgroup.

We have

$$\mathrm{Hom}_G(L^i, A) \cong \mathrm{Hom}_{G'}(L^i, A')$$

$$\| $$

$$\mathrm{Hom}_G(L^i, \mathrm{Hom}_{G'}(\mathbb{Z}[G], A'))$$

$\varphi \mapsto$ $b \mapsto (s \mapsto \varphi(bs^{-1}))$

$\psi \mapsto$ $b \mapsto \psi(b)(1) \in A'$.

$s' b \mapsto \psi(s'b)(1) = s'\psi(b)(1)$

Conclusion: Same complexes so same cohomology.

§3: Inf and Res.

$H \hookrightarrow G$

$H \leq G$ subgroup

A - G-module

Morphism of pairs: $(G, A) \longrightarrow (H, A)$ $\qquad H \hookrightarrow G$, $A \xleftarrow{id} A$

induces: $$H^q(G,A) \xrightarrow{\mathrm{res}_H} H^q(H,A).$$

"Restriction homomorphism".

E.g.: in terms of 1-cocycles

$$H^1(G,A) \xrightarrow{\mathrm{res}_H} H^1(H,A)$$
$$[f] \longmapsto [f|_H] \quad \text{literally "restriction".}$$

$f: G \to A$

$f(st) = f(s) + s f(t)$

Next suppose $H \trianglelefteq G$. Get $(G/H, A^H) \longrightarrow (G, A)$ $\qquad G \to G/H$, $A \hookleftarrow A^H$

Induces the inflation homomorphism:

$$H^q(G/H, A^H) \xrightarrow{\inf} H^q(G,A).$$

Theorem (for later):

$$0 \to H^1(G/H, A^H) \xrightarrow{\inf} H^1(G,A) \xrightarrow{\mathrm{res}} H^1(H,A)^{G/H} \xrightarrow{\mathrm{Tr}} H^2(G/H, A^H)$$

is exact.

Full Proof uses spectral sequences.

Lecture 5:

Errata: action on co-induced module.

§4 Inner Automorphisms:

Suppose $t \in G$

$s \mapsto tst^{-1}$ inner automorphism of G

Morphism of pairs: $(G,A) \longrightarrow (G,A)$

$$\begin{tikzcd} tst^{-1} & s \arrow[l, mapsto] \\ G & G \arrow[l] \\ a \arrow[r, mapsto] & t^{-1}a \\ A \arrow[r] & A \end{tikzcd}$$

Get $H^q(G,A) \longrightarrow H^q(G,A)$.

i.e. a natural action of G on $H^q(G,A)$!

$st^{-1}a$

$tst^{-1}a$

Prop: This action is trivial, i.e. map $t \in G$ induces identity map on $H^q(G,A)$.

Proof: Use dimension shifting.

$q=0$:

$$\begin{tikzcd} H^0(G,A) \arrow[d, phantom, "\cong"] \\ \mathrm{Hom}_G(\mathbb{Z}[G],A) \arrow[r, "d"] \arrow[d] & {} \\ \mathrm{Hom}_G(\mathbb{Z}[G],A) \arrow[r, "d"] \arrow[d, phantom, "\cong"] & {} \\ H^0(G,A) \end{tikzcd}$$

$\varphi: \mathbb{Z}[G] \to A$ with $\varphi \in \ker(d)$ so $\varphi(1) \in A^G$

$\downarrow$

$g \circ \varphi \circ f$ where $f: G \to G$, $f(s) = tst^{-1}$

$g: A \to A$, $g(a) = t^{-1}(a)$

so $(g \circ \varphi \circ f)(1)$
$= g(\varphi(t1t^{-1}))$
$= t^{-1}\varphi(t1t^{-1}) = t^{-1}\varphi(1)$
$= \varphi(1)$ since $\varphi(1)$ is G-invariant.

so map is id for $q=0$.

$q>0$: $0 \to A \longrightarrow A^* \longrightarrow A' \to 0$ with A^* co-induced.

$H^q(G,A) \cong H^{q-1}(G,A')$ for $q \geq 2$

and $H^1(G,A) \cong \mathrm{coker}(H^0(G,A^*) \to H^0(G,A'))$

functorial.

By induction, $H^{q-1}(G,A') \xrightarrow{\cong} H^{q-1}(G,A')$ by conj. by t, so prop follows.

§5. An application of inner automorphisms.

k field of char $\neq 2$.

$G = GL_n(k)$, $A = k^n$ equipped with natural G-action

Prop: $H^q(G,A) = 0$ for all q. // Finally, an example!

Proof: Let $t = \begin{pmatrix} -1 & & 0 \\ & \ddots & \\ 0 & & -1 \end{pmatrix} \in G$. Then

we proved above that $(G,A) \longrightarrow (G,A)$ induced by conj by t induces the identity map:

$$H^q(G,A) \longrightarrow H^q(G,A).$$

But in fact t induces mult. by -1 on A:

$$\begin{tikzcd} G \arrow[d, dash] & G \arrow[l] \arrow[d, dash] \\ A \arrow[r] & A \end{tikzcd} \qquad H^q(G,A) \xrightarrow{-1} H^q(G,A)$$

$s = t^{-1}st \longleftarrow\!\mapsto s$

$a \longmapsto t^{-1}a = -a$

Conclusion: On k-vector space, $H^q(G,A)$, we have $-1 = 1$, so $H^q(G,A) = 0$ for all q. $\blacksquare$

Remark: In the proof we can replace G by any subgroup of $GL_n(k)$ that contains a nonidentity scalar t.

Application Elliptic Curves.

$$H^q(\mathrm{Gal}(K(E[p])/K),\ E[p]) = 0 \quad \forall q$$

whenever $\bar{\rho}_{E,p} : G_K \twoheadrightarrow \mathrm{Aut}(E[p])$.

§6. The Restriction–Inflation Sequence

Prop: $H \triangleleft G$ normal subgroup

A G-module

Then get exact sequence:

$$0 \to H^1(G/H, A^H) \xrightarrow{\text{inf}} H^1(G,A) \xrightarrow{\text{res}} H^1(H,A)$$

Proof: Direct check on 1-cocycles, following book (Atiyah & Wall).

Exactness at $H^1(G/H, A^H)$, i.e. injectivity of inf.

$$\begin{array}{ccc} H^1(G/H, A^H) & \xrightarrow{\text{inf}} & H^1(G,A) \\ \Downarrow & & \Downarrow \\ [f] & \longmapsto & [\bar{f}] \end{array}$$

where

$$\begin{tikzcd} G \arrow[rr, "\bar{f}"] \arrow[dr, two heads] & & A \\ & G/H \arrow[r, "f"] & A^H \arrow[u, hook] \end{tikzcd}$$

$[\bar{f}] = 0 \Rightarrow \bar{f}$ is a coboundary

$\Rightarrow a \in A$ with $\bar{f}(s) = sa - a$, all $s \in G$.

$\bar{f}$ is constant on each coset of H in G, so

$$\bar{f}(s) = \bar{f}(st), \quad \text{all } t \in H$$

$$\underset{sa-a}{\overset{\|}{}} \qquad \underset{sta-a}{\overset{\|}{}} \quad \Rightarrow \quad sa = sta \Rightarrow a = ta \Rightarrow a \in A^H.$$

(all $t \in H$)

Thus f is also a coboundary (attached to $a \in A^H$), hence $[f] = 0$.

Res∘Inf = 0: $H^1(G/H, A^H) \longrightarrow H^1(G,A) \longrightarrow H^1(H,A)$

i.e. $\text{im}(\text{Inf}) \subseteq \ker(\text{res})$

$$[f] \longmapsto [\bar{f}] \longmapsto [\bar{f}|_H].$$

$f(1\cdot 1) = f(1) + 1\cdot f(1)$
$= f(1) + f(1)$
$\Rightarrow f(1) = 0.$

But $\bar{f}$ constant on cosets of H, so $\bar{f}|_H(t) = \bar{f}|_H(1) = 0.$

continued

Exactness at $H^1(G,A)$; i.e. $\ker(\mathrm{res}) \subseteq \mathrm{im}(\mathrm{inf})$

$$H^1(G,A) \longrightarrow H^1(H,A)$$
$$[f] \longmapsto [0]$$

$f : G \to A$, $\exists\, a \in A$ s.t. $f(t) = t.a - a$ all $t \in H$.

$[f] = [f - \varphi_a]$ where $\varphi_a(s) = s.a - a$, all $s \in G$.

So may assume $f|_H = 0$.

If $t \in H$, then

$f(st) = f(s) + s.f(t) = f(s)$ (so constant on cosets)

so f defines $\tilde{f} : G/H \longrightarrow A$; function.

If $s \in H$, $t \in G$, then

$f(t) = f(st) = 0 + s.f(t) \implies f(t) \in A^H$.

since constant on cosets

So $\tilde{f} : G/H \longrightarrow A^H$, so $[\tilde{f}] \longmapsto [f]$. $\blacksquare$

Prop: Let $q \geq 1$.

Suppose $H^i(H,A) = 0$ all $1 \leq i \leq q-1$.

Then $0 \to H^q(G/H, A^H) \xrightarrow{\mathrm{inf}} H^q(G,A) \xrightarrow{\mathrm{res}} H^q(H,A)$ is exact.

Proof: Dimension shifting. See book.

Next: Cohomology Sets, i.e. case when A is nonabelian.

See pg's 123–126 of Serre's "Local Fields".